The University of Sydney
School of Mathematics and Statistics
NSW 2006 Australia

SUMS Problem Competition 2006

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Entrants may use any source of information except other people. The problems will also be posted on the web page **http://www.maths.usyd.edu.au/u/SUMS/**.

Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years). Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, preference may be given to students in the earlier year of university.

Entries must be received by **Friday, September 8, 2006**. They may be posted to Dr Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or delivered in person to Room 805, Carslaw Building. Please mark your entry SUMS Problem Competition 2006, and include your name, university, student number, course and year, term address and telephone number. Prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr Henderson at the above address.

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1. For any positive real number x, let $\langle x\rangle$ denote the fractional part of x, i.e. the unique element of $[0,1)$ such that $x-\langle x\rangle$ is an integer. If N is a positive integer, the *scale* based on x and N is the set $\{0,\langle x\rangle,\langle 2x\rangle,\cdots,\langle Nx\rangle,1\}$. This has at most $N+2$ distinct elements, possibly fewer. If we list the distinct elements of the scale in order, $0=s_0<s_1<\cdots<s_k=1$, the *intervals* in the scale are the differences $s_1-s_0, s_2-s_1,\cdots,s_k-s_{k-1}$. Prove that there are at most three different intervals.

2. Find the volume of the region in $\mathbb{R}^3$ defined by the inequalities

$$|x|^{2/3}+|y|^{2/3}\le 1,\ |x|^{2/3}+|z|^{2/3}\le 1,\ |y|^{2/3}+|z|^{2/3}\le 1.$$

3. Let D be a regular dodecahedron with edges of length 1. Find the shortest possible length of a path on the surface of D starting at one vertex and finishing at the antipodal vertex.

4. In this problem, ‘number’ means positive integer. Suppose we consider two numbers to be *essentially equal* (written $\approx$) if they become the same when all zeroes are deleted from their decimal expression (for instance, $1023\approx 120030$). For consistency with multiplication, we had better extend the notion of essential equality so that

$$a\approx b \iff a\times c\approx b\times c,\ \text{ for any numbers } a,b,c.$$

(For instance, the fact that $2\times 6=12\approx 102=17\times 6$ implies that $2\approx 17$.) Of course, we also stipulate that $a\approx b$ and $b\approx c$ together imply $a\approx c$. Show that for any number a, there is another number b such that $a\times b\approx 1$.

5. Let n be a positive integer. Show that the average of the numbers $(\tan\frac{\pi}{2n+1})^2$, $(\tan\frac{2\pi}{2n+1})^2$, $\cdots$, $(\tan\frac{n\pi}{2n+1})^2$ equals their product.

6. Fix positive integers n, k such that $k \leq n-1$. A permutation $a_1, \cdots, a_n$ of the numbers $1, 2, \cdots, n$ is called a k-*shuffle* if $1, 2, \cdots, k$ occur in the correct order and $k+1, k+2, \cdots, n$ occur in the correct order. For example, the 2-shuffles of $1, 2, 3, 4$ are those permutations where 1 precedes 2 and 3 precedes 4, namely (omitting the commas) 1234, 1324, 1342, 3124, 3142, and 3412. For any distinct complex numbers $x_1, \cdots, x_n$, show that

$$\sum_{\substack{a_1,\cdots,a_n \\ \text{a } k\text{-shuffle}}} \frac{1}{(x_{a_1}-x_{a_2})(x_{a_2}-x_{a_3})\cdots(x_{a_{n-1}}-x_{a_n})} = 0.$$

7. Suppose we have m white balls and n black balls, indistinguishable apart from their colour. We put them in a bag to hide the colour, and then draw out b of the $m+n$ balls, chosen at random. For any a, let $P(a; b, m, n)$ denote the probability that at least a of these b balls are white. On the assumption that a and b are nonnegative integers satisfying $0 \leq b \leq m+n$, $0 \leq a \leq m$, and $0 \leq b-a \leq n$, prove that

$$P(a+1; b, m, n) < P(a+1; b+1, m+1, n+1) < P(a; b, m, n).$$

8. Let A be the set of rational numbers r such that $0 < r < 1$. It is well known that A is *countable*, i.e. the elements of A can be listed $r_1, r_2, r_3, \cdots$ so that every element appears exactly once on the list. Given such a listing, we define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{n\geq 1 \\ r_n \leq x}} 2^{-n}.$$

a) Show that there exists a listing of A for which the corresponding function f takes no rational values other than 0 and 1.

b) Show that there exists a listing of A for which f takes infinitely many rational values.

9. Fix a positive integer n and let $x_1, \cdots, x_n$ be indeterminates. For any permutation $a_1, \cdots, a_n$ of $1, \cdots, n$, define a polynomial in $x_1, \cdots, x_n$:

$$\Pi_{a_1,\cdots,a_n} = (x_{a_1}-x_{a_2})(x_{a_1}+x_{a_2}-x_{a_3})(x_{a_1}+x_{a_2}+x_{a_3}-x_{a_4})\cdots(x_{a_1}+x_{a_2}+\cdots+x_{a_{n-1}}-x_{a_n}).$$

Prove that each of these polynomials is a linear combination, with integer coefficients, of the polynomials attached to permutations where $a_1 = 1$.

10. Fix an integer $n \geq 2$. Determine for which real numbers c the following polynomial has n real roots (counting multiplicities):

$$x^n + cx^{n-1} + \binom{c}{2}x^{n-2} + \binom{c}{3}x^{n-3} + \cdots + \binom{c}{n},$$

where $\binom{c}{s}$ means $\frac{c(c-1)(c-2)\cdots(c-s+1)}{s!}$.