Sydney University Mathematical Society Problem Competition 2014

1. Let $A_1, A_2, \cdots, A_{2n}$ be the vertices of a convex $(2n)$-gon in the plane, listed in clockwise order (here $n \geq 3$). Suppose that opposite edges of the $(2n)$-gon are parallel (that is, A_1A_2 is parallel to $A_{n+2}A_{n+1}$, A_2A_3 is parallel to $A_{n+3}A_{n+2}$, and so on until A_nA_{n+1} is parallel to A_1A_{2n}). Prove that the main diagonals ($A_1A_{n+1}, A_2A_{n+2}, \cdots, A_nA_{2n}$) intersect in a single point if and only if opposite edges of the $(2n)$-gon have equal length.

Solution. For convenience, we define A_{2n+j} to equal A_j for $1 \leq j < 2n$.

For $1 \leq i \leq 2n$, let P_i be the point of intersection of the main diagonals A_iA_{n+i} and $A_{i+1}A_{n+i+1}$. (Thus, we actually have $P_{n+i} = P_i$ for $1 \leq i \leq n$.) Since A_iA_{i+1} is parallel to $A_{n+i+1}A_{n+i}$, the triangles $A_iA_{i+1}P_i$ and $A_{n+i}A_{n+i+1}P_i$ are similar, so

$$|A_iP_i|/|P_iA_{n+i}| = |A_{i+1}P_i|/|P_iA_{n+i+1}| = |A_iA_{i+1}|/|A_{n+i+1}A_{n+i}|. \tag{1}$$

Now the main diagonals intersect in a single point if and only if all P_i are equal. Since both P_i and P_{i+1} lie in the interval $A_{i+1}A_{n+i+1}$ (here we define P_{2n+1} to be P_1), this happens if and only if $|A_{i+1}P_i|/|P_iA_{n+i+1}| = |A_{i+1}P_{i+1}|/|P_{i+1}A_{n+i+1}|$ for all $1 \leq i \leq 2n$. Using (1), we conclude that the main diagonals intersect in a single point if and only if all the ratios $|A_iA_{i+1}|/|A_{n+i+1}A_{n+i}|$ are the same, for $1 \leq i \leq 2n$. The latter condition implies that, for $1 \leq i \leq n$, the ratio $|A_iA_{i+1}|/|A_{n+i+1}A_{n+i}|$ equals its inverse $|A_{n+i}A_{n+i+1}|/|A_{i+1}A_i|$, forcing both to equal 1. So the main diagonals intersect in a single point if and only if opposite edges A_iA_{i+1} and $A_{n+i+1}A_{n+i}$ have equal length. Note that, if this is the case, (1) shows that the diagonals bisect each other.

2. Anna is playing a mathematical computer game. The computer is hiding a polynomial P; the degree and coefficients of P are unknown to Anna, but she does know that the coefficients are strictly positive real numbers. In each move, Anna inputs a real number a and the computer outputs $P(a)$. This is repeated until Anna can determine what P must be.

For any strategy S used by Anna, denote by $S(P)$ the number of moves needed to determine P. Call a strategy S *optimal* if $S(P) \leq S'(P)$ for all possible strategies S' and all polynomials P with strictly positive real coefficients. Does there exist an optimal strategy?

Solution. Yes, the following strategy S is optimal: Choose the positive integers $1, 2, 3, \cdots$ in ascending order. Furthermore, this strategy gives $S(P) = \deg(P) + 2$ for all allowable polynomials P (where “allowable” is short for “with strictly positive real coefficients”). To prove this, it suffices to prove the following two claims:

(i) $S(P) \leq \deg(P) + 2$ for all allowable polynomials P,

(ii) $S'(P) \geq \deg(P) + 2$ for all possible strategies S' and all allowable polynomials P.

We prove (i) by induction on $n = \deg(P)$. In the base case where $n = 0$, P is a constant polynomial c for some $c > 0$, and our claim amounts to saying that if Q is an allowable polynomial such that $Q(1) = Q(2) = c$, then Q must be the constant polynomial c. This is clear, because if Q were not constant, then $Q(1) < Q(2)$ because all the coefficients are positive.

Now suppose that n is positive, and assume that (i) is true for polynomials of degree $n-1$. Our claim amounts to saying that if Q is an allowable polynomial such that $Q(i) = P(i)$ for all $i \in \{1, 2, \cdots, n+2\}$, then $Q = P$. Consider the polynomials ΔP and ΔQ defined by

$$\Delta P(x) = P(x+1) - P(x) \quad \text{and} \quad \Delta Q(x) = Q(x+1) - Q(x).$$

These are both allowable (this follows from the fact that $(x+1)^d - x^d$ has positive coefficients), and $\Delta Q(i) = \Delta P(i)$ for all $i \in \{1, 2, \cdots, n+1\}$. Since $\deg(\Delta P) = n-1$, we can conclude from the inductive hypothesis that $\Delta Q = \Delta P$. So $Q - P$ is a polynomial with the property that $(Q-P)(x+1) = (Q-P)(x)$ for all x, forcing $Q - P$ to be constant; since $Q(1) = P(1)$, we have $Q = P$ as required. This completes the proof of (i).

We prove (ii) by contradiction. Suppose that there was a strategy S' and an allowable polynomial P of degree n such that $S'(P) \le n+1$. Let the first $n+1$ moves of S' be $a_1, \cdots, a_{n+1}$. Since P has strictly positive coefficients, there exists some $\epsilon > 0$ such that the coefficients of

$$R(x) = P(x) + \epsilon(x - a_1) \cdots (x - a_{n+1})$$

are strictly positive. The polynomials P and R are different, but they agree on $a_1, \cdots, a_{n+1}$. This contradicts our assumption that P could be determined after the moves $a_1, \cdots, a_{n+1}$.

3. For any positive integer n, let $P_n(x)$ be the polynomial defined by

$$P_n(x) = x^n + 2x^{n-1} + 3x^{n-2} + \cdots + nx + (n+1).$$

Show that, if either $n+1$ or $n+2$ is prime, then $P_n(x)$ is irreducible (that is, it cannot be written as a product of two non-constant polynomials with integer coefficients).

Solution. Note that we have the following expressions:

$$\begin{aligned}(x-1)P_n(x) &= x^{n+1} + x^n + x^{n-1} + \cdots + x^2 + x - (n+1),\\ (x-1)^2 P_n(x) &= x^{n+2} - (n+2)x + (n+1).\end{aligned}$$

First suppose that $n+1$ is prime. For a contradiction, assume that $P_n(x) = Q(x)R(x)$ where $Q(x), R(x)$ are non-constant polynomials with integer coefficients. Then $n+1 = P_n(0) = Q(0)R(0)$, so without loss of generality we have $Q(0) = \pm 1$. Now $Q(0)$ is (up to sign) the product of the complex roots of $Q(x)$, so this means that there must be some complex root z of $Q(x)$ with $|z| \le 1$. This z is also a root of $P_n(x)$. But then the equation $(z-1)P_n(z) = 0$ gives

$$n+1 = |z^{n+1} + z^n + \cdots + z^2 + z| \le |z^{n+1}| + |z^n| + \cdots + |z^2| + |z| \le n+1,$$

where we first use the triangle inequality and then the fact that $|z| \le 1$. So both $\le$ signs are equalities: thus $|z| = 1$, and since equality holds in the use of the triangle inequality, all of $z, z^2, \cdots, z^{n+1}$ have the same argument, so $z = 1$. But clearly $P_n(1) = 1+2+\cdots+(n+1) \ne 0$, giving the desired contradiction.

Next suppose that $n+2$ is prime. Define a polynomial $S(x) = P_n(x+1)$; then it suffices to prove that $S(x)$ is irreducible. But

$$\begin{aligned}x^2 S(x) &= (x+1)^{n+2} - (n+2)(x+1) + (n+1)\\ &= (x+1)^{n+2} - (n+2)x - 1\\ &= \sum_{j=0}^{n} \binom{n+2}{j} x^{n+2-j},\end{aligned}$$

so $S(x) = \sum_{j=0}^{n} \binom{n+2}{j} x^{n-j}$. The leading coefficient 1 of $S(x)$ is not divisible by the prime $n+2$; every other coefficient $\binom{n+2}{j}$ (for $1 \leq j \leq n$) is divisible by $n+2$ (since $n+2$ divides $(n+2)!$ and not $j!(n+2-j)!$); and the constant term $\binom{n+2}{n} = \frac{(n+2)(n+1)}{2}$ is not divisible by $(n+2)^2$. Thus $S(x)$ is irreducible by Eisenstein's Criterion.

4. a) Show that the zero function is the only continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_a^{a^2} f(x)\,dx = 0, \quad \text{for all } a \in \mathbb{R}.$$

b) Show that there exists a nonzero continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_a^{a^2+1} f(x)\,dx = 0, \quad \text{for all } a \in \mathbb{R}.$$

Solution. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the integral condition in (a). By the Fundamental Theorem of Calculus, $g(y) = \int_0^y f(x)\,dx$ is a differentiable (and hence continuous) function of y with derivative $g'(y) = f(y)$. The assumption implies that

$$g(a^2) = g(a), \quad \text{for all } a \in \mathbb{R}. \tag{2}$$

Therefore, to prove (a), it suffices to prove that any continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying (2) is constant. Note that for any $a > 0$, repeated application of (2) gives that $g(a) = g(a^{1/2^n})$ for all positive integers n; since $a^{1/2^n} \to 1$ as $n \to \infty$, we conclude by continuity that $g(a) = g(1)$. Then from (2) it follows that $g(a) = g(1)$ for any $a < 0$, and continuity then implies that $g(0) = g(1)$ also. This completes the solution to part (a).

The solution to part (b) begins in the same way. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition in (b) if and only if its antiderivative function $g : \mathbb{R} \to \mathbb{R}$, defined by $g(y) = \int_0^y f(x)\,dx$, satisfies

$$g(a^2+1) = g(a), \quad \text{for all } a \in \mathbb{R}. \tag{3}$$

Since both sides of (3) are differentiable functions of a, they agree everywhere if and only if they agree at $a = 0$ and their derivatives agree everywhere. In other words, (3) is equivalent to the following two conditions on f:

$$\int_0^1 f(x)\,dx = 0 \quad \textbf{and} \quad 2af(a^2+1) = f(a), \ \text{for all } a \in \mathbb{R}. \tag{4}$$

The second condition in (4) clearly implies that $f(0) = 0$, that $f(-a) = -f(a)$ for all $a > 0$, and that $f(a^2+1) = \frac{f(a)}{2a}$ for all $a > 0$. Conversely, if $f : [0, \infty) \to \mathbb{R}$ is a continuous function such that $f(0) = 0$ and $f(a^2+1) = \frac{f(a)}{2a}$ for all $a > 0$, then we can extend it to a continuous function $f : \mathbb{R} \to \mathbb{R}$ by setting $f(-a) = -f(a)$ for all $a > 0$, and this will satisfy the second condition in (4). Therefore it suffices to construct a nonzero continuous function $f : [0, \infty) \to \mathbb{R}$ satisfying

$$\int_0^1 f(x)\,dx = 0, \quad f(0) = 0 \quad \textbf{and} \quad f(a^2+1) = \frac{f(a)}{2a}, \ \text{for all } a > 0. \tag{5}$$

Let a_0, a_1, a_2 be the (obviously increasing) sequence defined by $a_0 = 0$ and $a_n = a_{n-1}^2 + 1$ for all $n \geq 1$. Thus, $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, $a_4 = 26$, etc. For any $n \geq 1$, we have a continuous bijection from the interval $[a_{n-1}, a_n]$ to the interval $[a_n, a_{n+1}]$ given by sending x to x^2+1; its inverse, also continuous, sends x to $\sqrt{x-1}$.

If we have already defined f on the domain $[0, a_n]$ for some $n \geq 1$ and we want to extend it to a function on the domain $[0, a_{n+1}]$, we are forced to define f on the semi-closed interval $(a_n, a_{n+1}]$ by the rule

$$f(x) = \frac{f(\sqrt{x-1})}{2\sqrt{x-1}} \quad \text{for } a_n < x \leq a_{n+1},$$

in order to ensure that the third condition in (5) holds. (Note that $x > a_n$ implies $\sqrt{x-1} > 0$.) In this way we can start with a function f defined on $[0, 1]$ and recursively define f on the whole of $[0, \infty)$. If f is continuous on the domain $[0, a_n]$, then the extension to the domain $[0, a_{n+1}]$ is clearly continuous everywhere except possibly at the point a_n. For continuity at a_n, we require

$$\lim_{x \to a_n^+} \frac{f(\sqrt{x-1})}{2\sqrt{x-1}} = f(a_n). \tag{6}$$

Note that when $n \geq 2$, the left-hand side of (6) is $\frac{f(a_{n-1})}{2a_{n-1}} = f(a_n)$ by the assumed continuity of f on $[0, a_n]$ and the continuity of $x \mapsto \sqrt{x-1}$. So we only need to ensure (6) when $n = 1$.

The conclusion is that, to construct our desired $f : \mathbb{R} \to \mathbb{R}$, it suffices to construct a nonzero continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)\,dx = 0, \quad f(0) = 0 \quad \textbf{and} \quad \lim_{x \to 0^+} \frac{f(x)}{2x} = f(1). \tag{7}$$

Such functions are plentiful; one example is $f(x) = 8x^3 - 9x^2 + 2x$ for $0 \leq x \leq 1$.

5. Find all pairs (a, b) of positive integers such that $2a$ divides b^2+1 and b divides $2a^2+1$.

Solution. The set of solutions is

$$\{(P_{2n+1}, H_{2n}), (P_{2n+1}, H_{2n+2}) \mid n \geq 0\} \tag{8}$$

where P_n is the nth Pell number and H_n is the nth half-companion Pell number (following the terminology of Wikipedia). These numbers are the unique integers satisfying

$$(1+\sqrt{2})^n = H_n + P_n\sqrt{2} \quad \text{and hence also} \quad (1-\sqrt{2})^n = H_n - P_n\sqrt{2},$$

for all $n \geq 0$. Equivalently, they can be defined by the closed formulae:

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}, \quad H_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}.$$

Incidentally, the pairs (H_n, P_n) are the unique nonnegative integer solutions (x, y) of the equation $x^2 - 2y^2 = \pm 1$, but that fact is not used in the following proof.

Easy calculations from the above formulae show that

$$H_n^2 + (-1)^n = 2P_{n+1}P_{n-1} \text{ for all } n \geq 1, \tag{9}$$

and that

$$2P_n^2 - (-1)^n = H_{n+1}H_{n-1} \text{ for all } n \geq 1. \tag{10}$$

For each of the pairs (a, b) in the above list (8), the fact that $2a$ divides b^2+1 follows immediately from (9) while the fact that b divides $2a^2+1$ follows immediately from (10).

We must now show that any pair (a, b) satisfying the desired divisibility properties belongs to the list (8). We prove this by induction on the sum $a + b$. The smallest this sum can be is 2, when $a = b = 1$; the pair $(1, 1)$ satisfies the divisibility properties, and is indeed on the list, since $(1, 1) = (P_1, H_0)$. Having settled this base case, we can assume that $(a, b) \neq (1, 1)$.

We now construct a new pair (a', b') according to the following rules:

(i) if $b > \sqrt{2}a$, let $(a', b') = (a, (2a^2 + 1)/b)$;

(ii) if $b < \sqrt{2}a$, let $(a', b') = ((b^2 + 1)/(2a), b)$.

It is simple to check that in both cases the new pair (a', b') satisfies the required divisibility properties, i.e. that $2a'$ divides $b'^2 + 1$ and b' divides $2a'^2 + 1$. We claim that $a' + b' < a + b$. In case (i) this claim amounts to saying that $2a^2 + 1 < b^2$, and we have assumed that $2a^2 < b^2$, so the only possibility we have to rule out is that $2a^2 + 1 = b^2$; but this would imply that $2a$ divides $2a^2 + 2$, forcing $a = 1$ and then $b^2 = 3$, a contradiction. Similarly, in case (ii) the claim amounts to saying that $b^2 + 1 < 2a^2$, and the possibility we have to rule out is that $b^2 + 1 = 2a^2$; but this would imply that b divides $b^2 + 2$, forcing $b \in \{1, 2\}$ and a contradiction in either case.

By the induction hypothesis, we know that (a', b') belongs to the list (8). If we are in case (i) and $(a', b') = (P_{2n+1}, H_{2n+1\pm1})$, then $(a, b) = (P_{2n+1}, H_{2n+1\mp1})$ by (10). If we are in case (ii) and $(a', b') = (P_{2n\pm1}, H_{2n})$ for $n \geq 1$, then $(a, b) = (P_{2n\mp1}, H_{2n})$ by (9). If we are in case (ii) and $(a', b') = (P_1, H_0)$, then $(a, b) = (1, 1)$. Some of these conclusions contradict our previous assumptions (that is, not all cases can actually occur), but in any case it is now certain that (a, b) belongs to the list (8). This completes the inductive step.

6. For which connected finite simple graphs can one label each vertex v with a positive integer $f(v)$ in such a way that, for every v, the sum of the labels of the vertices adjacent to v is $2f(v) - 1$?

Solution. The list of solutions is as follows (here we replace the vertices by their labels):

$$\begin{tikzcd}
a \arrow[r, dash] & (2a-1) \arrow[r, dash] & \cdots \arrow[r, dash] & \left(ka-\binom{k}{2}\right) \arrow[r, dash] & \cdots \arrow[r, dash] & \left((2a)a-\binom{2a}{2}\right) \\
& & & k\text{th vertex} \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& & & & & & \frac{1}{2}\binom{a+1}{2} \\
a \arrow[r, dash] & (2a-1) \arrow[r, dash] & \cdots \arrow[r, dash] & \left(ka-\binom{k}{2}\right) \arrow[r, dash] & \cdots \arrow[r, dash] & \left((a-1)a-\binom{a-1}{2}\right) \arrow[ur, dash] \arrow[dr, dash] & \\
& & & k\text{th vertex} \arrow[u] & & & \frac{1}{2}\binom{a+1}{2}
\end{tikzcd}$$

$$\begin{tikzcd}
8 \arrow[r, dash] & 15 \arrow[r, dash] & 21 \arrow[r, dash] \arrow[d, dash] & 15 \arrow[r, dash] & 8 \\
& & 11 & &
\end{tikzcd}$$

and

$$\begin{array}{ccccccccccccc} 29 & — & 57 & — & 84 & — & 110 & — & 135 & — & 91 & — & 46 \\ &&&&&&&& | &&&& \\ &&&&&&&& 68 &&&& \end{array}$$

A graph of the first kind (a path) has $2a$ vertices, where a can be any positive integer. A graph of the second kind has $a+1$ vertices, where $a \geq 3$ is an integer such that either $a \equiv 3 \pmod 4$ or $a \equiv 0 \pmod 4$. (These conditions ensure that $\frac{1}{2}\binom{a+1}{2} = \frac{a(a+1)}{4}$ is an integer.) The third and fourth kinds only have the examples shown, with 6 and 8 vertices. These graphs are some of the so-called 'finite Dynkin diagrams', and their names as Dynkin diagrams are A_{2a}, D_{a+1}, E_6 and E_8 respectively (the subscript indicates the number of vertices). For each of these graphs, the labelling depicted above is the only one satisfying the stated property.

To prove these assertions, we suppose that we have a graph and a labelling with the stated property. We first note that the graph must be a tree, i.e. it cannot contain a cycle of vertices $v_1, \cdots, v_n$ ($n \geq 3$) where v_i is adjacent to $v_{i\pm1}$ for all $1 \leq i \leq n$ (subscripts taken modulo n). For if it did, and v_a was a vertex in the cycle with the minimum value of $f(v_a)$, then the sum of the labels of adjacent vertices would be at least $f(v_{a-1}) + f(v_{a+1}) \geq 2f(v_a)$, contrary to assumption.

Next, this tree can have at most one branch vertex (i.e. vertex of degree ≥ 3). For suppose there were two branch vertices v_1 and v_n, with $v_2, \cdots, v_{n-1}$ being the other vertices on the unique path joining v_1 and v_n (this path is unique since the graph is a tree). For the same reason as in the previous paragraph, the minimum value of $f(v_i)$ cannot be attained at any of $i = 2, \cdots, n-1$, so it must be attained at either $i = 1$ or $i = n$. In particular, we must have either $f(v_1) < f(v_2)$ or $f(v_n) < f(v_{n-1})$; without loss of generality (because we could just swap the numbering of v_1 and v_n otherwise), we can assume that $f(v_n) < f(v_{n-1})$. Since v_n is a branch vertex, it is adjacent to at least two other vertices v_{n+1} and v_{n+2} as well as to v_{n-1}. We then have $f(v_n) < 2f(v_{n+1})$ and $f(v_n) < 2f(v_{n+2})$ by considering the condition at these other vertices. So the sum of the labels of vertices adjacent to v_n is at least $f(v_{n-1}) + f(v_{n+1}) + f(v_{n+2}) > f(v_n) + \frac{1}{2}f(v_n) + \frac{1}{2}f(v_n) = 2f(v_n)$, contrary to assumption.

Furthermore, the tree cannot have a vertex of degree ≥ 4. For if v_1 were adjacent to all of v_2, v_3, v_4, v_5, then the sum of the labels of vertices adjacent to v_1 is at least $f(v_2) + f(v_3) + f(v_4) + f(v_5) > 4 \times \frac{1}{2}f(v_1) = 2f(v_1)$, contrary to assumption.

Note that so far we have not used the fact that the labels are integers, only that they are positive real numbers; and we have used the property on the labelling only in the weaker form that the sum of the labels of vertices adjacent to v is less than $2f(v)$. We could go further in this generality (eventually proving that the graph must be a finite Dynkin diagram of type A, D or E), but it is more convenient instead to use the actual property with $2f(v) - 1$.

If the tree has no branch vertices, then it is a path, and the vertices can be numbered $v_1, \cdots, v_n$ in order. If $f(v_1) = a$, then $f(v_2) = 2a - 1$; hence $f(v_3) = 2(2a-1) - 1 - a = 3a - 3$, and by an easy induction one shows that $f(v_k) = ka - \binom{k}{2}$ for all $k \leq n$. Considering the final vertex then gives $2na - 2\binom{n}{2} - 1 = (n-1)a - \binom{n-1}{2}$, which rearranges to give $n = 2a$. This gives the first (type A_{2a}) solution listed above. Notice that the labels are not only positive but symmetric under a reflection of the whole path, because $ka - \binom{k}{2} = \frac{k(2a-k+1)}{2}$.

If the tree has a branch vertex, we know that this branch vertex must have degree 3. Suppose that the three branches have lengths $k \geq \ell \geq m$, as measured by the number of vertices in each branch not including the branch vertex, and that the labels of the end vertices of the branches are a, b, c respectively. Then the labels of the other vertices of each branch can be determined in terms of a, b, c just as in the path case, and the label d of the branch vertex must be simultanously equal to

$$\frac{(k+1)(2a-k)}{2} = \frac{(\ell+1)(2b-\ell)}{2} = \frac{(m+1)(2c-m)}{2}.$$

We conclude that $k+1, \ell+1, m+1$ must all be divisors of $2d$, and that

$$a = \frac{k}{2} + \frac{d}{k+1}, \quad b = \frac{\ell}{2} + \frac{d}{\ell+1}, \quad c = \frac{m}{2} + \frac{d}{m+1}.$$

Moreover, the condition at the branch vertex is that

$$\begin{aligned} 2d-1 &= \frac{k(2a-k+1)}{2} + \frac{\ell(2b-\ell+1)}{2} + \frac{m(2c-m+1)}{2}, \\ &= \frac{k}{k+1}d + \frac{\ell}{\ell+1}d + \frac{m}{m+1}d + \frac{k+\ell+m}{2}. \end{aligned}$$

This implies that $\frac{k}{k+1} + \frac{\ell}{\ell+1} + \frac{m}{m+1} < 2$, which forces $\frac{m}{m+1} < \frac{2}{3}$ and hence $m = 1$. Then $\frac{k}{k+1} + \frac{\ell}{\ell+1} < \frac{3}{2}$ forces $\ell = 1$ or $\ell = 2$; if $\ell = 2$, we have $\frac{k}{k+1} < \frac{5}{6}$ which forces $k \in \{2, 3, 4\}$.

In the $\ell = 1$ case, the above equation becomes

$$2d-1 = \frac{k}{k+1}d + d + \frac{k+2}{2},$$

which rearranges to give $d = \frac{(k+1)(k+4)}{2}$. We then have $a = k+2$ and $b = c = \frac{1}{2}(d+1) = \frac{1}{2}\binom{k+3}{2} = \frac{1}{2}\binom{a+1}{2}$, which gives the second (type D_{a+1}) solution listed above, with $a \geq 3$ and either $a \equiv 3 \pmod 4$ or $a \equiv 0 \pmod 4$.

In the $\ell = 2$ case where $k \in \{2, 3, 4\}$, the above equation becomes

$$2d-1 = \frac{k}{k+1}d + \frac{7}{6}d + \frac{k+3}{2},$$

which rearranges to give $d = \frac{3(k+5)(k+1)}{5-k}$. If $k = 2$ we get $d = 21$ and then $a = b = 8$, $c = 11$, leading to the third (type E_6) solution listed above. If $k = 3$ we get $d = 48$ and then a contradiction because $a = \frac{27}{2}$ is not an integer. If $k = 4$ we get $d = 135$ and then $a = 29$, $b = 46$, $c = 68$, leading to the fourth (type E_8) solution listed above.

7. Say that a function $f : \mathbb{Z} \to \{1, 0, -1\}$ is a *perturbed sign function* if it satisfies the following properties for $a \gg 0$ (i.e., there is some $N \geq 0$ such that these equations hold for all $a \geq N$):

$$f(a) = 1, \quad f(-a) = -1, \quad \sum_{i=-a}^{a} f(i) = 0.$$

Given such a perturbed sign function f, let its *degeneracy* $d(f)$ be the number of $a \in \mathbb{Z}$ such that $f(a) = 0$, and let its *weight* $w(f)$ be the number of pairs $(a, b) \in \mathbb{Z}^2$ such that $a < b$ and $f(a) > f(b)$. Find a formula for the power series $\sum x^{w(f)}$ where the sum is over all perturbed sign functions f of fixed degeneracy d.

Solution. First note that a perturbed sign function f as defined in this problem must have odd degeneracy, because for $a \gg 0$ the set $[-a, a]$ contains all $d(f)$ numbers b such that $f(b) = 0$, and hence $\sum_{i=-a}^{a} f(i)$ has parity equal to that of $2a + 1 - d(f)$. So the answer to the question would be 0 if d is even. However, for the following solution it is convenient to give a different definition which allows even degeneracy.

Instead of functions $f : \mathbb{Z} \to \{1, 0, -1\}$, we want to consider doubly infinite sequences of 1s, 0s and (-1)s without a specification of which is the 0th term, which is the 1st term, and so on; more formally, this means that we consider equivalence classes of functions $f : \mathbb{Z} \to \{1, 0, -1\}$ where two functions f, f' are equivalent if there is some $m \in \mathbb{Z}$ such that $f'(a) = f(a+m)$ for all $a \in \mathbb{Z}$. We say that such a sequence S is a *perturbed sign sequence* if the sequence ends with an infinite sequence of 1s and begins with an infinite sequence of (-1)s; in terms of equivalence classes of functions, this means that for some (hence any) representative $f : \mathbb{Z} \to \{1, 0, -1\}$ of the equivalence class we have $f(a) = 1$ and $f(-a) = -1$ for all $a \gg 0$. Define the *degeneracy* $d(S)$ of such a sequence to be the number of 0s (this is clearly finite), and define its *weight* $w(S)$ to be the number of pairs of terms of the sequence where the first is bigger than the second.

With this definition, it is easy to see that if $d(S)$ is odd, then S has a unique representative function $f : \mathbb{Z} \to \{1, 0, -1\}$ satisfying the condition in the question that $\sum_{i=-a}^{a} f(i) = 0$ for all $a \gg 0$ (in other words, there is a unique way to specify which is the 0th term of the sequence so that the signs are balanced around that term). So for odd degeneracy, the concepts of perturbed sign function and perturbed sign sequence are effectively the same. We will thus be solving a slightly more general problem if we find a formula for the power series $\sum x^{w(S)}$ where the sum is over all perturbed sign sequences S of fixed degeneracy d, and d is any nonnegative integer.

We consider the $d = 0$ case first. In this case we are considering a sequence of 1s and -1s only. Every 1 that occurs in the sequence before a -1 contributes to the weight; its contribution is the number of (-1)s to the right of it. Reading these contributions from left to right gives a weakly decreasing sequence of positive integers, or in other words a *partition*; the weight of the sequence is the *size* (sum of the parts) of the partition. It is easy to see that this gives a bijection between perturbed sign sequences of degeneracy 0 and partitions, so the power series we seek is Euler's generating function for partitions:

$$\sum_{\substack{S \\ d(S)=0}} x^{w(S)} = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = \prod_{m=1}^{\infty} \frac{1}{1-x^m}.$$

Recall that Euler's formula is proved by expanding $\frac{1}{1-x^m}$ as $x^{0m} + x^{1m} + x^{2m} + x^{3m} + \cdots$; the choice of the x^{im} term from this factor of the product specifies that the partition should have i parts equal to m.

Now consider the general case. Given a perturbed sign sequence S of arbitrary degeneracy, we can create two perturbed sign sequences S', S'' of degeneracy 0, as follows. We obtain S' from S simply by changing all 0s to 1s. We obtain S'' from S by deleting all (-1)s so that only 0s and 1s remain and the sequence is infinite only to the right, then changing all 0s to (-1)s and making the sequence doubly infinite once more by starting with an infinite sequence of (-1)s. This construction is designed so that the weight of S is the sum of the weights of S' and S'': the sequence S' captures the contributions to the weight of S made by pairs of terms where the first is a 1, and the sequence S'' captures the contribution made by pairs of terms where the first is a 0. Notice that if S has degeneracy d, then every 1 in the sequence S'' has at most d (-1)s to the right of it (because these (-1)s must originally have been 0s of S). Thus, S' and S'' each correspond to some partition, and the partition corresponding to S'' has no parts bigger than d.

Conversely, suppose we are given two perturbed sign sequences S' and S'' of degeneracy 0, and a nonnegative integer d such that every 1 in the sequence S'' has at most d (-1)s to the right of it. We claim that there is a unique perturbed sign sequence S of degeneracy d which gives rise to S' and S'' in the manner described above. Indeed, one is forced to construct S from S' by changing d of the 1s to 0s, where the positions among all 1s in S' of those 1s which must be changed to 0 are the same as the positions among the 1s in S'' of the d right-most (-1)s.

We conclude that perturbed sign sequences of degeneracy d are in bijection with ordered pairs of partitions of which the second has no parts bigger than d, and this bijection is such that the weight of the sequence is the sum of the sizes of the partitions. Using Euler's formula and its obvious modification to handle partitions with parts bounded above by d, we find the desired formula:

$$\sum_{\substack{S \\ d(S)=d}} x^{w(S)} = \prod_{m=1}^{\infty} \frac{1}{1-x^m} \prod_{m=1}^{d} \frac{1}{1-x^m}.$$

8. Let n denote a positive integer. The symmetric group S_n is the group of permutations of the set $\{1, 2, \cdots, n\}$. This group acts naturally on the set $\mathcal{P}_n$ of all subsets of $\{1, 2, \cdots, n\}$: if $\sigma \in S_n$ and $I \in \mathcal{P}_n$, then $\sigma(I)$ has the usual meaning of $\{\sigma(i) \mid i \in I\}$. We regard S_n as a subgroup of S_{n+1} or of S_{n+2} in the obvious way, i.e. S_n is identified with the subgroup of S_{n+1} consisting of permutations that fix $n+1$, and with the subgroup of S_{n+2} consisting of permutations that fix $n+1$ and $n+2$.

a) Show that the action of S_n on $\mathcal{P}_n$ extends to S_{n+1} (that is, there is an action of S_{n+1} on $\mathcal{P}_n$ which, when restricted to S_n, coincides with the aforementioned action of S_n on $\mathcal{P}_n$).

b) For which n does the action of S_n on $\mathcal{P}_n$ extend to S_{n+2}?

Solution. In this solution, we denote the set $\{1, 2, \cdots, n\}$ by $[n]$.

Consider the set $\mathcal{Q}_{n+1}$ of unordered pairs $\{I, J\}$ of subsets of $[n+1]$ such that $I \cap J = \emptyset$ and $I \cup J = [n+1]$. The group S_{n+1} naturally acts on $\mathcal{Q}_{n+1}$: if $\tau \in S_{n+1}$ and $\{I, J\} \in \mathcal{Q}_{n+1}$, then $\tau(\{I, J\})$ has the obvious meaning $\{\tau(I), \tau(J)\}$. There is a map $f : \mathcal{Q}_{n+1} \to \mathcal{P}_n$ sending $\{I, J\}$ to the unique one of the subsets I and J that does not contain $n+1$. This map is a bijection: its inverse $f^{-1} : \mathcal{P}_n \to \mathcal{Q}_{n+1}$ sends I to $\{I, [n+1] \setminus I\}$. So we can transport the S_{n+1}-action on $\mathcal{Q}_{n+1}$ to $\mathcal{P}_n$ via f: explicitly, if $\tau \in S_{n+1}$ and $I \in \mathcal{P}_n$, we define $\tau \cdot I = f(\tau(f^{-1}(I)))$, or in other words

$$\tau \cdot I = \begin{cases} \tau(I), & \text{if } n+1 \notin \tau(I), \\ \tau([n+1] \setminus I), & \text{if } n+1 \in \tau(I). \end{cases}$$

It is clear that this S_{n+1}-action on $\mathcal{P}_n$ extends the original S_n-action, finishing part (a).

The answer to (b) is that the S_n-action on $\mathcal{P}_n$ extends to S_{n+2} if and only if $n = 1$ or n is even. Our proof of this will actually determine the existence of an extension to S_{n+k} for any positive integer k. Namely, we will show that:

- if $n = 1$ or 2, the S_n-action on $\mathcal{P}_n$ extends to S_{n+k} for all positive integers k;
- if $n \geq 3$ and n is odd, the S_n-action on $\mathcal{P}_n$ extends to S_{n+1} but not to S_{n+2} (and hence not to S_{n+k} for any $k \geq 3$);
- if $n \geq 4$ and n is even, the S_n-action on $\mathcal{P}_n$ extends to S_{n+2} but not to S_{n+3} (and hence not to S_{n+k} for any $k \geq 4$).

The $n = 1$ case is trivial: S_1 is the trivial group, so the claim is just that S_{k+1} has an action on the (two-element) set $\mathcal{P}_1$, which is obvious. In fact, there are two possibilities for the S_{k+1}-action on $\mathcal{P}_1$: it could either be the trivial action, or the action in which odd permutations swap the two elements and even permutations fix them. (Henceforth, we will not attempt to classify all the extended actions; we just consider whether they exist.)

The $n = 2$ case is also easy: S_2 is the group with two elements, and the nontrivial element $(1\ 2)$ fixes the elements $\emptyset$ and $[2]$ of $\mathcal{P}_2$, while swapping the elements $\{1\}$ and $\{2\}$. We can extend this to S_{k+2} by letting odd permutations act in the same way as $(1\ 2)$ while even permutations act trivially.

We have already seen a general way to extend the S_n-action on $\mathcal{P}_n$ to S_{n+1}. If n is even, we can extend this action further to S_{n+2}, as follows. The key observation is that in the set $\mathcal{Q}_{n+1}$ defined above, every element $\{I, J\}$ consists of one subset of even size and one subset of odd size, since $n+1$ is odd. We thus have a bijection between $\mathcal{Q}_{n+1}$ and the set $\mathcal{R}_{n+2}$ of unordered pairs $\{I, J\}$ of subsets of $[n+2]$ such that $I \cap J = \emptyset$, $I \cup J = [n+2]$ and $|I|$ and $|J|$ are both even: the map $g : \mathcal{R}_{n+2} \to \mathcal{Q}_{n+1}$ simply deletes the element $n+2$, and the inverse $g^{-1} : \mathcal{Q}_{n+1} \to \mathcal{R}_{n+2}$ adds the element $n+2$ to whichever of the two subsets has odd size. So we can transport the natural S_{n+2}-action on $\mathcal{R}_{n+2}$ to $\mathcal{P}_n$ via $f \circ g$: that is, if $\tau \in S_{n+2}$ and $I \in \mathcal{P}_n$, we define $\tau \cdot I = f(g(\tau(g^{-1}(f^{-1}(I)))))$. Since the bijection $f : \mathcal{Q}_{n+1} \to \mathcal{P}_n$ is S_n-equivariant (checked in part (a) above), and the bijection $g : \mathcal{R}_{n+2} \to \mathcal{Q}_{n+1}$ is S_{n+1}-equivariant (just as clear), the bijection $f \circ g : \mathcal{R}_{n+2} \to \mathcal{P}_n$ is S_n-equivariant, or in other words the S_{n+2}-action on $\mathcal{P}_n$ we have defined does indeed extend the original S_n-action. One can easily write out an explicit formula for this S_{n+2}-action on $\mathcal{P}_n$ like the above formula for the S_{n+1}-action, but since it has eight cases rather than two, we will omit it. (It is worth noting that, instead of $\mathcal{R}_{n+2}$, we could have used the set $\mathcal{R}'_{n+2}$ defined similarly but with $|I|$ and $|J|$ required to be odd.)

Now suppose that $n \geq 3$ and we have an S_{n+2}-action on $\mathcal{P}_n$ extending the natural S_n-action; we want to show that n must be even. The proof we will give focuses on the transposition $\tau := (n+1\ n+2) \in S_{n+2}$, which has the useful property that it commutes with every element of S_n. If $I \in \mathcal{P}_n$, the stabilizer of I in S_n is the subgroup of S_n consisting of permutations that preserve I and thus also preserve its complement $\bar{I} := [n] \setminus I$ (if $|I| = k$, this stabilizer is isomorphic to $S_k \times S_{n-k}$). Apart from I, the only other element of $\mathcal{P}_n$ with this same stabilizer in S_n is $\bar{I}$. However, from the fact that τ commutes with S_n, it easily follows that $\tau \cdot I$ and I have the same stabilizer in S_n; we conclude that, for all $I \in \mathcal{P}_n$, either $\tau \cdot I = I$ or $\tau \cdot I = \bar{I}$. Moreover, two subsets $I, I' \in \mathcal{P}_n$ with the same number of elements are in the same S_n-orbit; again using the fact that τ commutes with S_n, it follows that $\tau \cdot I = I$ if and only if $\tau \cdot I' = I'$. So there is some subset K of $\{0, 1, 2, \cdots, n\}$ such that for all $I \in \mathcal{P}_n$,

$$\tau \cdot I = \begin{cases} I, & \text{if } |I| \in K, \\ \bar{I}, & \text{if } |I| \notin K. \end{cases}$$

Moreover, since τ is self-inverse, $\tau \cdot I = \bar{I}$ if and only if $\tau \cdot \bar{I} = \bar{\bar{I}}$, so $i \in K$ if and only if $n - i \in K$. For notational convenience, let $\delta_K : \{0, 1, \cdots, n\} \to \{0, 1\}$ be the function defined by $\delta_K(i) = 1$ if $i \in K$ and 0 otherwise; then we have $\delta_K(n-i) = \delta_K(i)$.

Now let j be any positive integer with $j \leq n-2$, and let σ denote the transposition $(n-1\ n) \in S_n$. We have a subgroup $S_j \times \langle \sigma \rangle$ of S_n, and the fixed points of this subgroup in its action on $\mathcal{P}_n$ are the subsets $I \in \mathcal{P}_n$ such that either I contains all elements of $[j]$ or it contains none of them, and either I contains both of $n-1, n$ or it contains neither of them; this means that there are $4 \times 2^{n-2-j} = 2^{n-j}$ such fixed points. Now in the larger group S_{n+2}, the subgroup $S_j \times \langle \sigma \rangle$ is conjugate to $S_j \times \langle \tau \rangle$, and it follows that $S_j \times \langle \tau \rangle$ also has 2^{n-j} fixed points in $\mathcal{P}_n$. But the fixed points of $S_j \times \langle \tau \rangle$ are the subsets $I \in \mathcal{P}_n$ such that either I contains all elements of $[j]$ or it contains none of them, and $|I| \in K$. Hence we have

$$\sum_{i \in K} \binom{n-j}{i-j} + \binom{n-j}{i} = 2^{n-j}, \tag{11}$$

where we use the standard convention that $\binom{n-j}{a} = 0$ if $a < 0$ or $a > n - j$. The left-hand side of (11) can be rewritten as

$$\sum_{j\le i\le n}\delta_K(i)\binom{n-j}{i-j}+\sum_{0\le i\le n-j}\delta_K(i)\binom{n-j}{i}.$$

Making the change of variables $i\leftrightarrow n-i$ in the first sum and using $\delta_K(n-i)=\delta_K(i)$ and the fact that $\binom{n-j}{a}=\binom{n-j}{n-j-a}$, it becomes equal to the second sum. So we have

$$\sum_{0\le i\le n-j}\delta_K(i)\binom{n-j}{i}=2^{n-j-1}. \tag{12}$$

Setting $j=n-2$ in (12) gives that

$$\delta_K(0)+2\delta_K(1)+\delta_K(2)=2,$$

which obviously forces one of the following two cases: either $\delta_K(0)=\delta_K(2)=1$ and $\delta_K(1)=0$, or $\delta_K(0)=\delta_K(2)=0$ and $\delta_K(1)=1$. In either case, successively setting $j=n-3$, $j=n-4$, …, $j=1$ in (12) allows the unique determination of the unknowns $\delta_K(3)$, $\delta_K(4)$, …, $\delta_K(n-1)$ in turn, so there can be at most two functions $\delta_K:\{0,1,\cdots,n-1\}\to\{0,1\}$ that satisfy the equations (12). But two such functions are well known: considering the binomial theorem applied to $(1+(-1))^{n-j}=0$, we have

$$\sum_{\substack{0\le i\le n-j\\ i\text{ even}}}\binom{n-j}{i}=\sum_{\substack{0\le i\le n-j\\ i\text{ odd}}}\binom{n-j}{i}=2^{n-j-1}.$$

We conclude that, for $i\in\{0,1,\cdots,n-1\}$, either $\delta_K(i)=1$ if i is even and 0 if i is odd, or vice versa. For this to be consistent with $\delta_K(n-1)=\delta_K(1)$, n must be even, as claimed.

Now suppose that $n\ge 4$ and n is even, and assume for a contradiction that we have an S_{n+3}-action on $\mathcal{P}_n$ extending the natural S_n-action. The preceding argument determines the action of $\tau=(n+1\ n+2)$ up to a choice of two possibilities: we must have

$$\textbf{either}\quad \tau\cdot I=\begin{cases} I, & \text{if } |I| \text{ is even},\\ \bar{I}, & \text{if } |I| \text{ is odd},\end{cases}\qquad \textbf{or}\quad \tau\cdot I=\begin{cases} I, & \text{if } |I| \text{ is odd},\\ \bar{I}, & \text{if } |I| \text{ is even}.\end{cases}$$

Exactly the same applies to $\tau'=(n+2\ n+3)$. If $\tau'\cdot I$ were given by the opposite formula to $\tau\cdot I$, then we would have $(\tau\tau')\cdot I=\bar{I}$ for all I, which is impossible since $\tau\tau'=(n+1\ n+2\ n+3)$ has order 3 and cannot act as an involution. Hence $\tau'\cdot I$ is given by the same formula as $\tau\cdot I$, so the fixed points in $\mathcal{P}_n$ for the subgroup $\langle\tau,\tau'\rangle$ are either the subsets of even size or the subsets of odd size; in either case, there are 2^{n-1} fixed points. But this subgroup is conjugate in S_{n+3} to the subgroup S_3, which has 2^{n-2} fixed points in $\mathcal{P}_n$, giving the desired contradiction.