Vortex filaments and leapfrogging phenomena for Euler equations

Monica Musso

University of Bath

with J. Dávila, M. del Pino, J. Wei

Asia-Pacific Analysis and PDEs Seminar, October 2021

We consider the Euler equations for inviscid incompressible fluids in $\mathbb{R}^n$, $n = 2, 3$

$$
\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} &= \nabla p \quad \text{in} \quad \mathbb{R}^n \times [0, T) \\
\nabla \cdot \mathbf{u} &= 0 \\
\mathbf{u}(x, 0) &= \mathbf{u}_0(x)
\end{aligned}
$$

where the velocity field $\mathbf{u}$ and the pressure p are unknowns.

Stream-vorticity formulation

Let

$$\omega = \nabla \times \mathbf{u}, \quad \mathbf{u} = \nabla \times \psi$$

If $n = 2$

$$\begin{aligned} &\omega_t + (\mathbf{u} \cdot \nabla)\omega = 0 \qquad \text{in} \quad \mathbb{R}^2 \times [0, T), \\ &\psi(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x - y|} \omega(y, t)\, dy \end{aligned} \qquad (SV)$$

If $n = 3$

$$\begin{aligned} &\omega_t + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} = 0 \qquad \text{in} \quad \mathbb{R}^3 \times [0, T), \\ &\psi(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \omega(y, t)\, dy \end{aligned} \qquad (SV)$$

Classical well-posedness results

If $n = 2$: global existence and uniqueness for initial data $\omega_0 \in L^\infty$.
Yudovich (1963), Wolibner (1933), Di Perna-Majda (1987), Delort (1990).

If $n = 3$: for initial data in the space $C^{1,\alpha}$ the Euler equations are well-posed for short time in $C^{1,\alpha}$. Moreover, the solution conserves energy.
Lichtenstein (1925).

The same result holds the context of the Sobolev spaces H^s, $s > \frac{5}{2}$.
Ebin-Marsden, Kato-Lai, Temam.

Aim. Construct solutions with **concentrated vorticity**

$$\nabla \times \mathbf{u} \sim \delta_\Gamma,$$

$$\Gamma = \{p_1, \dots, p_k\} \quad (\text{in} \quad \mathbb{R}^2) \quad \text{or} \quad \Gamma = \gamma(s) \quad (\text{in} \quad \mathbb{R}^3)$$

via **gluing methods**.

I. Concentrated vorticities in 2D Euler flows

Let $\omega_\varepsilon(x,t)$ be a smooth solution, ε-concentrated around a finite number of points $\xi_1(t),\dots,\xi_k(t)$, in the sense

$$\omega_\varepsilon(x,t) \rightharpoonup \omega^s(x,t) = \sum_{j=1}^{k} \kappa_j \delta_{\xi_j(t)}, \quad \text{as} \quad \varepsilon \to 0,$$

then

$$(K) \qquad \dot{\xi}_j = \frac{1}{2\pi} \sum_{i\neq j} \frac{(\xi_i - \xi_j)^\perp}{|\xi_i - \xi_j|^2}, \quad t \in [0,T]$$

Helmholtz (1858), Kirchhoff (1876), Routh (1881), Lagally (1921), C.C. Lin (1941).

Formal derivation of (K)

Assume $\omega^s(x,t) = \sum_{j=1}^k \kappa_j \delta(x - \xi_j(t))$. Then

$$\Psi^s(x,t) = \sum_{j=1}^k \frac{\kappa_j}{4\pi} \log \frac{1}{|x - \xi_j(t)|^2}, \quad -\Delta\Psi^s = \omega^s$$

Compute, with $\mathbf{u} = \nabla \times \Psi^s = \nabla^\perp \Psi^s$,

$$\partial_t \omega^s + \nabla^\perp \Psi^s \cdot \nabla \omega^s = \sum_{j=1}^k \kappa_j \dot{\xi}_j \cdot \nabla \delta_{\xi_j}(x)$$

$$- \Big(\sum_{i=1}^k \frac{\kappa_i}{2\pi} \frac{(x - \xi_i)}{|x - \xi_i|^2}\Big)^\perp \cdot \Big(\sum_{j=1}^k \kappa_j \nabla \delta_{\xi_j}(x)\Big)$$

$$= \sum_{j=1}^{k} \kappa_j \left[\dot{\xi}_j - \sum_{i=1}^{k} \frac{\kappa_i}{2\pi} \frac{(x - \xi_i)^\perp}{|x - \xi_i|^2} \right] \cdot \nabla \delta_{\xi_j}$$

Then $\partial_t \omega^s + \nabla^\perp \Psi^s \cdot \nabla \omega^s \sim 0$ at $x = \xi_j$ if

$$\left[\dot{\xi}_j - \sum_{i \neq j} \frac{\kappa_i}{2\pi} \frac{(\xi_j - \xi_i)^\perp}{|\xi_j - \xi_i|^2} \right] = 0 \quad \forall t \in [0, T) \qquad (K)$$

k-Vortex Desingularization Problem

Given a colissionless solution $\xi(t)$ of System (K) is there a solution $(\omega_\varepsilon, \Psi_\varepsilon)$ of (SV) with

$$\omega_\varepsilon(x,t) \rightharpoonup \omega^s(x,t) = \sum_{j=1}^{k} \kappa_j \delta(x - \xi_i(t)), \quad \varepsilon \to 0$$

YES: Marchioro-Pulvirenti (1983), (1993) following the trajectories
Dávila, del Pino, Musso, Wei (2020) via gluing methods

Dávila, del Pino, Musso, Wei (2020)

We find a solution in the form

$$\Psi_\varepsilon = \Psi_{0\varepsilon} + \psi_\varepsilon, \quad \omega_\varepsilon = \omega_{0\varepsilon} + \phi_\varepsilon, \quad -\Delta\Psi_{0\varepsilon} = \omega_{0\varepsilon}$$

where $\omega_{0\varepsilon}$ and $\Psi_{0\varepsilon}$ are explicit ε-regularizations of

$$\omega^s(x,t) = \sum_{j=1}^{k} \kappa_j \delta(x - \xi_j(t))$$

$$\Psi^s(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{4\pi} \log \frac{1}{|x - \xi_j(t)|^2}, \quad -\Delta\Psi^s = \omega^s$$

and we have control on the ε-smallness of ψ_ε and ϕ_ε in stronger norms.

We choose the regularization

$$\Psi_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{4\pi} \underbrace{\log \frac{1}{|x-\xi_j(t)|^2+\varepsilon^2}}_{=\Gamma_0(\frac{x-\xi_j(t)}{\varepsilon})-4\log\varepsilon},$$

Observe that Γ_0 solves the Liouville equation

$$\Delta_y \Gamma_0 + e^{\Gamma_0} = 0 \quad \text{in} \quad \mathbb{R}^2, \quad U_0(y) = e^{\Gamma_0}$$

with $\int_{\mathbb{R}^2} U_0 = 1$.

Define

$$\omega_{0\varepsilon}(x,t) = \sum_{j=1}^{k} \frac{\kappa_j}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \quad U_0(y) = \frac{1}{\pi(1+|y|^2)^2}\ .$$

to have $\omega_{0\varepsilon} = -\Delta\Psi_{0\varepsilon}$ and

$$\omega_{0\varepsilon} \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta(x-\xi_i(t)), \quad \frac{1}{|\log\varepsilon|} |\nabla^{\perp}\Psi_{0\varepsilon}|^2 \rightharpoonup \sum_{j=1}^{k} \kappa_j^2 \delta(x-\xi_j(t)).$$

U_0 is the *Kaufmann-Scully vortex*.

Theorem (Dávila, del Pino, Musso, Wei, 2020)

Let $\xi(t) = (\xi_1(t), \dots, \xi_k(t))$ *be a colisionless solution of System* **(K)**. *There exists a solution* $(\omega_\varepsilon, \Psi_\varepsilon)$ *of Problem* **(SV)** *of the form*

$$\boxed{\begin{aligned} \omega_\varepsilon(x,t) =& \omega_{0\varepsilon}(x,t) + \phi_\varepsilon(x,t) \\ \Psi_\varepsilon(x,t) =& \Psi_{0\varepsilon}(x,t) + \psi_\varepsilon(x,t) \end{aligned}}$$

where for some $0 < \sigma < 1$ *and all* $(x,t) \in \mathbb{R}^2 \times (0,T)$ *we have*

$$\boxed{\begin{aligned} |\phi_\varepsilon(x,t)| \ &\le\ \varepsilon^\sigma \sum_{j=1}^k \frac{1}{\varepsilon^2} U_0\left(\frac{x-\xi_j}{\varepsilon}\right), \\ |\psi_\varepsilon(x,t)| + \varepsilon |D_x \psi_\varepsilon(x,t)| \ &\le\ \varepsilon^2. \end{aligned}}$$

The question whether or not $T = \infty$ is allowed in the above theorem is an open question, and it strongly related to the stability of concentrated vortices.

Bedrossian-Masmoudi (2015): Couette flow.
Ionescu-Jia (2020): Stability of singular solution $\omega = \delta_0$.
Bedrossian-Coti Zelati-Vicol (2020): linear stability of $\omega = \frac{1}{(1+|y|^2)^2}$.
Ao-Davila-del Pino-Musso-Wei (2020): exact traveling wave solutions

$$\omega_\varepsilon(x,t) \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta(x - (\xi_{j,1} - \alpha t, \xi_{j,2})),$$

$(\xi_1, ..., \xi_k)$ roots of Adler-Moser polynomials.

II. Concentrated vorticity in 3D Euler Flows

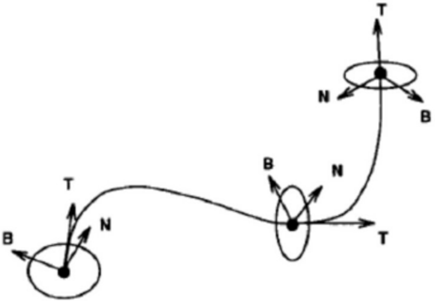
Stream-vorticity formulation ($\omega = \nabla \times \mathbf{u}$ in $\mathbb{R}^3$)

$$\begin{cases} \omega_t + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \quad , \\ \mathbf{u} = \nabla \times \psi, \quad -\Delta\psi = \omega. \end{cases} \tag{SV}$$

Assume the vorticity is concentrated in an ε-neighbourhood of a time evolving curve (filament) $\Gamma(t)$ parametrized by arclength as $\gamma(s,t)$ in $\mathbb{R}^3$

$$\omega(x,t) \sim c\, \delta_{\Gamma(t)}\, \mathbf{T}_{\Gamma(t)}$$

Let us consider a Frenet frame for $\gamma(\cdot, t)$,



$$\gamma_{ss} = \kappa\,\mathbf{N}, \quad \mathbf{B} = \gamma_s \times \mathbf{N}, \quad \gamma_s = \mathbf{T}.$$

N normal and **B** binormal vectors. κ curvature.

da Rios' formal computation (1904): γ evolves by binormal flow

$$\gamma_t = 2\,c|\log\varepsilon|\;(\gamma_s \times \gamma_{ss}) = 2\;c|\log\varepsilon|\,\kappa\,\mathbf{B} \qquad (B)$$

Equivalently, $t = |\log\varepsilon|^{-1}\tau$,

$$\gamma_\tau = 2\;c\,\kappa\,\mathbf{B} \qquad (B)$$

Levi-Civita (1908), Ricca (1991), Betchov (1965), Arms-Hana (1965), Ting-Klein (1991)

Binormal flow: Hasimoto (1972), Banica-Vega (2013-2015), de la Hoz-Kumar-Vega (2020), Gutierrez-Riva-Vega (2003)

Vortex Filament Conjecture

Given a solution to the binormal flow

$$\gamma_\tau = 2\, c\, \kappa\, \mathbf{B} \quad \text{in} \quad [0, T]$$

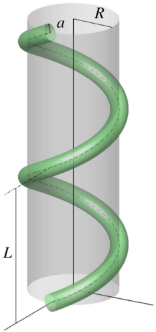
Find a true solution of 3D Euler Flow satisfying

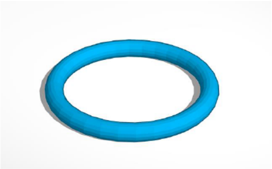
$$\vec{\omega}_\varepsilon(\cdot, |\log \varepsilon|^{-1}\tau) \rightharpoonup c\, \delta_{\Gamma(\tau)} \mathbf{T}_{\Gamma(\tau)}, \quad 0 \le \tau \le T$$

Helmholtz, Kelvin, Da Rios
Benedetto-Caglioti-Marchioro (2015), Jerrard-Seis (2017)
Jerrard-Seis, 2017: **If** vorticities are concentrated around tubes $\gamma(t, s)$, they evolve in weak sense by binormal flow.

This Conjecture is unknown except for some **special cases**.

Examples: a helix whose horizontal section rotates at a constant angular speed and a vertically translating circle are solutions of the bi-normal flow of curves.





Exact solutions for 3D Euler with Helical Symmetry

One known solution of the binormal flow that does not change its form in time is the rotating-translating helix, the curve $\Gamma(\tau)$ parametrized as

$$\gamma(s,\tau) = \begin{pmatrix} R\cos\left(\frac{s-a_1\tau}{\sqrt{h^2+R^2}}\right) \\ R\sin\left(\frac{s-a_1\tau}{\sqrt{h^2+R^2}}\right) \\ \frac{hs+b_1\tau}{\sqrt{h^2+R^2}} \end{pmatrix}, \quad a_1 = \frac{2ch}{h^2+R^2}, \quad b_1 = \frac{2cR^2}{h^2+R^2}.$$

Theorem (Davila-del Pino-Musso-Wei (arXiv 2020))

Let $\Gamma(\tau)$ be the helix. Then there exists a smooth solution $\vec{\omega}_\varepsilon(x,t)$ to 3D Euler, defined for $t \in (-\infty, \infty)$ that does not change form and follows the helix, such that for all τ,

$$\vec{\omega}_\varepsilon(x, \tau |\log \varepsilon|^{-1}) \rightharpoonup c\delta_{\gamma(s,\tau)}\gamma_s(s,\tau)\varepsilon \to 0.$$

This result extends to the situation of several helices symmetrically arranged: $\cup_{j=1}^{k}[R_{2\pi\frac{j-1}{k}}\gamma(s,\tau)]$, $k \geq 2$.

Ettinger-Titi, 2009 : Solutions $\bar{\omega}(x, y, z, t)$ of 3d-Euler with **Helicoidal symmetry** and **velocity orthogonal to the symmetry lines** of the Helix can be obtained by screw motion of vectors formed from a two-variable scalar function $\omega(x + iy, t)$ in the form

$$\bar{\omega}(x, y, z, t) = \omega(e^{i\frac{z}{h}}(x + iy), t) \begin{bmatrix} i(x + iy) \\ h \end{bmatrix}$$

where, for $t = \tau |\log \varepsilon|^{-1}$,

$$|\log \varepsilon|\omega_t + \nabla^{\perp}\psi \cdot \nabla\omega = 0, \qquad -\nabla \cdot (K\nabla\psi) = \omega$$

and

$$K(x, y) = \frac{1}{h^2 + x^2 + y^2} \begin{pmatrix} h^2 + y^2 & -xy \\ -xy & h^2 + x^2 \end{pmatrix}$$

Rotating helicoidal solutions

$$\omega(x+iy,t)=\omega(e^{-i\alpha t}(x+iy)),\quad \psi(x+iy,t)=\psi(e^{-i\alpha t}(x+iy))$$

The problem becomes

$$\begin{cases} \nabla\omega\cdot\nabla\left(\psi-\frac{\alpha}{2}|\log\varepsilon|(x^2+y^2)\right)^{\perp}=0 \\ -\nabla\cdot(K\nabla\psi)=\omega \end{cases}$$

Take $\boxed{\omega=f(\psi-\frac{\alpha}{2}|\log\varepsilon|(x^2+y^2))}$, for some f.

The problem reduces to the elliptic equation

$$-\nabla\cdot(K\nabla\psi)=f(\psi-\frac{\alpha}{2}|\log\varepsilon|(x^2+y^2))=\omega \quad \text{in } \mathbb{R}^2$$

We take $f(s) = \varepsilon^2 e^s$ and look for solutions to

$$S(\Psi) = \nabla \cdot (K\nabla\Psi) + \varepsilon^2 e^{(\Psi - \frac{\alpha}{2}|\log\varepsilon||x|^2)} = 0 \quad \text{in} \quad \mathbb{R}^2,$$

concentrating near a fixed point $q_0 = (x_0, 0)$ with $x_0 > 0$ (corresponds to R in the helices). By homogeneity, we can take $h = 1$. For $r = \sqrt{x_1^2 + x_2^2}$,

$$\begin{aligned} L := \nabla \cdot (K\nabla\Psi) = & \frac{1+x_2^2}{1+r^2}\partial_{x_1x_1} + \frac{1+x_1^2}{1+r^2}\partial_{x_2x_2} - 2\frac{x_1x_2}{1+r^2}\partial_{x_1x_2} \\ & + \left(\partial_{x_1}\left(\frac{1+x_2^2}{1+r^2}\right) - \partial_{x_2}\left(\frac{x_1x_2}{1+r^2}\right)\right)\partial_{x_1} \\ & + \left(\partial_{x_2}\left(\frac{1+x_1^2}{1+r^2}\right) - \partial_{x_1}\left(\frac{x_1x_2}{1+r^2}\right)\right)\partial_{x_2}. \end{aligned}$$

Consider the change of variables

$$x_1 = x_0 + z_1, \quad x_2 = \sqrt{1+x_0^2}\, z_2, \quad z = \varepsilon y.$$

The problem becomes

$$\begin{aligned}(1+x_0^2) S[\Psi](z) = {} & (\partial_{z_1 z_1} + \partial_{z_2 z_2})\Psi + B_0[\Psi] \\ & + \varepsilon^2 e^{\Psi} (1+x_0^2) e^{-\frac{\alpha}{2}|\log \varepsilon| x_0^2} e^{-\frac{\alpha}{2}|\log\varepsilon| \ell_0(z)},\end{aligned}$$

where

$$\begin{aligned} B_0 = {} & -\left(\frac{2x_0}{1+x_0^2} z_1 + O(|z|^2) \right) \partial_{z_1 z_1} + O(|z|^2) \partial_{z_2 z_2} \\ & - \left(2x_0 z_2 + O(|z|^2) \right) \partial_{z_1 z_2} \\ & - \left(x_0 (1 + \frac{2}{1+x_0^2}) + O(|z|) \right) \partial_{z_1} + O(|z|) \partial_{z_2} \\ \ell_0(z) = {} & 2x_0 z_1 + z_1^2 + (1+x_0^2) z_2^2. \end{aligned}$$

For a fixed $\delta > 0$, we define in $|z| < \delta$

$$\Psi_{1\varepsilon}(x) = \frac{\alpha}{2} |\log \varepsilon| x_0^2 - \log(1 + x_0^2) + \Gamma_\varepsilon(z)\,(1 + c_1 z_1)$$

where

$$\Gamma_\varepsilon(z) = \Gamma_0(\frac{z}{\varepsilon}) - 4 \log \varepsilon, \quad \Gamma_0(y) = \log \frac{8}{(1 + |y|^2)^2},$$

is the Liouville profile $\Delta\Gamma_\varepsilon + \varepsilon^2 e^{\Gamma_\varepsilon} = 0$ in $\mathbb{R}^2$. We choose c_1 to eliminate the main part of the error $\frac{4x_0}{1+x_0^2} \frac{z_1}{\varepsilon^2+|z|^2}$

$$c_1 = \frac{1}{2} \frac{x_0}{1 + x_0^2}.$$

We can construct a global approximation Ψ_α, $\Psi_\alpha(x_1, x_2) = \Psi_\alpha(x_1, -x_2)$: in $|z| < \delta$

$$(1+x_0^2)S[\Psi_\alpha](z) = (-\alpha x_0 + 4c_1)|\log\varepsilon| \frac{8\varepsilon^2 z_1}{(\varepsilon^2+|z|^2)^2} + \frac{\varepsilon^2}{(\varepsilon^2+|z|^2)^2} O\left(|z|\log(2+|\frac{z}{\varepsilon}|)\right).$$

and in the region $|z| > \delta$,

$$|(1+x_0^2)S[\Psi_\alpha](z)| \leq C\frac{\varepsilon^2}{(\varepsilon^2+|z|^2)^2} e^{-|z|^2},$$

for some constant $C > 0$.

Inner-outer gluing

We look for Ψ of the form

$$\Psi(x) = \Psi_\alpha(x) + \varphi(x).$$

We choose φ of the form

$$\varphi(x) = \eta_\delta(x)\phi^i\left(\frac{z}{\varepsilon}\right) + \phi^o(x),$$

and $\eta_\delta(x) = \eta\left(\frac{|z|}{\delta}\right)$. Recall that $z_1 = x_1 - x_0, \quad z_2 = \frac{x_2}{\sqrt{1+x_0^2}}$
The equation becomes

$$S[\Psi_\alpha + \varphi] = \mathcal{L}_{\Psi_\alpha}[\varphi] + N_{\Psi_\alpha}[\varphi] + E_\alpha = 0$$

with $E_\alpha = S[\Psi_\alpha]$.

Thus Ψ is a solution if the pair (ϕ^i, ϕ^o) satisfies the system of equations

$$L_x[\phi^i] + \varepsilon^2 f'(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2)\varphi + E_\alpha + N_{\Psi_\alpha}(\varphi) = 0, \quad |z| < 2\delta,$$

and

$$L_x[\phi^o] + (1-\eta_\delta)\left[\varepsilon^2 f'(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2)\phi^o + N_{\Psi_\alpha}(\varphi) + E_\alpha\right]$$
$$+ L_x[\eta_\delta]\phi^i + K_{ij}(x)\partial_{x_i}\eta_\delta\partial_{x_j}\phi^i = 0 \quad \text{in} \quad \mathbb{R}^2,$$

The first problem expressed in $y = \frac{z}{\varepsilon}$

$$\Delta_y\phi^i + f'(\Gamma_0)\phi^i + B[\phi^i] + \mathcal{N}(\varphi) + E_\alpha + (f'(\Gamma_0) + b_0)\phi^o = 0$$

in $B_{\frac{2\delta}{\varepsilon}}$, with $B[\phi] = \partial_{y_i}(b^0_{ij}(\varepsilon y)\partial_j\phi) + b_0(y)\phi$.

We solve this equation, coupled with the outer problem in such a way that ϕ^i has the size of the error E_α with two powers less of decay in y, say

$$(1+|y|)|D_y\phi^i(y)| + |\phi^i(y)| \le \frac{C\varepsilon|\log\varepsilon|}{1+|y|^{1-a}},$$

provided E_α satisfies an orthogonality condition

$$\int E_\alpha \partial_{y_1}\Gamma_0(y)\, dy \sim 0.$$

This condition gives

$$\alpha \sim \frac{2}{1+x_0^2}$$

This choice of α is precisely the number that makes the "rotating helix" a solution of the binormal flow.

Outer Linear Problem:

$$L[\psi^o] + g(x) = 0 \quad \text{in} \quad \mathbb{R}^2,$$

for a bounded function g. We prove that if $\|g\|_\nu := \sup_{x\in\mathbb{R}^2}(1+|x|)^\nu |g(x)| < +\infty$, where $\nu > 2$, then

$$|\psi^o(x)| \le C\|g\|_\nu(1+|x|^2),$$

We write L in polar coordinates

$$L[\psi] = \frac{1}{1+r^2}\left(\frac{1}{r^2}+1\right)\partial_\theta^2\psi + \frac{1}{r}\partial_r\left(\frac{r}{1+r^2}\partial_r\psi\right).$$

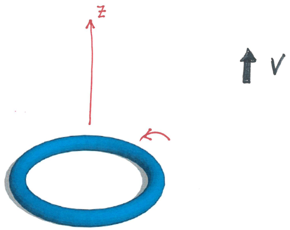
To solve Equation $L(\Psi) = g$ we decompose g and ψ into Fourier modes as

$$g(x) = \sum_{j=-\infty}^{\infty} g_j(r)e^{ji\theta}, \quad \psi(x) = \sum_{j=-\infty}^{\infty} \psi_j(r)e^{ji\theta}, \quad x = re^{i\theta}.$$

Vortex rings

Another known solution of the binormal flow that does not change its form in time is the vortex rings.

$$X(s,\tau) = \begin{pmatrix} R\cos\frac{s}{R} \\ R\sin\frac{s}{R} \\ \frac{2}{R}\tau \end{pmatrix}, \quad 0 < s \le 2\pi R, \tau \ge 0$$



Helmholtz 1858 paper on Vortex Ring

Helmoltz (1858): Here in a circular vortex-filament of very small section in an infinitely extended fluid. the center of gravity of the section has, from the commencement, an approximately and very great velocity parallel to the axis of the vortex-ring, and this is directed towards the side to which the fluid flows through the ring

Fraenkel (1970-1972): exact travelling ring solution

Axisymmetric Euler equations:

$$u(r,z,t) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z$$

$$e_r = \frac{1}{r}(x,y,0)^T, e_\theta = \frac{1}{r}(-y,x,0)^T, e_z = (0,0,1)^T$$

the 3D Euler becomes

$$\begin{cases} |\log\varepsilon| u_t^\theta + u^r u_r^\theta + u^z u_z^\theta = -\frac{1}{r} u^r u^\theta \\ |\log\varepsilon| \omega_t^\theta + u^r \omega_r^\theta + u^z \omega_z^\theta = \frac{2}{r} u^\theta u_z^\theta + \frac{1}{r} u^r \omega^\theta \\ -[\Delta - \frac{1}{r^2}]\psi^\theta = \omega^\theta \end{cases}$$

where ω^θ ans ψ^θ are the θ-component of the vorticity and the stream function.

Introducing new variables

$$U = \frac{u^\theta}{r}, \quad W = \frac{\omega^\theta}{r}, \quad \Psi = \frac{\psi^\theta}{r},$$

$$\begin{cases} |\log \varepsilon| U_t + u^r U_r + u^z U_z = 2U\Psi_z \\ |\log \varepsilon| W_t + u^r W_r + u^z W_z = (U^2)_z \\ -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi = W \quad (r,z) \in (0,\infty) \times \mathbb{R}, \end{cases}$$

where

$$u^r = -r\Psi_z, \; u^z = 2\Psi + r\Psi_r$$

Axisymmetric with no-swirl

Take $U = 0$ so we have the following system

$$\boxed{\begin{array}{c} |\log \varepsilon| r W_t + \nabla_x^\perp (r^2 \Psi) \cdot \nabla_x W = 0, \quad -\Delta_5 \Psi = W, \\[2ex] \Delta_5 := \partial_{rr}^2 + \frac{3}{r}\partial_r + \partial_{zz}^2 \qquad\qquad x = (r, z) \end{array}}$$

Fraenkel (1970-1972), Helmholtz (1858)

Exact traveling ring solutions

$$W(r, z, t) = W(r, z - \alpha t)$$

solve

$$\nabla^\perp \left[r^2 (\psi - \frac{\alpha}{2} |\log \epsilon|) \right] \cdot \nabla W = 0, \quad -\Delta_5 \psi = W$$

Take $W = f(r^2(\psi - \frac{\alpha}{2}|\log \epsilon|))$, then

$$\nabla^\perp \left[r^2(\psi - \frac{\alpha}{2}|\log \epsilon|) \right] \cdot \nabla W$$
$$= \underbrace{\nabla^\perp \left[r^2(\psi - \frac{\alpha}{2}|\log \epsilon|) \right] \cdot \nabla \left[r^2(\psi - \frac{\alpha}{2}|\log \epsilon|) \right]}_{=0} f'(\left[r^2(\psi - \frac{\alpha}{2}|\log \epsilon|) \right]) = 0$$

The problem reduces to

$$-\Delta_5 \psi = f(r^2(\psi - \frac{\alpha}{2}|\log \epsilon|)) = W$$

For a vortex-ring solution we want $\boxed{rW \sim 8\pi \delta_{P_0}}$. Take $P_0 = (r_0, 0)$

For $x = (r, z)$, the Green's function

$$-\Delta_5 G(x, P_0) = 8\pi\delta_{P_0}, \quad G(x, P_0) \to 0 \quad |x| \to \infty$$

can be expanded as

$$G(x, P_0) = \log \frac{1}{|x - P_0|^4}\left(1 - \frac{3}{2r_0}(r - r_0) + H(x; P_0)\right) + K(x; P_0).$$

where

$$\Delta_5\left(\log \frac{1}{|x - P_0|^4} H(x)\right) = -30\frac{(r - r_0)^2}{rr_0|x - P_0|^2} + \frac{9}{2rr_0}\log \frac{1}{|x - P_0|^4},$$

and

$$\Delta_5 K(x) = 0.$$

Fraenkel's solution $(\Psi_\varepsilon, W_\varepsilon)$: in $|x - P_0| < \delta$

$$r_0\Psi_\varepsilon(x) \sim \underbrace{\log \frac{1}{(\varepsilon^2 + |x - P_0|^2)^2}}_{=\Gamma_0(\frac{x-P_0}{\varepsilon}) - 4\log\varepsilon} \left(1 - \frac{3}{2r_0}(r - r_0) + H(x)\right) + K(x).$$

Taking $f(s) = \varepsilon^{2-\alpha r_0} e^{\frac{s}{r_0}}$

$$r_0 W_\varepsilon(x) \sim \frac{1}{\varepsilon^2} U_0\left(\frac{x - P_0}{\varepsilon}\right) \quad U_0(y) = \frac{1}{(1 + |y|^2)^2}$$

Then, in the expanded variables $x = P_0 + \varepsilon y$

$$E_\varepsilon := \varepsilon^2 \left[\Delta_5 \Psi_\varepsilon + f(r^2(\Psi_\varepsilon - \frac{\alpha}{2} |\log \epsilon|)) \right]$$

$$= \varepsilon y_1 U_0 [\frac{3}{2} + \frac{1}{2r_0}(\Gamma_0(y) - 4(2 - \alpha r_0) \log \varepsilon)] + O(\varepsilon^2 (1 + |y|)^{-3})$$

To reduce the error, we write

$$\Psi = \Psi_\varepsilon + \psi(y), \quad y = \frac{x - P_0}{\varepsilon}$$

and we linearize

$$\varepsilon^2 \left[\Delta_5 \Psi + f(r^2(\Psi - \frac{\alpha}{2} |\log \epsilon|)) \right] = \Delta_y \psi + U_0(y)\psi + E_\varepsilon + l.o.t$$

Consider the problem

$$\Delta_y \psi + U_0(y)\psi + E_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^2. \qquad (*)$$

The functions

$$Z_0(y) = 2\frac{1-|y|^2}{1+|y|^2}, \quad Z_1(y) = \frac{y_1}{1+|y|^2}, \quad Z_2(y) = \frac{y_2}{1+|y|^2}$$

solve $\Delta_y \psi + U_0(y)\psi = 0$.

You can solve $(*)$ for ψ in a space of decaying function provided

$$\int_{\mathbb{R}^2} E_\varepsilon(y) Z_j(y)\, dy = 0 \qquad \forall \quad j = 0, 1, 2.$$

Recall

$$E_\varepsilon = \varepsilon y_1 U_0[\frac{3}{2} + \frac{1}{2r_0}(\Gamma_0(y) - 4(2 - \alpha r_0)\log\varepsilon)] + O(\frac{\varepsilon^2}{(1+|y|)^3})$$

Then $\int_{\mathbb{R}^2} E_\varepsilon(y) Z_j(y)\, dy = 0$ automatically $j = 0, 2$.

The orthogonality condition for $j = 1$ gives

$$\alpha = \frac{2}{r_0} + \frac{\beta_\varepsilon}{|\log\varepsilon|}, \quad \beta_\varepsilon = O(1) \quad \text{as} \quad \varepsilon \to 0$$

Single traveling vortex ring solutions

- Existence of a single vortex-ring solution via constrained variational method:Arnold (1964), Fraenkel-Berger (1974), Benjaman (1976), Friedman-Turkington (1981),Burton (1983), Ambrosetti-Struwe (1989), Benedetto-Caglioti-Marchioro (2000)
- The speed and the radius of the vortex ring

$$\alpha \sim \frac{1}{r_0}$$

Nearly Parallel Vortex-Rings

Helmholtz 1858: We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

$$\text{speed} \sim \frac{1}{\text{radius}}$$

Leap-frogging

Aim: mathematically justify the leap-frogging dynamics for 3D axisymmetric Euler flow without swirls.

$$|\log \varepsilon| r W_t + \nabla^{\perp}(r^2\psi)\cdot\nabla W = 0, \quad -\Delta_5\psi = W$$

Formal Derivation of Leap-Frogging

Take two "Fraenkel solutions"

$$W^0_\varepsilon(x,t) = \sum_{j=1}^{2} \frac{1}{\varepsilon_j^2 r_j} U_0\Big(\frac{x-P_j}{\varepsilon_j}\Big), \quad x = (r,z)$$

$$\Psi^0_\varepsilon(x,t) = \sum_{j=1}^{2} \frac{1}{r_j} \log \frac{1}{(\varepsilon_j^2 + |x-P_j|^2)^2} \left[1 - \frac{3}{2r_j}(r-r_j) + H(x,P_j)\right] + \frac{G(x;P_j)}{r_j}$$

where

$$P_j = P_j(t) = (r_j(t), z_j(t)), \quad \varepsilon_j = \varepsilon_j(t)$$

One has

$$\Delta_5 \Psi^0_\varepsilon - W^0_\varepsilon \sim 0.$$

Consider now

$$S[W,\psi] := |\log\varepsilon| r W_t + \nabla^\perp (r^2\psi)\cdot\nabla W$$

We choose

$$\varepsilon_j^2(t) r_j(t) = \varepsilon^2 \quad \forall \quad j = 1,2.$$

At $x = P_1 + \epsilon_1 y$, we have

$$\varepsilon_1^4 S[W_\varepsilon^0, \psi_\varepsilon^0] = \varepsilon_j \left[-|\log \varepsilon| \partial_t P_1 + \Theta_1(P_1, P_2) \right] \cdot \nabla U_0(y)$$

$$+ O(\frac{\varepsilon^2 |\log \varepsilon|}{1 + |y|^4})$$

where

$$\Theta_1(P) = -4 \frac{(P_1 - P_2)^\perp}{|P_1 - P_2|^2} + 2 \frac{P_1 \cdot \mathbf{e_1}}{r_1^2} |\log \varepsilon| \begin{pmatrix} 0 \\ 1 \end{pmatrix} + R_j(P)$$

with

$$|R_j(P)|(t) = O(1), \quad \text{as} \quad \varepsilon \to 0.$$

We obtain

$$\partial_t P_1 - 2\frac{P_1 \cdot \mathbf{e_1}}{r_1^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{4}{|\log \epsilon|}\frac{(P_1 - P_2)^\perp}{|P_1 - P_2|^2} + O(\frac{1}{|\log \epsilon|}) = 0$$

$$\partial_t P_2 - 2\frac{P_2 \cdot \mathbf{e_1}}{r_2^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{4}{|\log \epsilon|}\frac{(P_2 - P_1)^\perp}{|P_1 - P_2|^2} + O(\frac{1}{|\log \epsilon|}) = 0$$

Fraenkel's (1972) single-ring traveling:

$$\partial_t P_1 - 2\frac{P_1 \cdot \mathbf{e_1}}{r_1^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \Longrightarrow \begin{array}{l} \partial_t r_1 = 0 \\ \partial_t z_1 = 2\frac{P_1 \cdot \mathbf{e_1}}{r_1^2} \end{array}$$

$$r_1 = r_0, \quad z_1 = z_0 + \frac{2t}{r_0}$$

Let

$$P_i = \left(r_0 + \frac{r(b_i(t))}{\sqrt{|\log \epsilon|}},\ z_0 + \frac{t}{r_0} + \frac{z(b_i(t))}{\sqrt{|\log \epsilon|}} \right)$$

We obtain that $b_i(t) = (r(b_i(t)), z(b_i(t)))$ satisfies the following LeapFrogging dynamics

$$(LeapFrog) \qquad \begin{cases} \dot{b}_i(t) = \sum_{j \neq i} \frac{(b_i - b_j)^\perp}{\|b_i - b_j\|^2} - \frac{r(b_i)}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ b_i(0) = b_i^0 \end{cases}$$

Theorem [Dávila, del Pino, Musso, Wei, 2021] Let $P(t) = (P_1(t), \dots, P_N(t))$ be a collision-less solution of System *(LeapFrog)*

$$P_i = \left(r_0 + \frac{r(b_i(t))}{\sqrt{|\log \epsilon|}}, z_0 + \frac{t}{r_0} + \frac{z(b_i(t))}{\sqrt{|\log \epsilon|}} \right) \quad \text{in} \quad (0, T)$$

Then there exists a solution ω_ε of 3D axi-symmetric Euler flow (without swirl) of the form

$$\boxed{\begin{aligned} W_\varepsilon(x,t) &\sim \sum_{j=1}^{N} \frac{1}{r_j \varepsilon_j^2} U_0\left(\frac{(r,z) - P_j}{\varepsilon_j} \right) \\ \Psi_\varepsilon(x,t) &\sim \sum_{j=1}^{N} \frac{1}{r_j} \log \frac{1}{(\varepsilon_j^2 + |x - P_j|^2)^2} \left[1 - \frac{3}{2r_j}(r - r_j)\right] \end{aligned}}$$

Ingredients in the construction:

• Improvement of the approximation in powers of ε using elliptic and transport equations.

• Setting up the problem as a coupled system of inner problems near the singularities and and an outer problem more regular (the inner-outer gluing scheme)

• A priori estimates to solve by a continuation (degree) argument.

Remarks

1. Klein-Majda-Damodaran (1995) formally derived the LeapFrogging dynamics.

2. Jerrard-Smets (2018): gave the first mathematical justification of leapfrogging in three-dimensional Gross-Pitaeskii equation

$$iu_t - \Delta u = \frac{1}{\varepsilon^2}(1 - |u|^2)u \quad \text{in} \quad \mathbb{R}^3$$

$u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2$

3. The gluing approach we developed will be useful for the Vortex Filament Conjecture.

Related results: Gallay-Smets (2018) (spectrum analysis for the vortex line filament)

Thanks for your attention