SUMS PROBLEM COMPETITION, 2002

SOLUTIONS

1. Let us write

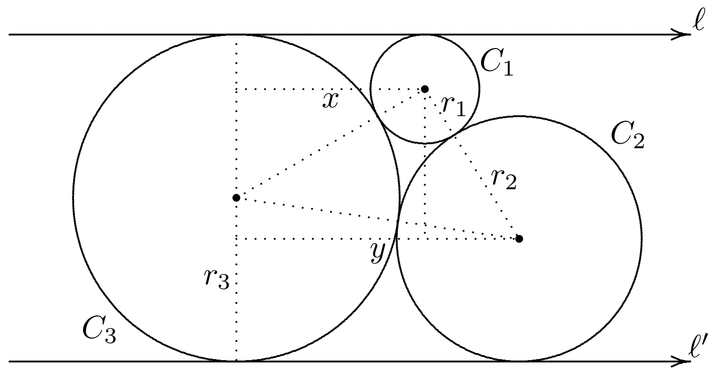
$$x_n = 9^{9^{\cdot^{\cdot^{\cdot^{9}}}}} \qquad (n\ 9'\text{s}) \quad \text{and} \quad y_n = 10^{10^{\cdot^{\cdot^{\cdot^{10}}}}} \qquad (n\ 10'\text{s}).$$

We claim that $x_{10} > y_9$. In fact, we'll show that $x_{n+1} > y_n$ for all $n \geq 1$. To do this, we'll show that $x_{n+1} \geq y_n^2$ for all $n \geq 1$, which implies the desired result, because $y_n \geq 10 > 1$ obviously holds for all $n \geq 1$. We prove that $x_{n+1} \geq y_n^2$ by induction on n. If $n = 1$, then $x_{n+1} = 9^9 = 387420489 > 100 = y_n^2$. Now let $n > 1$ and assume that $x_n \geq y_{n-1}^2$. Then

$$x_{n+1} = 9^{x_n} = e^{x_n \ln(9)} \quad \text{and} \quad y_n^2 = \left(10^{y_{n-1}}\right)^2 = 10^{2y_{n-1}} = e^{2y_{n-1}\ln(10)}.$$

So what we need to prove is that $x_n \ln(9) \geq 2y_{n-1}\ln(10)$. But the induction hypothesis tells us that $x_n \geq y_{n-1}^2$. So it is enough to show that $y_{n-1}^2 \ln(9) \geq 2y_{n-1}\ln(10)$, or equivalently, that $y_{n-1} \geq 2\ln(10)/\ln(9) = 2.095\cdots$. But this is clear, because $y_{n-1} \geq 10$.

2. Consider the following diagram, and draw the indicated dotted lines:



By Pythagoras's Theorem we have

$$(r_1 + r_3)^2 = (r_3 - r_1)^2 + x^2,$$

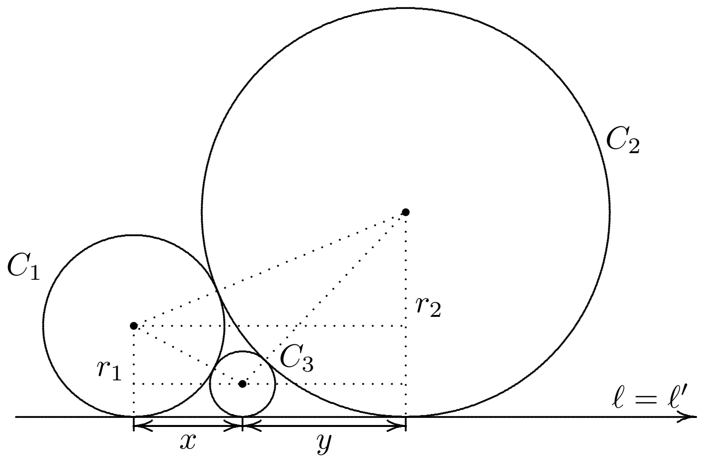
so that $x^2 = 4r_1r_3$. Similarly, $(r_2 + r_3)^2 = (r_3 - r_2)^2 + y^2$, so that $y^2 = 4r_2r_3$. Also,

$$(r_1 + r_2)^2 = (2r_3 - r_1 - r_2)^2 + (y - x)^2,$$

from which, using $x^2 = 4r_1r_3$ and $y^2 = 4r_2r_3$, we obtain the equation $xy = 2r_3^2$. From these last three equations we get

$$r_3 = 2\sqrt{r_1r_2}.$$

There is a second configuration of the three circles when $\ell = \ell'$:



By Pythagoras's Theorem again, applied to three right triangles indicated by dotted lines, we have

$$x^2 + (r_1 - r_3)^2 = (r_1 + r_3)^2$$
$$y^2 + (r_2 - r_3)^2 = (r_2 + r_3)^2$$
$$(x + y)^2 + (r_2 - r_1)^2 = (r_1 + r_2)^2$$

The first and second of these tell us that $x = 2\sqrt{r_1 r_3}$ and $y = 2\sqrt{r_2 r_3}$. Substituting these into the third equation, a little algebra shows that

$$r_3 = \frac{r_1 r_2}{\left(\sqrt{r_1} + \sqrt{r_2}\right)^2} \quad \text{or} \quad r_3 = \frac{r_1 r_2}{\left(\sqrt{r_1} - \sqrt{r_2}\right)^2},$$

where the second possibility only occurs if $r_1 \neq r_2$.

3. Given a circle C with centre O and radius R, we define the *power* of a point P with respect to C to be the quantity $d^2 - R^2$, where $d = |OP|$. It is easy to see that if a line through P meets C at points A and B, then $|PA|\,|PB| = |d^2 - R^2|$.

Now suppose that C_1 and C_2 are two different circles in a plane, with radii R_1 and R_2, respectively. Then the locus of points P with equal power with respect to both circles is a straight line perpendicular to the line joining the centres of C_1 and C_2. For we may assume that our plane is coordinatized, and that the centre of C_j is at the point $(a_j, 0)$ of the x-axis. The condition on the coordinates (x, y) of P is that

$$(x - a_1)^2 + y^2 - R_1^2 = (x - a_2)^2 + y^2 - R_2^2.$$

A little algebra reduces this equation to

$$x = \frac{R_1^2 - R_2^2 - a_1^2 + a_2^2}{2(a_2 - a_1)},$$

the equation of a line perpendicular to the x-axis.

Notice that when the two circles C_1 and C_2 intersect, then the line of the previous paragraph must pass through the points of intersection, because such points have power 0 with respect to both circles.

We now solve the stated problem. Let D be the point on the line $\overrightarrow{BC}$ through B and C such that $\overrightarrow{AD}$ is perpendicular to BC. Similarly, let E be the point on AC such that $\overrightarrow{BE}$ is perpendicular to $\overrightarrow{AC}$. Then AD and BE meet at H. Then the triangles $\triangle AEH$ and $\triangle BDH$ are clearly similar, and so

$$|HA|\,|HD| = |HE|\,|HB|. \tag{1}$$

Now the circle C_1 with diameter AX passes through D (as well as A and X) since $\angle ADX$ is a right angle. Similarly, the circle C_2 with diameter BY passes through E, B and Y. Equation (1) says that the powers of H with respect to C_1 and C_2 are equal. By the observation in the previous paragraph, P and Q have the same property. So P, Q and H are collinear, by the fact proved in the second paragraph.

4. Here is a somewhat informal solution of the somewhat informally stated problem. The probability that the pair is relatively prime is

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right), \tag{1}$$

since $1-1/p^2$ is the proportion of pairs (m, n) with p not a common factor. It is well-known that the value of this infinite product is $1/\zeta(2)$, where $\zeta(z)$ is the Riemann zeta function. It is also well-known that $\zeta(2) = \pi^2/6 < 2$. So the stated probability is $6/\pi^2 > 1/2$, and the statement is true.

One can show that the value of the product (1) is greater than $1/2$ without any knowledge of the zeta function as follows. The first three factors in (1) multiply to

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right) = \frac{16}{25}.$$

The product of the remaining factors is

$$\prod_{\substack{p \text{ prime} \\ p \geq 7}} \left(1 - \frac{1}{p^2}\right) \geq \prod_{n=7}^{\infty}\left(1 - \frac{1}{n^2}\right) = \frac{6}{7},$$

since the last product is a "collapsing" or "telescoping" product. Hence the value of the product (1) greater than $(16/25)(6/7) = 96/175 = 0.548\cdots > 1/2$.

The question was stated rather imprecisely, because there is no uniform distribution on the set of integers, and it is unclear what it means that the integers are picked "at random". To make the question more precise, let $n \geq 1$ be an integer, and let $P(n)$ be the probability that two integers, chosen at random from $\{1, \ldots, n\}$. One can show that

$$P(n) = \frac{1}{\zeta(2)} + O\big(\frac{\ln n}{n}\big)$$

(see J.E. Nyman, Journal of Number Theory, (1972) pp 469–473). Hence the precise assertion is that $\lim_{n\to\infty} P(n) = 1/\zeta(2) > 1/2$.

5. We claim that the nonzero polynomials with the property

$$P(X)P(-X) = P(X^2) \tag{1}$$

are precisely those of the form

$$X^m \prod_{\substack{n\geq 1:\\ n \text{ odd}}} (\Phi_n(X))^{r_n}, \tag{2}$$

where $m \geq 0$ is even and where $\Phi_n(X)$ be the n-th cyclotomic polynomial, and the r_n's are nonnegative integers, only finitely many of which are nonzero. Here we define

$$\Phi_n(X) = (X-\alpha_1)\cdots(X-\alpha_r),$$

where $\alpha_1, \dots, \alpha_r$ are the primitive n-th roots of unity, that is, the complex numbers α which satisfy $\alpha^n = 1$, but $\alpha^k \neq 1$ if $1 \leq k < n$. It is easy to see that the α_r's are just the numbers

$$e^{2\pi i j/n}, \quad \text{where } 1 \leq j < n \text{ and } \gcd(j,n) = 1.$$

For example, $\Phi_1(X) = X - 1$, $\Phi_2(X) = X + 1$, and

$$\Phi_3(X) = (X - e^{2\pi i/3})(X - e^{4\pi i/3}) = X^2 + X + 1.$$

It is well-known that $\Phi_n(X)$ has integer coefficients, and is irreducible over the rationals. The degree of $\Phi_n(X)$, denoted r above, is $\varphi(n)$, the number of integers j satisfying $1 \leq j < n$ and $\gcd(j,n) = 1$.

Let us first show that if n is odd, then $P(X) = \Phi_n(X)$ has property (1).

Step 1: The degree $\varphi(n)$ of $\Phi_n(X)$ is even. For if $n = p_1^{m_1}\cdots p_r^{m_r}$ is the prime decomposition of n, then the p_i's are odd. Since $\varphi(p^m) = p^{m-1}(p-1)$ and

$$\varphi(n) = \varphi(p_1^{m_1})\cdots\varphi(p_r^{m_r}),$$

we see that $\varphi(n)$ is even.

Step 2: If $\alpha_1, \dots, \alpha_r$ are the distinct primitive n-th roots of 1, then $\alpha_1^2, \dots, \alpha_r^2$ are also these roots, in some order. For if $\alpha = e^{2\pi i j/n}$ where $\gcd(j,n) = 1$, then $\alpha^2 = e^{2\pi i(2j)/n}$, and $\gcd(2j,n) = 1$. If $\alpha_j^2 = \alpha_k^2$, say, then $\alpha_j = \alpha_k$ or $\alpha_j = -\alpha_k$. The latter possibility cannot happen, as then $1 = \alpha_j^n = (-\alpha_k)^n = (-1)^n = -1$. So the numbers $\alpha_1^2, \dots, \alpha_r^2$ are distinct, and so they are the distinct primitive n-th roots of 1.

Step 3: If $P(X) = \Phi_n(X)$, n odd, then by Steps 1 and 2,

$$P(X)P(-X) = \prod_{j=1}^{r}(X-\alpha_j)\prod_{j=1}^{r}(-X-\alpha_j) = (-1)^r\prod_{j=1}^{r}(X^2-\alpha_j^2) = \prod_{j=1}^{r}(X^2-\alpha_j) = P(X^2).$$

Now we are ready to prove our claim that all polynomials $P(X)$ with property (1) have the form (2). It is clear that X^m has the property (1) if m is even. Also, if $P(X)$ and $Q(X)$ satisfy (1), then so does $P(X)Q(X)$. It follows that any polynomial of the form (2) has property (1).

Conversely, if $P(X)$ has property (1), we show that $P(X)$ has the form (2) by induction on the degree of $P(X)$. If this degree is zero, then $P(X)$ is a constant c, say, and $c^2 = c$, so that $c = 1$ (as we are assuming that $P(X)$ is not the zero polynomial). Thus $P(X)$ has the form (2), with $m = 0$ and all the r_n's equal to 0.

Now suppose that $P(X)$ is nonconstant, and let α be a complex root of $P(X)$. Then $P(\alpha^2) = P(\alpha)P(-\alpha) = 0$, so that α^2 is a root too. Repeating this, we see that

$$\alpha, \alpha^2, \alpha^4, \ldots$$

are all roots of $P(X)$. But polynomials only have finitely many roots, and so there must be $1 \leq j < k$ so that $\alpha^{2^j} = \alpha^{2^k}$. Hence

$$(\alpha^{2^j})^{2^{k-j}-1} = 1.$$

Let $\beta = \alpha^{2^j}$. Then β is a root of $P(X)$ and $\beta^n = 1$ for some odd number n, namely $n = 2^{k-j} - 1$. Let n be the smallest positive odd integer such that $\beta^n = 1$. Then β is a primitive n-th root of 1. For if there is a k such that $1 \leq k < n$ and $\beta^k = 1$, then $\beta^{n-k} = 1$ too. As either k or $n - k$ is odd, and both are less than n, we would have a contradiction to the definition of n. So β is a root of $\Phi_n(X)$. Since $\Phi_n(X)$ is irreducible over the rationals, it is the minimal polynomial of β and so $\Phi_n(X)$ divides $P(X)$. Writing $P(X) = \Phi_n(X)Q(X)$, it is easy to check that $Q(X)$ satisfies property (1), and so, by induction, $Q(X)$ has the form (2). Thus $P(X)$ has the form (2), with one more copy of $\Phi_n(X)$ than $Q(X)$ has.

6. We show more generally that if we have n squares $S_1, \ldots, S_n$ of side lengths $x_1 \geq \cdots \geq x_n$, respectively, and if

$$\sum_{j=1}^{n} x_j^2 \geq ab + (a+b)x_1,$$

then the squares can cover a rectangle of size $a \times b$, where $a \leq b$, say. This is done by induction on n. If $n = 1$, then $x_1^2 \geq ab + (a+b)x_1$ shows that $(x_1 - a)(x_1 - b) \geq 2ab > 0$ and so $x_1 > b$ and the statement is obvious.

Suppose that $n > 1$ and that the result has been proved for any rectangle and for any smaller n. Imagine a rectangle R of base b and height a. Let j be the smallest index such that $x_1 + x_2 + \cdots + x_j > a$. Imagine the j squares $S_1, \ldots, S_j$ stacked on top of each other, with their left sides on the left side of R. The part of R not covered by $S_1, \ldots, S_j$ is contained in a rectangle R' of size $(b - x_j) \times a$. We claim that the remaining squares $S_{j+1}, \ldots, S_n$ cover R'. By the induction hypothesis, it is enough to show that

$$\sum_{i=j+1}^{n} x_i^2 \geq a(b - x_j) + (a + b - x_j)x_{j+1}.$$

To see this, note that

$$x_1^2 + \cdots + x_{j-1}^2 \leq (x_1 + \cdots + x_{j-1})x_1 \leq ax_1,$$

so by our hypothesis,

$$\sum_{i=j+1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 - x_j^2 - \sum_{i=1}^{j-1} x_i^2 \geq ab + (a+b)x_1 - x_j^2 - ax_1$$
$$\begin{aligned} &= ab + bx_1 - x_j^2 \\ &\geq ab + bx_j - x_j^2 \\ &= a(b - x_j) + (a + b - x_j)x_j \\ &\geq a(b - x_j) + (a + b - x_j)x_{j+1}. \end{aligned}$$

7. A simple induction shows that for $x \geq 0$ and $n = 1, 2, \ldots$,

$$\varphi(x+n) = \varphi(x) + \frac{1}{x+1} + \cdots + \frac{1}{x+n},$$

and so in particular

$$\varphi(n) = \frac{1}{1} + \cdots + \frac{1}{n}.$$

Next we show that $\varphi(x)$ is an increasing function. For if $0 \leq x < y$, choose an integer $n \geq 1$ so that $y < x + n$. Then y is between x and $x + n$, and so $y = tx + (1 - t)(x + n)$ for some $t \in (0, 1)$. Thus

$$\begin{aligned} \varphi(y) &\geq t\varphi(x) + (1-t)\varphi(x+n) \\ &= t\varphi(x) + (1-t)\big(\varphi(x) + \frac{1}{x+1} + \cdots + \frac{1}{x+n}\big) \\ &= \varphi(x) + (1-t)\big(\frac{1}{x+1} + \cdots + \frac{1}{x+n}\big) \\ &> \varphi(x). \end{aligned}$$

We make no further use of the convexity of φ. For $n = 0, 1, \ldots$, let

$$I_n = \int_n^{n+1} \varphi(x)\, dx \quad \text{and} \quad J_n = \int_n^{n+1} x\varphi(x)\, dx$$

We need to evaluate I_0 and J_0. Now

$$\begin{aligned} I_{n+1} = \int_{n+1}^{n+2} \varphi(x)\, dx = \int_n^{n+1} \varphi(x+1)\, dx &= \int_n^{n+1} \big(\varphi(x) + \frac{1}{x+1}\big)\, dx \\ &= I_n + \ln(n+2) - \ln(n+1). \end{aligned}$$

A simple induction now shows that $I_n = I_0 + \ln(n + 1)$ for all n. Also, since $\varphi(x)$ is increasing, we have

$$\varphi(n) = \int_n^{n+1} \varphi(n)\, dx \leq \int_n^{n+1} \varphi(x)\, dx \leq \int_n^{n+1} \varphi(n+1)\, dx = \varphi(n+1).$$

Subtracting $\ln(n + 1)$ throughout, we see that $I_0 = I_n - \ln(n + 1)$ satisfies

$$\varphi(n) - \ln(n+1) \leq I_0 \leq \varphi(n+1) - \ln(n+1). \tag{1}$$

But it is well-known that $\varphi(n) - \ln(n) \to \gamma$ as $n \to \infty$, and since $\ln(n + 1) - \ln(n) = \ln(1 + 1/n) \to 0$, it follows that both left and right extremes of (1) tend to γ as $n \to \infty$. Thus $I_0 = \gamma$.

To evaluate J_0, start by observing that

$$\begin{aligned} J_{n+1} = \int_{n+1}^{n+2} x\varphi(x)\,dx = \int_n^{n+1} (x+1)\varphi(x+1)\,dx &= \int_n^{n+1} (x+1)\big(\varphi(x) + \frac{1}{x+1}\big)\,dx \\ &= J_n + I_n + 1. \end{aligned}$$

Using $I_n = I_0 + \ln(n+1) = \gamma + \ln(n+1)$, a simple induction shows that

$$J_n = J_0 + n\gamma + n + \ln(n!) \quad \text{for } n = 0, 1, \dots \tag{2}$$

But

$$J_n = \int_n^{n+1} x\varphi(x)\,dx = \int_n^{n+1} (x-n)\varphi(x)\,dx + nI_n,$$

and, since $\varphi(x)$ is increasing,

$$\begin{aligned} \frac{1}{2}\varphi(n) = \varphi(n)\int_n^{n+1}(x-n)\,dx \le \int_n^{n+1}(x-n)\varphi(x)\,dx \le \varphi(n+1)\int_n^{n+1}(x-n)\,dx& \\ = \frac{1}{2}\varphi(n+1)& \\ = \frac{1}{2}\big(\varphi(n) + \frac{1}{n+1}\big).& \end{aligned}$$

Thus

$$\frac{1}{2}\varphi(n) + nI_n \le J_n \le \frac{1}{2}\varphi(n) + nI_n + \frac{1}{2(n+1)},$$

so that $J_n = \frac{1}{2}\varphi(n) + nI_n + e_n$ for some e_n which tends to 0 as $n \to \infty$. So

$$J_n = \frac{1}{2}\varphi(n) + n\big(\gamma + \ln(n+1)\big) + e_n = \frac{1}{2}\varphi(n) + n\big(\gamma + \ln(n)\big) + 1 + e_n', \tag{3}$$

where $e_n' \to 0$ as $n \to \infty$.

By Stirling's Formula, we can write

$$\ln(n!) = \frac{1}{2}\ln(2\pi) - n + \big(n + \frac{1}{2}\big)\ln(n) + e_n'',$$

where e_n'' also tends to 0 as $n \to \infty$. Combining this with (2) and (3), we get

$$\begin{aligned} J_0 &= \Big(\frac{1}{2}\varphi(n) + n\big(\gamma + \ln(n)\big) + 1 + e_n'\Big) - \Big(n\gamma + n + \frac{1}{2}\ln(2\pi) - n + \big(n + \frac{1}{2}\big)\ln(n) + e_n''\Big) \\ &= \frac{1}{2}\big(\varphi(n) - \ln(n)\big) + 1 - \frac{1}{2}\ln(2\pi) + e_n' - e_n'' \\ &\to \frac{\gamma}{2} + 1 - \frac{1}{2}\ln(2\pi) \quad \text{as } n \to \infty. \end{aligned}$$

Hence

$$J_0 = \frac{\gamma}{2} + 1 - \frac{1}{2}\ln(2\pi).$$

8. (i) An example of face transitive polygon which is not vertex transitive is a "dipyramid" X, obtained from taking two pyramids whoses bases are regular n-gons, $n \neq 4$, and joining them at their bases. The faces are all isosceles triangles of the same size, and so either a rotation about the axes joining the top and bottom vertices, T and B, say, or a reflection in the joined bases of the pyramids, followed by such a rotation, maps any face to any other. So X is face transitive. But it is not vertex transitive, because the valency (number of neighbouring vertices) of both T and B is n, but the valency of all the vertices on the common base is 4, and so no symmetry could send T or B to one of these latter vertices.

(ii) We can make a "prism" by taking two regular n-gons, one above the other, and joining corresponding vertices by vertical edges. Then this polyhedron is vertex transitive, but if $n \neq 4$, it is not face transitive, because two of its faces have n sides, while the others have 4 sides.

(iii) Consider a (non-regular) tetrahedron X in which opposite sides have the same length. So two opposite sides have length a, say, two have length b, and two have length c. Then each face is a triangle having one side of each of the lengths a, b and c. If a, b and c are not all equal, then X cannot be edge-transitive. On the other hand, X is both vertex- and face-transitive. A concrete realization of such a tetrahedron is as follows: take the vertices $A = (-1, 0, 1)$, $B = (-1, 0, -1)$, $C = (1, 1, 0)$ and $D = (1, -1, 0)$ in $\mathbb{R}^3$. Here we have $|AB| = |CD| = 2$, and the other four edge lengths equal to $\sqrt{6}$. Using the linear transformations given by the matrices

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we can easily see that X is both vertex and face transitive.

(iv) Let X be a polyhedron, with vertex, edge and face sets $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$, respectively. Let G be the group of symmetries of X. We assume that X is edge-transitive, but not vertex- or face-transitive.

Step 1. Pick any edge e, with vertices v_1 and v_2. We first show that any vertex v is in the G-orbit of v_1 or of v_2, but not both. To see this, let d be an edge containing v, and let w be the other vertex of d. By edge-transitivity, there is a $g \in G$ which maps e to d. So $g\{v_1, v_2\} = \{v, w\}$. So $v = gv_1$ or $v = gv_2$, and v is either in the G-orbit of v_1 or of v_2.

Suppose that v is in both the G orbit of v_1 and of v_2. Write $v = g_1 v_1$ and $v = g_2 v_2$. Since X is not vertex transitive, there is a vertex v' which is not in the G-orbit of v_1. Let e' be an edge containing v' and let w' be the other vertex of e'. Let $g' \in G$ map e to e'. Then $g\{v_1, v_2\} = \{v', w'\}$, and since gv_1 cannot equal v', we must have $gv_2 = v'$. But then $v' = gv_2 = gg_2^{-1} v_1$ is in the G-orbit of v_1, contrary to hypothesis.

Because of Step 1, the vertex set $\mathcal{V}$ is the disjoint union $\mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_j$ is the G-orbit of v_j. Since the edge e we started with was arbitrary, we see that any edge has one vertex in $\mathcal{V}_1$ and one vertex in $\mathcal{V}_2$. Moreover, if v is any vertex, in $\mathcal{V}_1$ say, then all the vertices joined to v by an edge must be in $\mathcal{V}_2$. Let m_v denoted the number of such vertices (the valency of v). Then $m_v = m_w$ if v and w are in the same G-orbit. Let m_j be the common valency of the vertices in $\mathcal{V}_j$. Since each edge e contains exactly one vertex in $\mathcal{V}_1$, we have

$$\mathcal{E} = \bigcup_{v \in \mathcal{V}_1} \{e : v \text{ is a vertex of } e\},$$

and therefore $|\mathcal{E}| = |\mathcal{V}_1|\, m_1$. Similarly, $|\mathcal{E}| = |\mathcal{V}_2|\, m_2$.

Another consequence of Step 1 is that each face F must have an even number of edges. For if we start at some vertex v of F and start working round the edges of F, the vertices must alternatingly be in $\mathcal{V}_1$ and in $\mathcal{V}_2$, and so we must take an even number of steps to get back to v.

Step 2. Let F_1 and F_2 be two adjacent faces of X, and let e be their common edge. Then any face F of X is in the G-orbit of F_1 or of F_2, but not both.

One proves Step 2 in exactly the same was as Step 1. Corresponding consequences of Step 2 are that $\mathcal{F}$ is the disjoint union $\mathcal{F}_1 \cup \mathcal{F}_2$ of the G-orbits of F_1 and of F_2, that any two adjacent faces are in different sets $\mathcal{F}_j$, and that $|\mathcal{E}| = |\mathcal{F}_1|\, n_1 = |\mathcal{F}_2|\, n_2$, where n_j is the number of edges on any face belonging to $\mathcal{F}_j$. We saw as a consequence of Step 1 that n_1 and n_2 are even. In the same way, we see from Step 2 that the valencies m_1 and m_2 are both even.

By definition of a vertex, the valency of any vertex must be at least 3, and so being even, each m_j is at least 4. Similarly, each n_j is at least 4. Hence

$$|\mathcal{V}| = |\mathcal{V}_1| + |\mathcal{V}_2| = \frac{|\mathcal{E}|}{n_1} + \frac{|\mathcal{E}|}{n_2} \le \frac{|\mathcal{E}|}{4} + \frac{|\mathcal{E}|}{4} = \frac{|\mathcal{E}|}{2}.$$

Similarly, $|\mathcal{F}| \le |\mathcal{E}|/2$. Hence by Euler's Formula,

$$2 = |\mathcal{V}| - |\mathcal{E}| + |\mathcal{F}| \le \frac{|\mathcal{E}|}{2} - |\mathcal{E}| + \frac{|\mathcal{E}|}{2} = 0,$$

a contradiction.

9. Suppose that n is even. Then

$$1 = (-1)^n = (1-2)^n = \sum_{k=0}^{n} \binom{n}{k} (-2)^k,$$

so that, subtracting 1 from both sides and dividing by n, we get

$$\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} (-2)^k = 0. \tag{1}$$

Now

$$\frac{1}{n}\binom{n}{k} = \frac{(n-1)\cdots(n-(k-1))}{k!} = \frac{(-1)^{k-1}(k-1)! + n m_{n,k}}{k!} = \frac{(-1)^{k-1}}{k} + n\frac{m_{n,k}}{k!}.$$

for some integer $m_{n,k}$. Hence (1) can be written

$$\sum_{k=1}^{n} \frac{2^k}{k} = n \sum_{k=1}^{n} \frac{m_{n,k}(-1)^k 2^k}{k!}. \tag{2}$$

The 2-adic valuation $\mathrm{ord}_2(r)$ of a nonzero rational number r is by definition v $(\in \mathbb{Z})$ if $r = 2^v a/b$, where $a, b \in \mathbb{Z}$ are odd integers. In other words, $\mathrm{ord}_2(r)$ is the number of times r is divisible by 2, counting negatively for factors of 2 in the denominator.

The formula

$$\operatorname{ord}_2(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{2^j} \right\rfloor \tag{3}$$

is well-known (see Niven and Zuckerman, *An Introduction to the Theory of Numbers*, Theorem 4.2, for example). Since

$$\left\lfloor \frac{n}{2^j} \right\rfloor \le \frac{n}{2^j},$$

we see that $\operatorname{ord}_2(n!) \le n$. So the terms $2^k/k!$ appearing on the right in (2) have 2-adic valuation at least 0. That is, they are rational numbers which may be expressed as fractions with odd denominators. So the same is true for the sum on the right in (2). Hence $x_n = nr_n$, where $\operatorname{ord}_2(r_n) \ge 0$. Thus

$$\operatorname{ord}_2(x_n) \ge \operatorname{ord}_2(n) \quad \text{if } n \text{ is even.}$$

In particular, $\operatorname{ord}_2(x_{2^k}) \ge k$. It follows easily from this that $\operatorname{ord}_2(x_n) \to \infty$ as $n \to \infty$.

The estimate $\operatorname{ord}_2(x_{2^k}) \ge k$ is very weak. Here is another solution to the problem which gives a better estimate.

Solution 2. We first derive the formula

$$x_n = \frac{2^n}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}}. \tag{4}$$

Replacing n by $n+1$ and multiplying both sides by $n!$, this is equivalent to showing that

$$\sum_{k=0}^{n} (n-k)!k! = (n+1)! \sum_{k=0}^{n} \frac{1}{(k+1)2^{n-k}}. \tag{5}$$

We leave it to the reader to verify that both sides of (5) satisfy the recurrence relation

$$y_{n+1} = \frac{n+2}{2} y_n + (n+1)!.$$

Since both sides of (5) are equal to 1 when $n = 0$, the identity (5), and therefore (4), holds for all $n \ge 0$.

We use (3) to estimate the 2-adic valuation of the denominators on the right in (4). Indeed,

$$\operatorname{ord}_2\left(\binom{n-1}{k}\right) = \sum_{j=1}^{\infty} \left(\left\lfloor \frac{n-1}{2^j} \right\rfloor - \left\lfloor \frac{k}{2^j} \right\rfloor - \left\lfloor \frac{n-1-k}{2^j} \right\rfloor \right). \tag{6}$$

If we write $r = (n-1)/2^j$ and

$$m = \left\lfloor \frac{k}{2^j} \right\rfloor + \left\lfloor \frac{n-1-k}{2^j} \right\rfloor,$$

then

$$m \le \frac{k}{2^j} + \frac{n-1-k}{2^j} = r,$$

so that $m \leq \lfloor r \rfloor$. Hence the summands on the right in (6) are non-negative. Also,

$$\left\lfloor \frac{k}{2^j} \right\rfloor > \frac{k}{2^j} - 1 \quad \text{and} \quad \left\lfloor \frac{n-1-k}{2^j} \right\rfloor > \frac{n-1-k}{2^j} - 1,$$

so that $m > r-2$. Hence $m \leq \lfloor r \rfloor \leq r < m+2$. Hence $0 \leq \lfloor r \rfloor - m < 2$. Since $\lfloor r \rfloor - m \in \mathbb{Z}$, it must be 0 or 1. That is, the j-th summand on the right in (6) is 0 or 1. Now suppose that $2^r \leq n-1 < 2^{r+1}$. Then the j-th term in (6) is zero once $j \geq r+1$. So the sum in (6) is a sum of r terms, each of which is 0 or 1. Hence

$$\operatorname{ord}_2\left(\binom{n-1}{k}\right) \leq r \quad \text{if } 2^r + 1 \leq n \leq 2^{r+1}.$$

So we may write the formula (4) as

$$x_n = \frac{2^{n-r}}{n+1} \sum_{k=0}^{n-1} \frac{2^r}{\binom{n-1}{k}}$$

and each summand in the sum on the right has non-negative 2-adic valuation. When n is even, so that $n+1$ is odd, we therefore have shown that

$$\operatorname{ord}_2(x_n) \geq n - r \quad \text{if } 2^r + 2 \leq n \leq 2^{r+1} \text{ is even.}$$

In particular, we have

$$\operatorname{ord}_2(x_{2^k}) \geq 2^k - k + 1.$$

10. For general n, this problem was rather harder than SUMS thought. Sorry about that. We refer the interested reader to an article by N. Bergeron and A.M. Garsia "On certain spaces of harmonic polynomials", Contemporary Mathematics Volume 138 (1992), pp. 51–86. We shall give here a complete solution for the case of the Van der Monde determinant in 3 variables, but only part of the solution for the case of general n, giving the main ideas at least.

Fix an integer $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be indeterminates. Let

$$V_n = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}$$

be the Vandermonde determinant. Using row and column operations it is easy to check that

$$V_n = (x_n - x_1)(x_n - x_2) \ldots (x_n - x_{n-1}) V_{n-1}, \tag{1}$$

and then a simple induction leads to the well known formula

$$V_n = \prod_{1 \leq i < j \leq n} (x_j - x_i). \tag{2}$$

Note that V_{n-1} is a polynomial in the variables $x_1, x_2, \ldots, x_{n-1}$; in particular, V_{n-1} does not depend on x_n.

For convenience, let $\partial_i = \frac{\partial}{\partial x_i}$, and more generally, $\partial_i^k = \frac{\partial^k}{\partial x_i^k}$, for $i = 1, 2, \ldots, n$ and $k \geq 0$. It is well known that for all i and j, $\partial_i \partial_j F = \partial_j \partial_i F$ for any reasonable function F of $x_1, \ldots, x_n$, and in particular this is true for any polynomial in the x_k's, which is all we are dealing with here. So any function obtained from V_n by repeated partial differentiations, in any order, can be written as

$$\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_n^{k_n} V_n, \qquad \text{where } k_1, k_2, \ldots, k_n \geq 0. \tag{3}$$

In fact, we can also assume that $0 \leq k_j < n$, for all j. For (2) shows that for any j, V_n is a polynomial of degree $n-1$ in x_j. Hence $\partial_i^n V_n = 0$. There are therefore at most n^n nonzero partial derivatives (3).

We want to show that these partial derivatives span a vector space of functions which has dimension $n!$. We would like to show that any partial derivative (3) is a unique linear combination of partial derivatives

$$\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_n^{k_n} V_n, \qquad \text{where } 0 \leq k_j < j \text{ for each } j. \tag{4}$$

There are $n!$ such partial derivatives. For example, if $n = 2$ then $V_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$; so the only nonzero partial derivatives are

$$V_2 = \partial_1^0 \partial_2^0 V_2, \quad 1 = \partial_1^0 \partial_2^1 V_2, \quad \text{and} \quad -1 = \partial_1^1 \partial_2^0 V_2.$$

The third of these is -1 times the second one, and so a linear combination of the first two, which are the ones of the form (4).

Now suppose that $n = 3$. Let us write simply ∂_{ijk} in place of $\partial_1^i \partial_2^j \partial_3^k V_3$. Then the $3! = 6$ partial derivatives (4) are

$$\begin{aligned} &\partial_{000} = V_3, \quad \partial_{001} = (x_1 - x_2)(x_1 + x_2 - 2x_3), \quad \partial_{002} = -2(x_1 - x_2), \\ &\partial_{010} = -(x_1 - x_3)(x_1 + x_3 - 2x_2), \quad \partial_{011} = -2(x_2 - x_3) \quad \text{and} \quad \partial_{012} = 2. \end{aligned}$$

All 27 ∂_{ijk}'s, $0 \leq i, j, k \leq 2$, are linear combinations of these 6 partial derivatives. Indeed,

$$\begin{aligned} &\partial_{020} = 2(x_1 - x_3) = -\partial_{011} - \partial_{002}, \quad \partial_{021} = -2 = -\partial_{012}, \quad \partial_{100} = -\partial_{010} - \partial_{001} \\ &\partial_{101} = -\partial_{011} - \partial_{002}, \quad \partial_{102} = -\partial_{012}, \quad \partial_{110} = \partial_{002}, \quad \partial_{120} = \partial_{012}, \\ &\partial_{200} = \partial_{011}, \quad \partial_{201} = \partial_{012}, \quad \partial_{210} = -\partial_{012} \end{aligned}$$

and the other partial derivatives ∂_{ijk} are all zero.

We now show that, for each $n \geq 2$, the partial derivatives (4) are linearly independent. We prove this by induction on n. We have already checked this when $n = 2$, so assume now that $n > 2$ and that the claim holds for V_{n-1}. Let $0 \leq k_j < j$ for $j = 1, \ldots, n$ and write k in place of k_n and write

$$\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_{n-1}^{k_{n-1}} = \partial^{k'}$$

Then since V_{n-1} is independent of x_n,

$$\begin{aligned}\partial_1^{k_1}\partial_2^{k_2}\dots\partial_n^{k_n}V_n = \partial^{k'}\partial_n^k(V_n) &= \partial^{k'}\partial_n^k\left(V_{n-1}\prod_{j=1}^{n-1}(x_n-x_j)\right)\\ &= \partial^{k'}\left(V_{n-1}\,\partial_n^k\prod_{j=1}^{n-1}(x_n-x_j)\right)\\ &= \partial^{k'}\bigl(V_{n-1}p_k\bigr), \quad \text{say.}\end{aligned}$$

Notice that p_k is a polynomial in $x_1,\dots,x_n$ which is of degree $n-1-k$ in x_n. Now by repeated use of the product rule for differentiation (in the various variables $x_1,\dots,x_{n-1}$), we have

$$\partial^{k'}\bigl(V_{n-1}p_k\bigr) = p_k\partial^{k'}\bigl(V_{n-1}\bigr) + \text{terms of degree } < n-1-k \text{ in } x_n. \tag{5}$$

Suppose that we have an equation

$$\sum t_{k_1,\dots,k_n}\,\partial_1^{k_1}\partial_2^{k_2}\dots\partial_n^{k_n}V_n \equiv 0, \tag{6}$$

where the sum is over all n-tuples of k_j's such that $0 \le k_j < j$ for each j, and where the $t_{k_1,\dots,k_n}$'s are constants.

The left hand side of (6) is a polynomial in the x_j's. Consider the terms which are of degree $n-1$ in x_n. From (5) we see that the only way to get such a term in (6) is by taking $k_n = 0$, and that these terms are

$$p_0\sum t_{k_1,\dots,k_{n-1},0}\,\partial_1^{k_1}\partial_2^{k_2}\dots\partial_{n-1}^{k_{n-1}}V_{n-1},$$

where $p_0 = \prod_{j=1}^{n-1}(x_n-x_j)$. The coefficient of x_n^{n-1} in (6) is therefore

$$\sum t_{k_1,\dots,k_{n-1},0}\,\partial_1^{k_1}\partial_2^{k_2}\dots\partial_{n-1}^{k_{n-1}}V_{n-1},$$

By the induction hypothesis, this implies that the coefficients $t_{k_1,\dots,k_{n-1},0}$ are all zero.

Next we look at the terms of degree $n-2$ in x_n. Since all the coefficients $t_{k_1,\dots,k_{n-1},0}$ are zero, from (5) we see that the only way of getting such terms is from terms in (6) in which $k_n = 1$, and that these terms are

$$p_1\sum t_{k_1,\dots,k_{n-1},1}\partial_1^{k_1}\partial_2^{k_2}\dots\partial_{n-1}^{k_{n-1}}V_{n-1},$$

where $p_1 = \partial_n p_0 = (n-1)x_n^{n-2}+$ terms of lower degree in x_n. The coefficient of x_n^{n-2} in (6) is therefore

$$(n-1)\sum t_{k_1,\dots,k_{n-1},1}\partial_1^{k_1}\partial_2^{k_2}\dots\partial_{n-1}^{k_{n-1}}V_{n-1},$$

By the induction hypothesis again, this implies that the coefficients $t_{k_1,\dots,k_{n-1},1}$ are all zero. Continuing in this way we see that all the coefficients $t_{k_1,\dots,k_{n-1},k_n}$ are zero, and linear independence is proved.

The proof that every partial derivative (3) is a linear combination of the partial derivatives (4) has been given above for $n = 2, 3$. For general n, we refer the reader to the paper cited above, but at least we indicate the ideas involved here:

We first show that

$$\sum_{j=1}^{n} \partial_j V_n = 0, \quad \sum_{j=1}^{n} \partial_j^2 V_n = 0, \quad \text{etc,} \tag{7}$$

and more generally, that

$$P(\partial_1, \ldots, \partial_n) V_n = 0$$

for any symmetric polynomial P in n variables with zero constant term. The reason for this is that V_n is an alternating polynomial in $x_1, \ldots, x_n$ (that is, if you interchange two of the variables in V_n, you get $-V_n$). For a symmetric P it is easy to see that $P(\partial_1, \ldots, \partial_n)V_n$ is also alternating. But alternating polynomials are all divisible by each $x_i - x_j$, $i \neq j$, and hence by V_n. The hypothesis that P has no constant term means that the degree of $P(\partial_1, \ldots, \partial_n)V_n$ is less than that of V_n, and so the only way it can be divisible by V_n is if it is zero.

Now we use the first of the relations (7):

$$\partial_1 V_n = -(\partial_2 + \cdots + \partial_n) V_n,$$

to express any partial derivative (3) without using ∂_1. This also tells us that

$$\partial_1^2 V_n = -(\partial_2 + \cdots + \partial_n)^2 V_n = \partial_2^2 V_n + \sum_{k=3}^{n} \partial_2 \partial_k V_n + \sum_{j,k=3}^{n} \partial_j \partial_k V_n.$$

Using this and the second of the relations (7):

$$\partial_1^2 V_n + \partial_2^2 V_n + \cdots + \partial_n^2 V_n = 0,$$

we see that

$$\partial_2^2 V_n = -\frac{1}{2}\Big(\sum_{k=3}^{n} \partial_2 \partial_k V_n + \sum_{k=3}^{n} \partial_k^2 V_n + \sum_{j,k=3}^{n} \partial_j \partial_k V_n \Big),$$

which expresses $\partial_2^2 V_n$ in terms of partial derivatives (3) in which there are no ∂_1's, and ∂_2 only appears to the first power. To see how to continue this procedure, we refer the reader to the paper cited above (equation (3.11)).