Super Pentagon Relations

Nezhla Aghaei

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Sydney

February 2023

Super Pentagon Relations

based on joint work:

Quantization of Super Teichmüller spaces,
N.A, Joerg Teschner, and Michal Pawelkiewicz,
Commun. Math. Phys. 353 (2017) no.2, 597-631, arXiv:1512.02617.

Towards Super Teichmüller Spin TQFT,
N.A, Masahito Yamazaki and Michal Pawelkiewicz,
Advances in Theoretical and Mathematical Physics Vol.26, no.2 , arXiv:2008.09829.

Heisenberg double and Drinfeld double of the quantum superplane,
N.A, Michal Pawelkiewicz, arXiv:1909.04565.

MARIE CURIE ACTIONS

QM

SDU

UNIVERSITÉ DE GENÈVE

Main Question in Mathematical Physics

Physics: Contemporary QFT is a very active subject, and it seems to work in Physics (Standard Model, Higgs boson in CERN,…)

Mathematics: Precise definition of QFT uses infinite-dimensional integral that we don’t have a precise mathematical definition for.

Main Question in Mathematical Physics

Physics: Contemporary QFT is a very active subject, and it seems to work in Physics (Standard Model, Higgs boson in CERN,…)

It seems its true in physics.

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Main Question in Mathematical Physics

Physics: Contemporary QFT is a very active subject, and it seems to work in Physics (Standard Model, Higgs boson in CERN,…)

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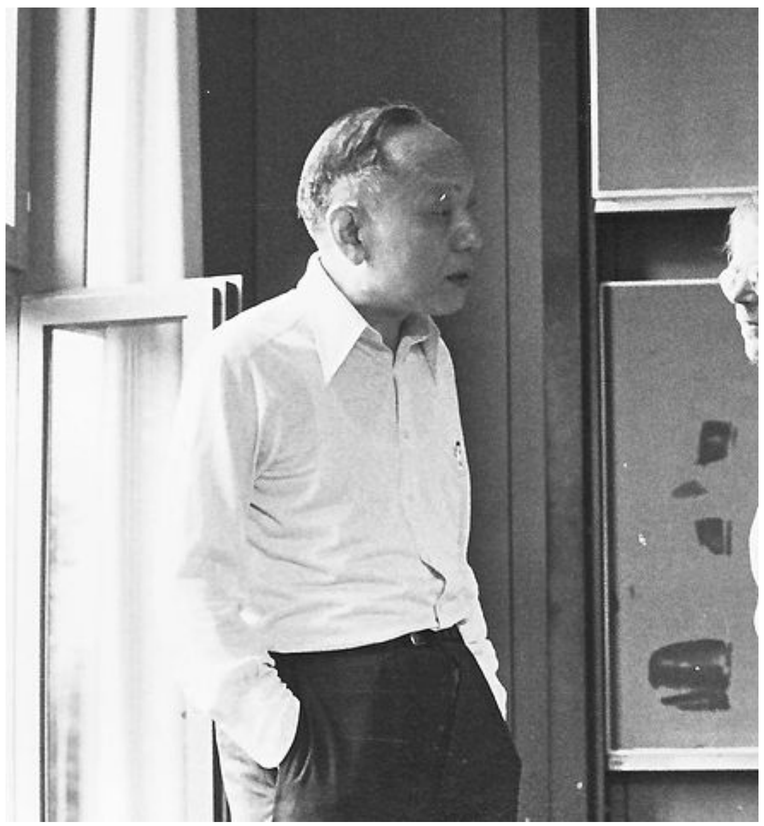
Mathematics: Precise definition of QFT uses infinite-dimensional integral that we don't have a precise mathematical definition for.

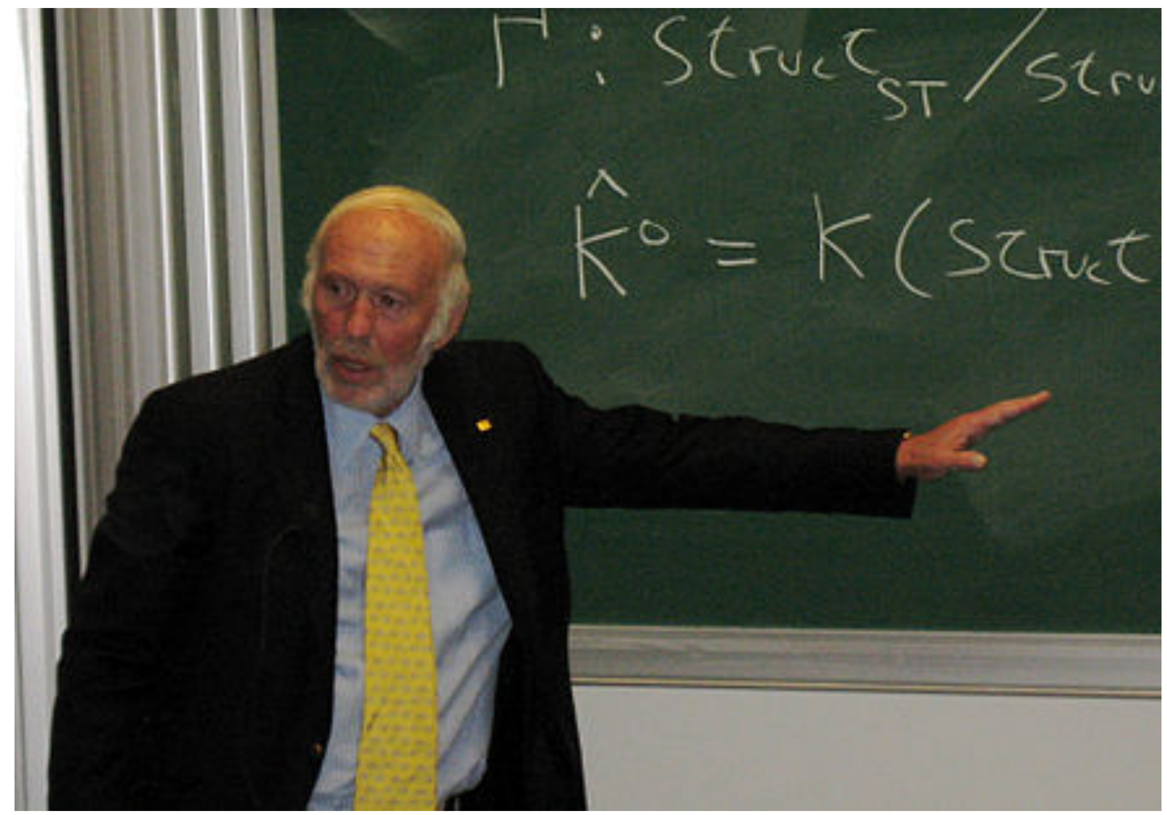
We need a machinery which seems would be revolutionary

Try to answer the question

Particular type of QFT, called quantum Chern-Simons theory,



Shiing-Shen Chern



James Simons

contributed to the development of string theory by providing a theoretical framework to combine geometry and topology with quantum field theory.

Try to answer the question

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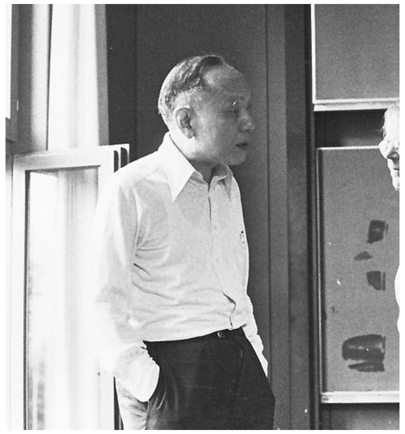

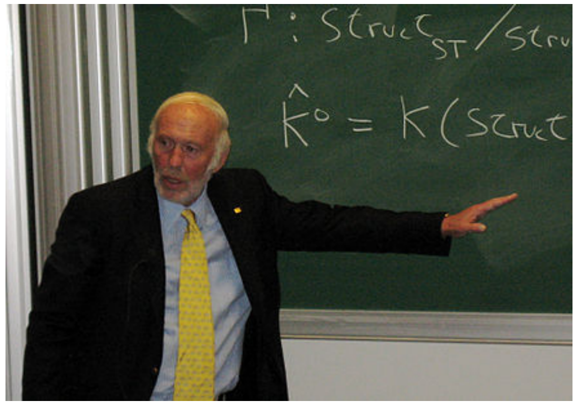

Main Properties:

There is no dynamics. The only dynamic is when the **topology** change.

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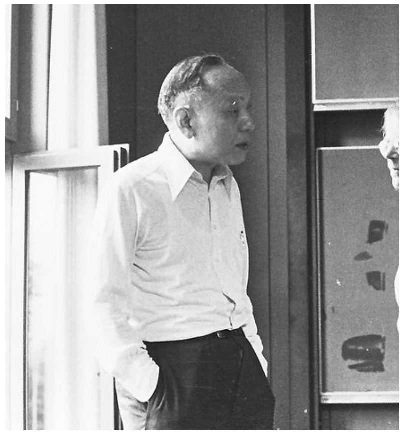
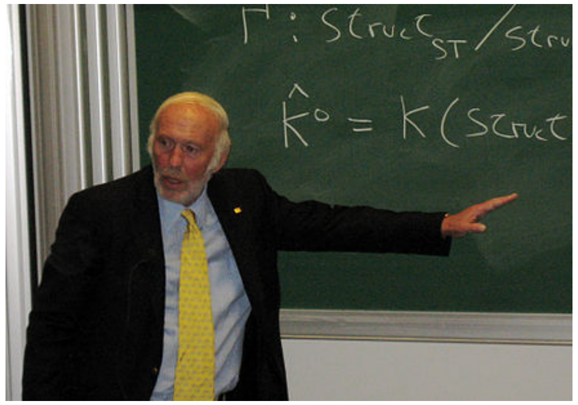




Main Properties:

There is no dynamics. The only dynamic is when the **topology** change.

In 1988 Witten wrote a foundational paper:

This path integrals can produce important invariants in topology

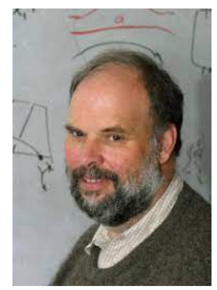
Take qCs for certain compact Lie group SU(2). Take observables ———————— Knot embedded in 3 space.

Not any invariant but the Jones polynomial of a given knot

The Big Picture

In 1988 Witten wrote a foundational paper.
This paper answered a major question posed by Michael Atiyah:

"What is the physical interpretation of the Jones polynomial?"



Vaghaun Jones

Commun. Math. Phys. 121, 351–399 (1989)

Communications in Mathematical Physics

Quantum Field Theory and the Jones Polynomial *

Edward Witten **

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

Abstract. It is shown that 2 + 1 dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from S^3 to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in 1 + 1 dimensions.

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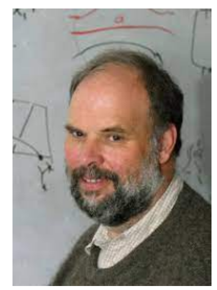

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In 1991, Reshetikhin–Turaev used language of representation theory of quantum group (and Modular tensor category) to construct mathematical precisely the output of this path integral.

3d- CS was the first and most important example of a topological quantum field theory (TQFT) called WRT-TQFT and birth of quantum topology.

The Big Picture

$U_q(G)$

TQFT's

- Compact, Semisimple

$$\boxed{G = SU(2)}$$

Witten-Reshetikhin-Turaev

Turaev-Viro

The Big Picture

$U_q(G)$

TQFT's

- Compact, Semisimple

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Witten-Reshetikhin-Turaev

Turaev-Viro

- NonCompact, Semisimple

$$\boxed{G = SL(2, \mathbb{C})}$$

Quantum Teichmüller →

Andersen-Kashaev
(Teichmüller)TQFT

The Big Picture

$U_q(G)$ | TQFT's

- Compact, Semisimple

$\boxed{G = SU(2)}$

Witten-Reshetikhin-Turaev
Turaev-Viro

- NonCompact, Semisimple

$\boxed{G = SL(2,\mathbb{C})}$ —Quantum Teichmüller→ Andersen-Kashaev (Teichmüller)TQFT

My talk in Hobart: Combinatorial qCS

- Compact, NonSemisimple

$\boxed{G = GL(1|1)}$

Non-Semisimple TQFT

[Gukov,Dimofte,....] [Blanchet, Geer,....]

The Big Picture

$U_q(G)$

TQFT's

- Compact, Semisimple

$G = SU(2)$

Witten-Reshetikhin-Turaev

Turaev-Viro

- NonCompact, Semisimple

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Quantum Teichmüller →

Andersen-Kashaev (Teichmüller)TQFT

My talk Today:

- NonCompact, Semisimple

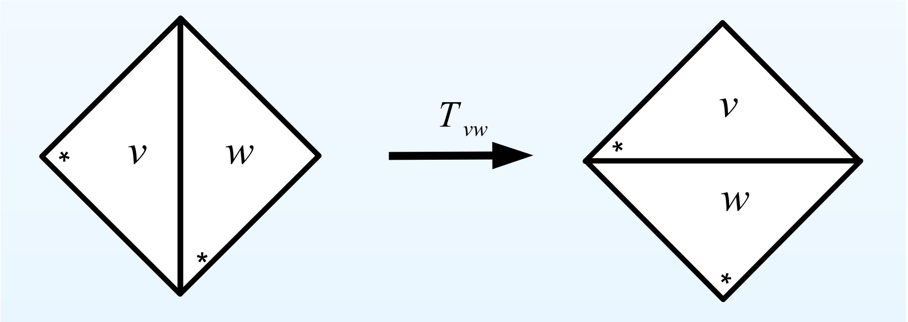
$G = OSP(1|2)$

Super Quantum Teichmüller →

Super Teichmüller TQFT

I) Play a game

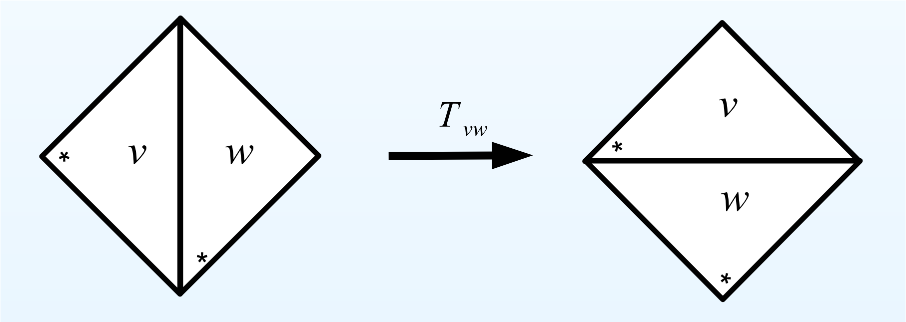
Pentagon Relation



$$T_{vw}$$

Flip

Pentagon Relation



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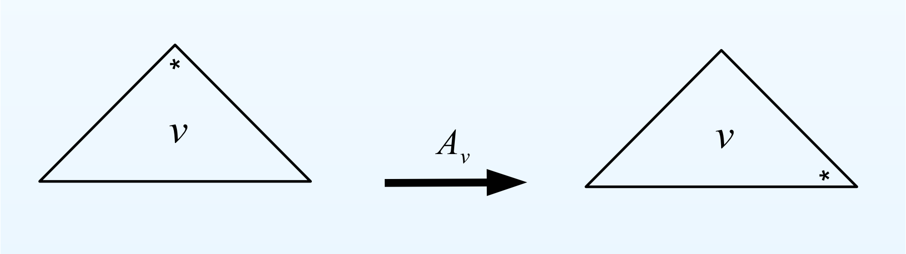
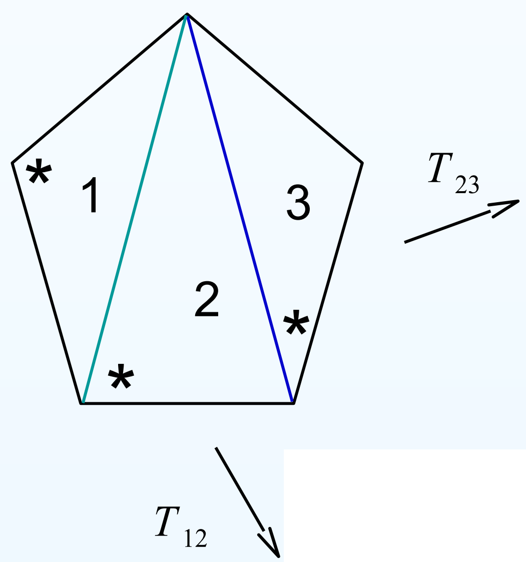
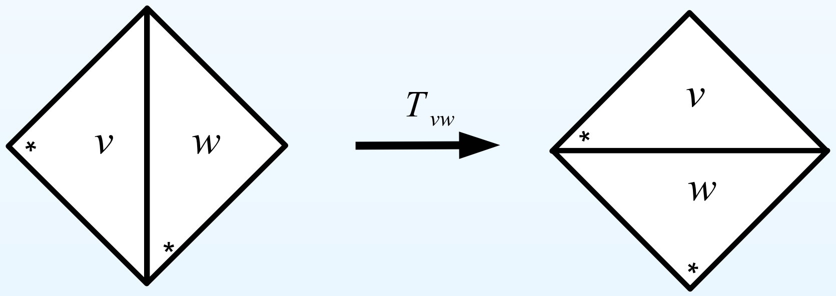


$$A^3 = 1$$

Pentagon Relation





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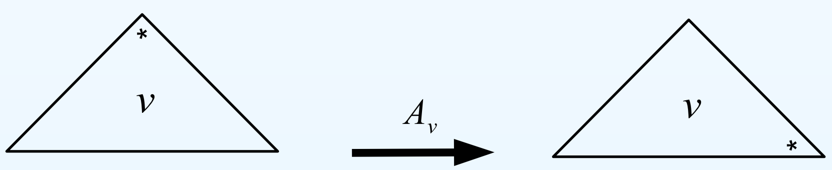

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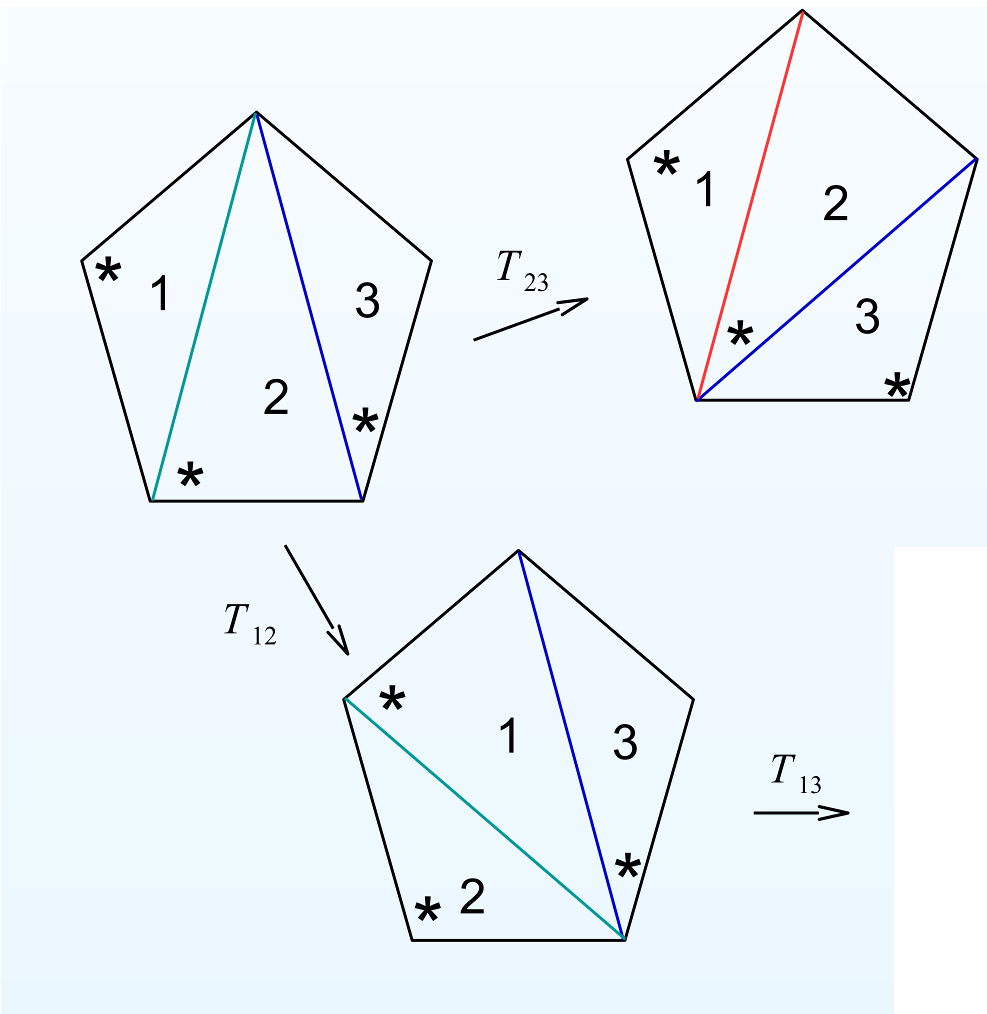

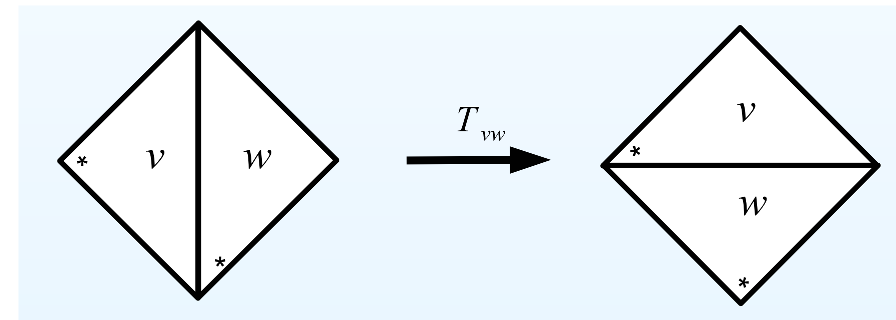

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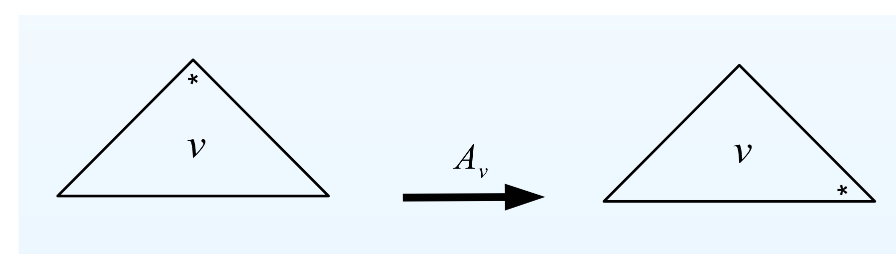
Flip



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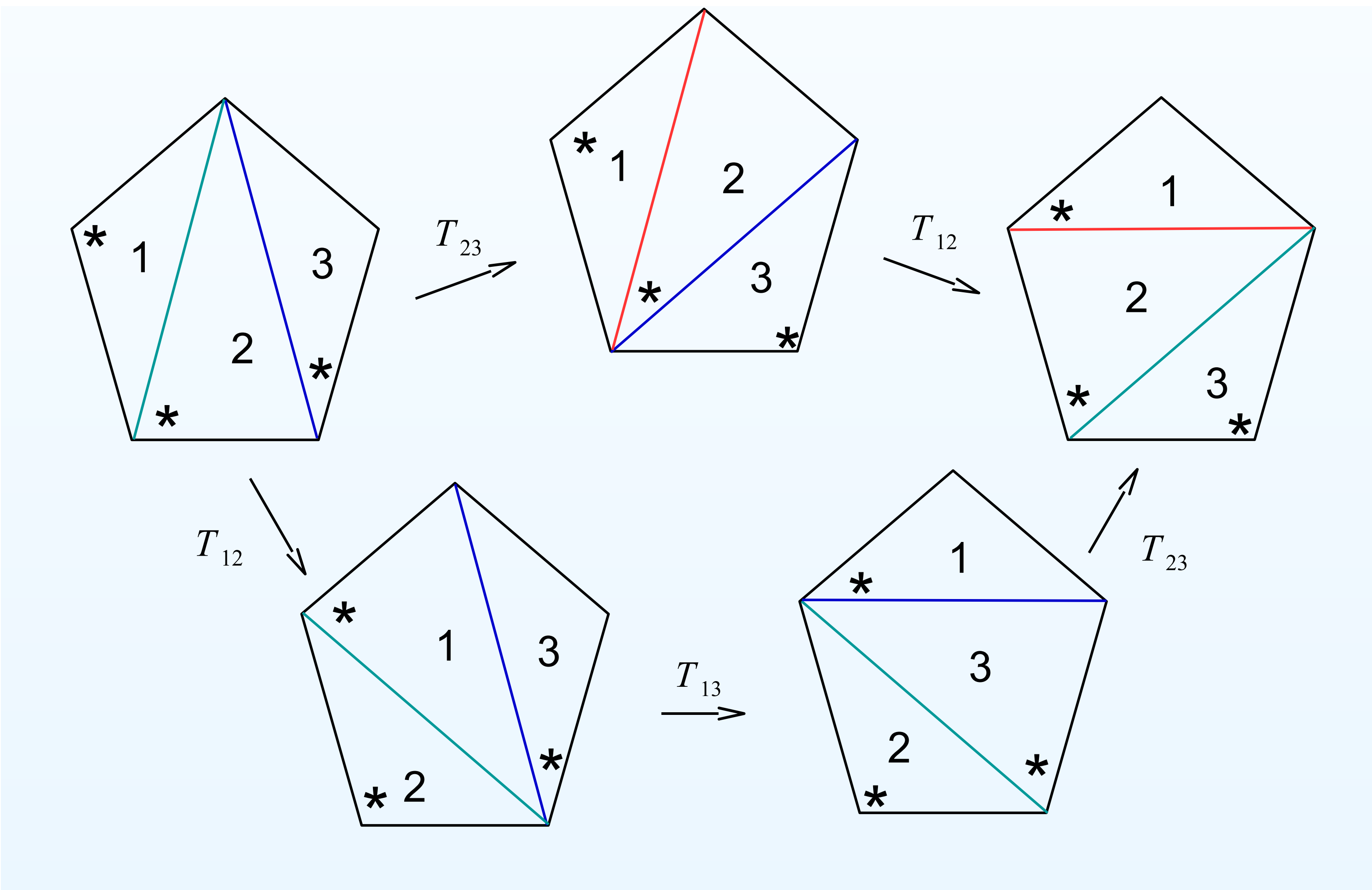
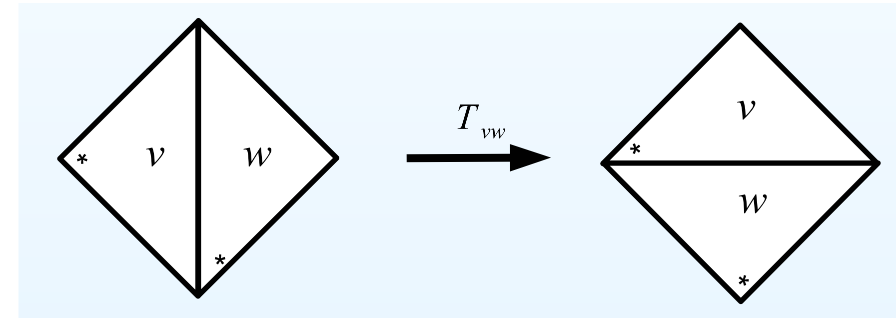
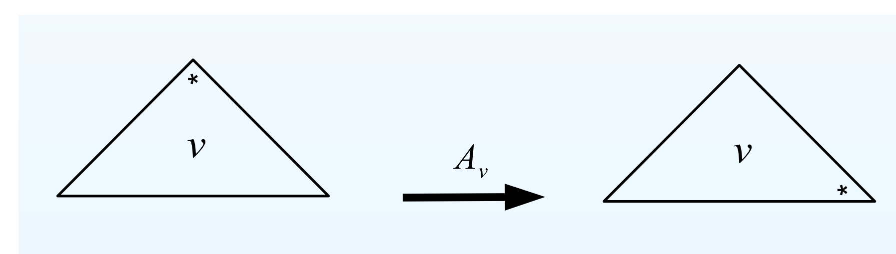




T_{vw}

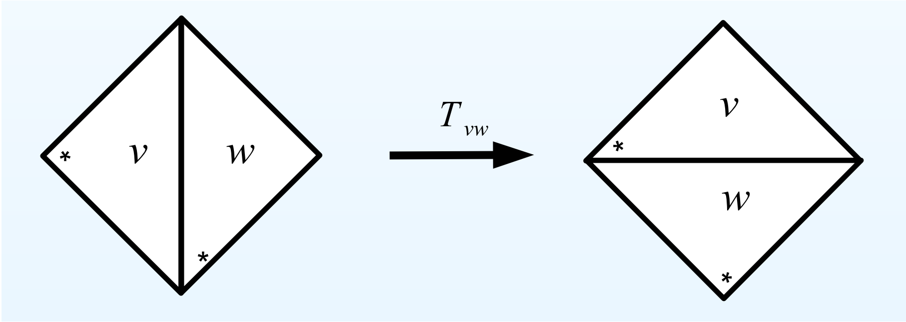
Flip



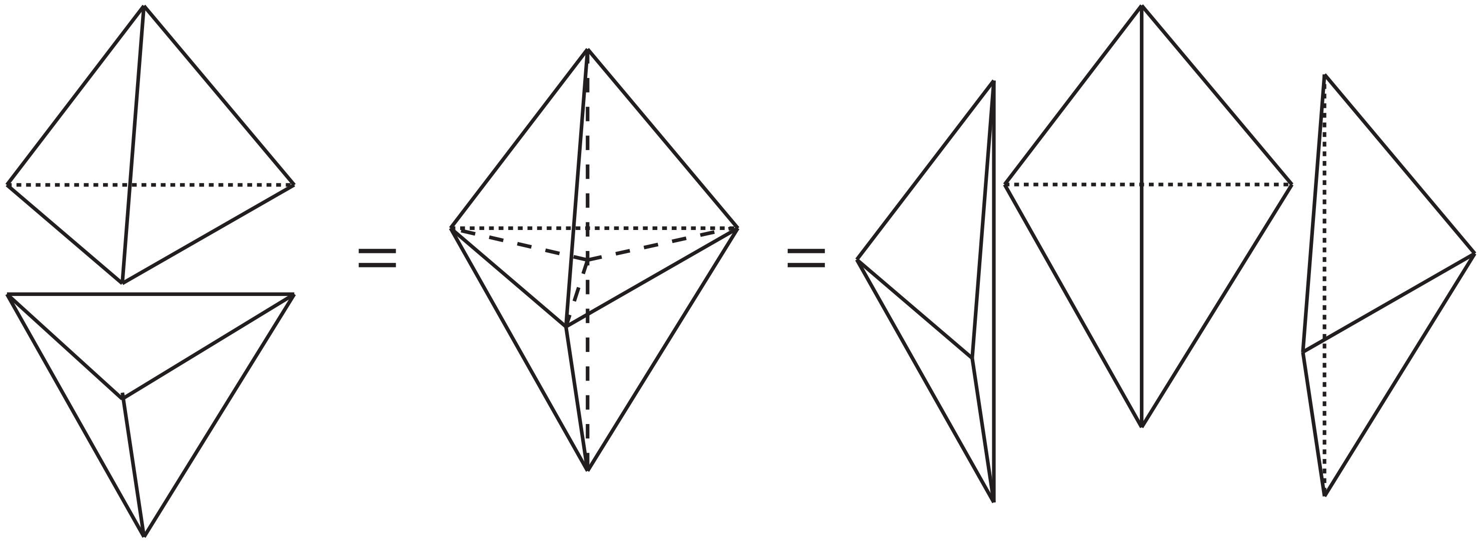
$$T_{23}T_{12} = T_{12}T_{13}T_{23}$$

$$A^3 = 1$$

Pentagon Relation



T_{vw}



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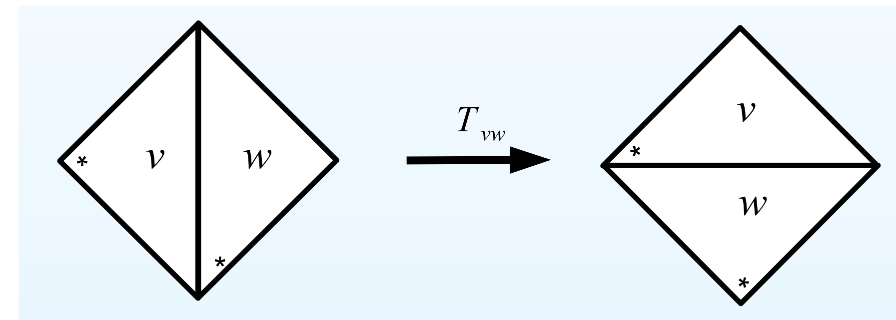
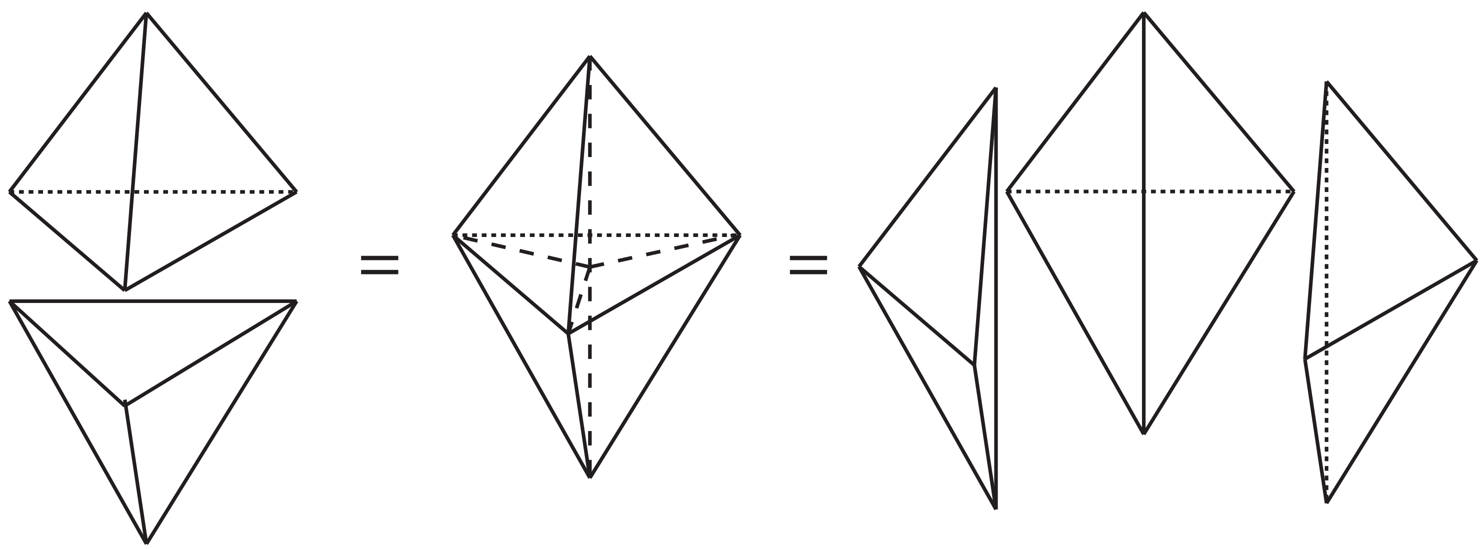
Pentagon Relation

2-3 Pachner moves



$$T_{vw}$$



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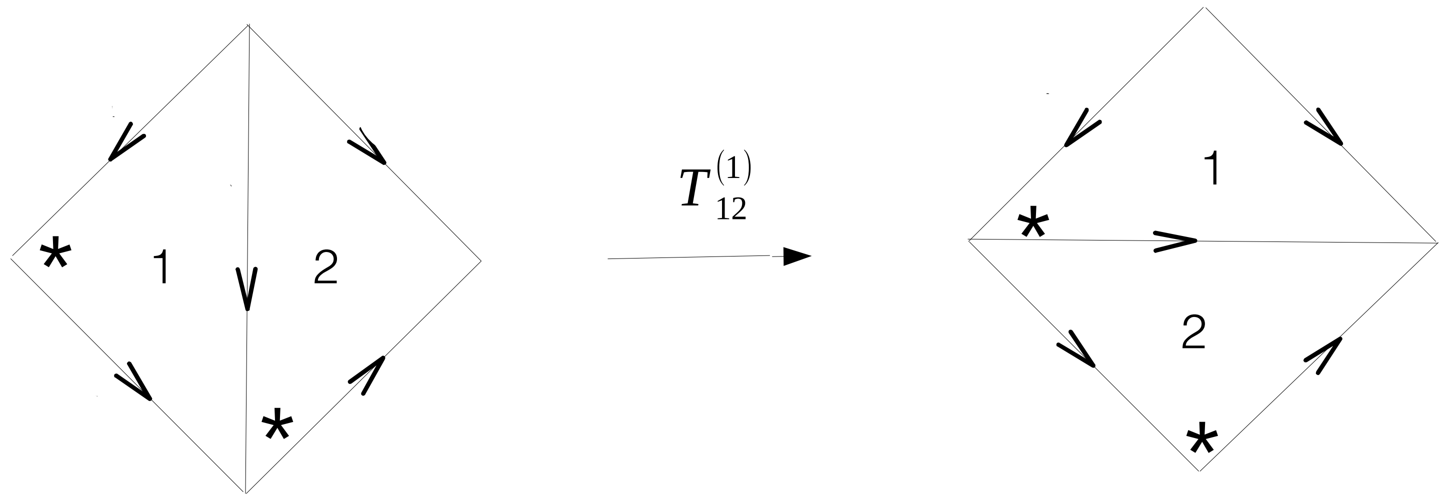
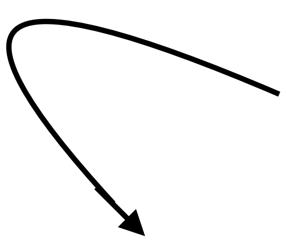
Super Flip

Superflip



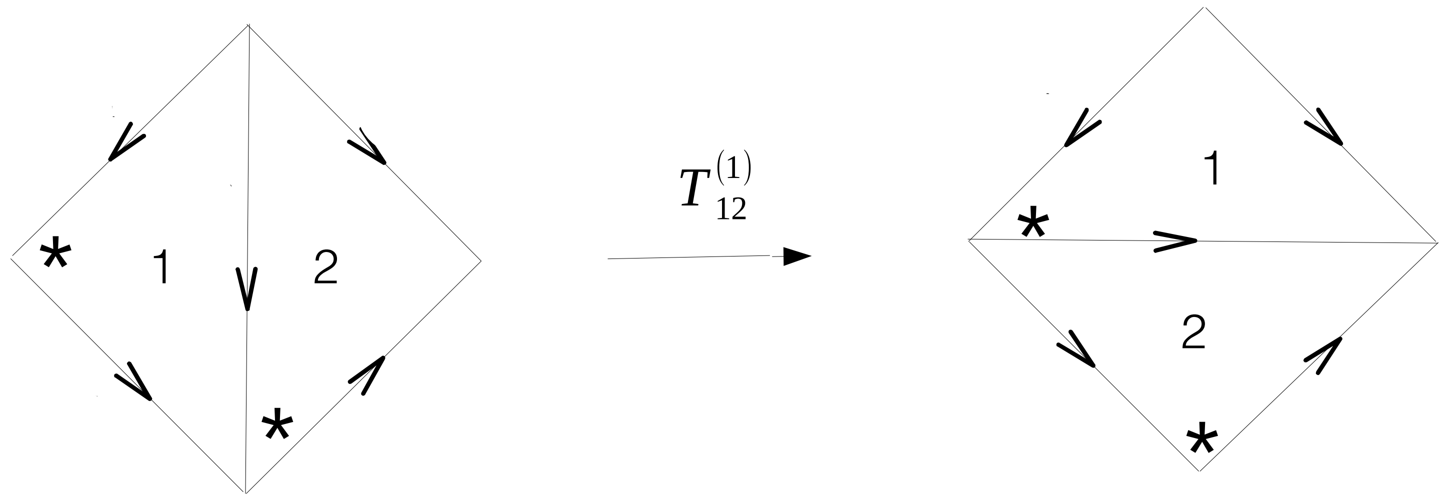
Rule: Just allow to have one or three arrows agains the background

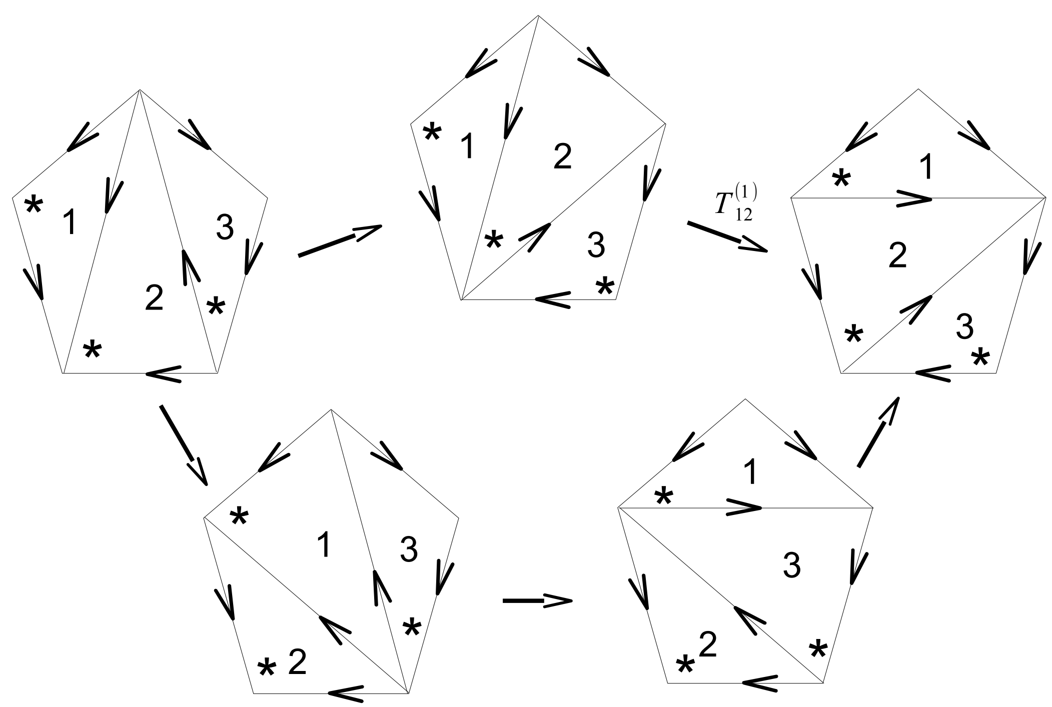


Background orientation

Super Pentagon Relation

Superflip

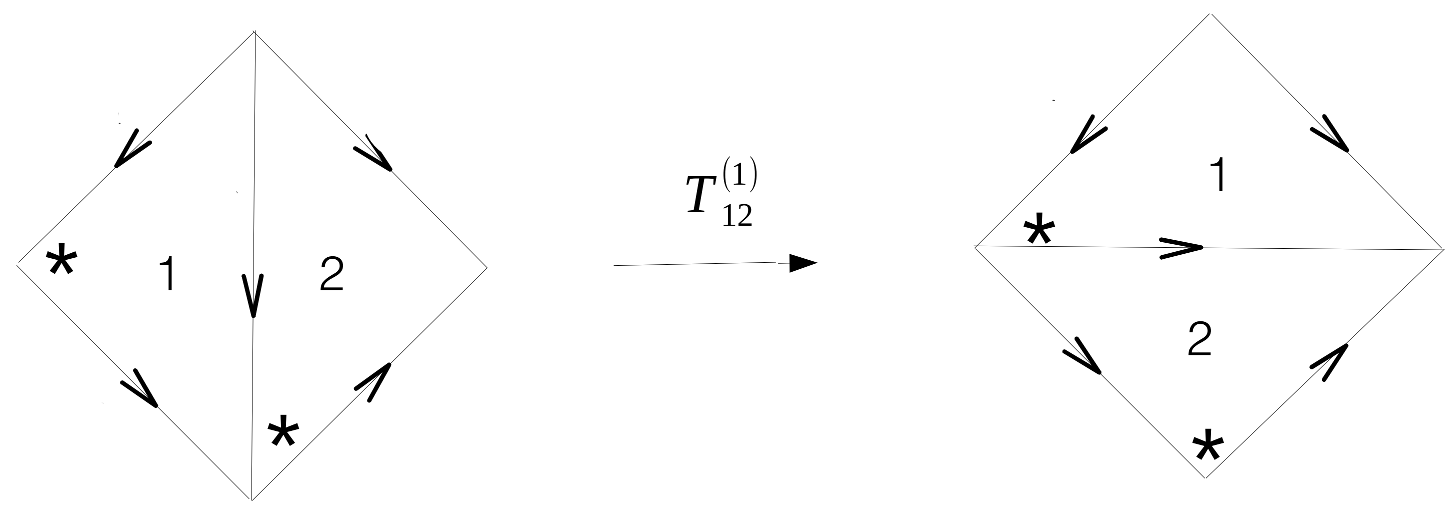




$$T^{(1)}_{23}T^{(1)}_{12} = T^{(1)}_{12}T^{(1)}_{13}T^{(1)}_{23}$$

More Super Flip

Superflip



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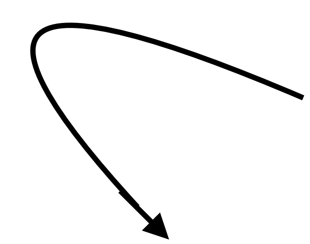
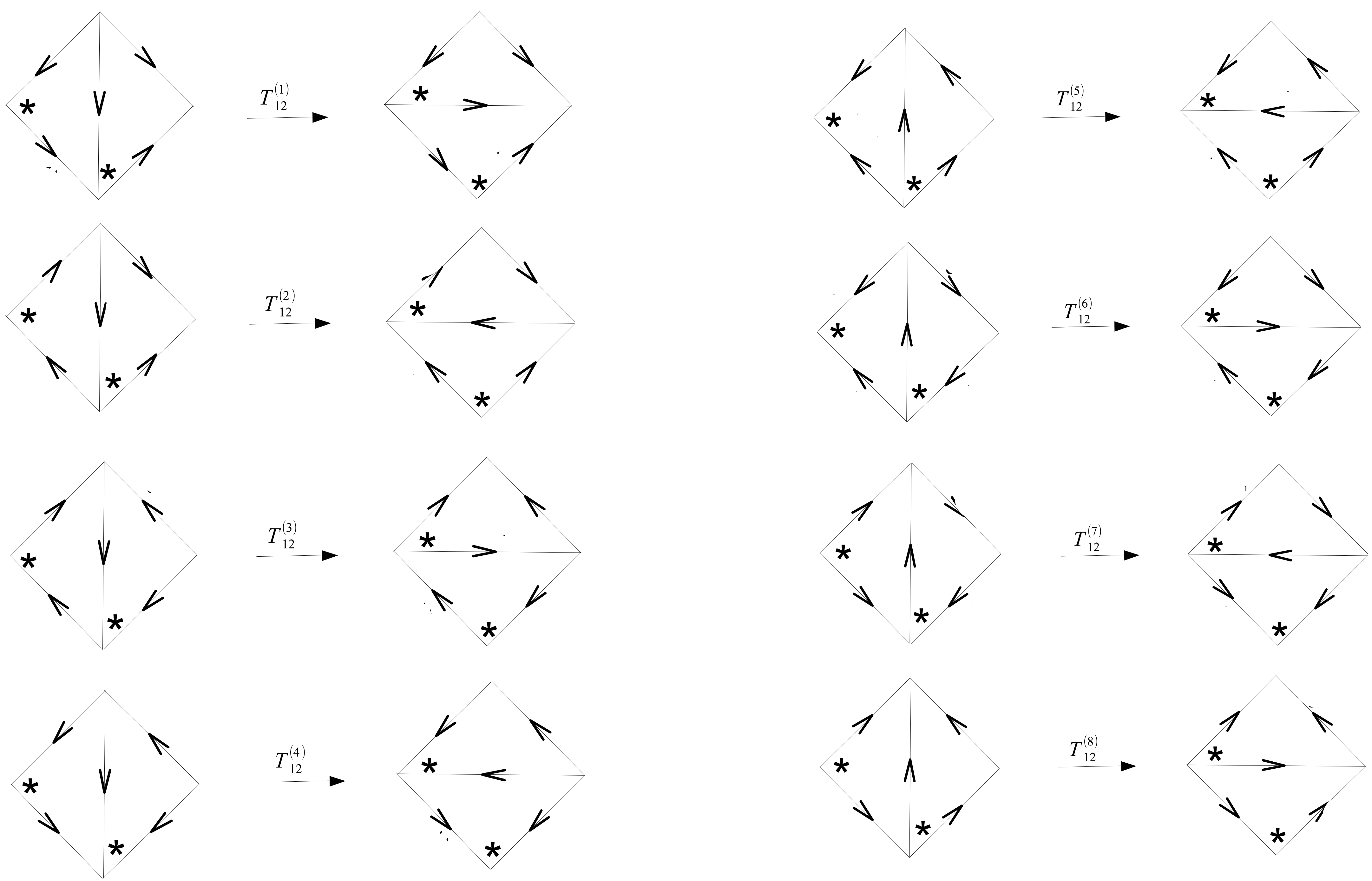


Background orientation

There are different types of flip depends on the orientation $T^{(i)}_{12}$

Super Flips

There are different types of flip depends on the orientation $T_{12}^{(i)}$



Super Pentagon Relation

- Question: How many super pentagon relation we can get?

$$T_{23}^{(k)}T_{13}^{(j)}T_{12}^{(i)} = T_{12}^{(m)}T_{23}^{(l)}$$

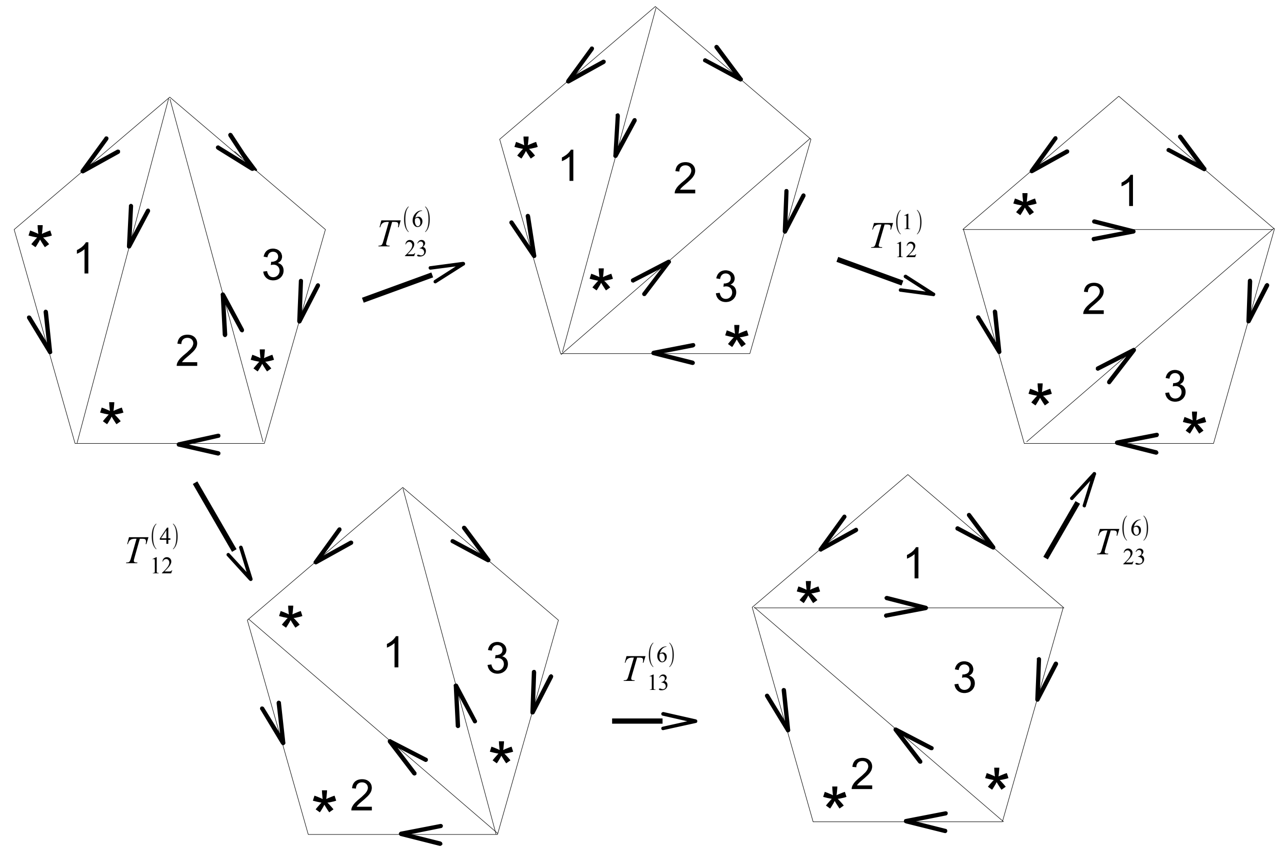
Super Pentagon Relation

- There are 16 pentagon relation.

$$T_{23}^{(k)}T_{13}^{(j)}T_{12}^{(i)} = T_{12}^{(m)}T_{23}^{(l)}$$

- $T^{(1)}$ satisfy pentagon by itself.

- All the other combinations reduced to the pentagon $T_{12}^{(i)} -> T_{12}^{(j)}$ satisfy pentagon by itself.

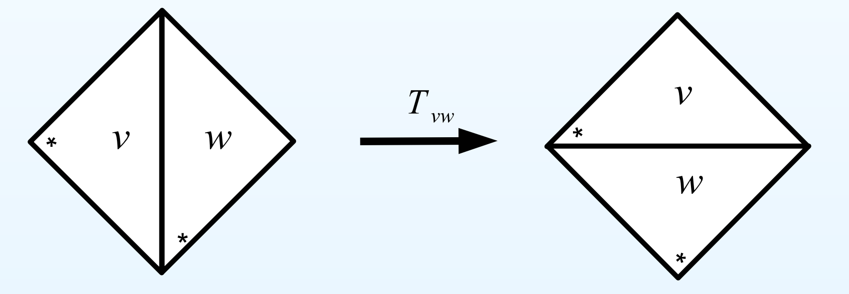


$$T_{23}^{(6)}T_{13}^{(6)}T_{12}^{(4)} = T_{12}^{(1)}T_{23}^{6)}$$

[N.A-Pawelkiewicz-Teschner ’15]

Algebraic Relation

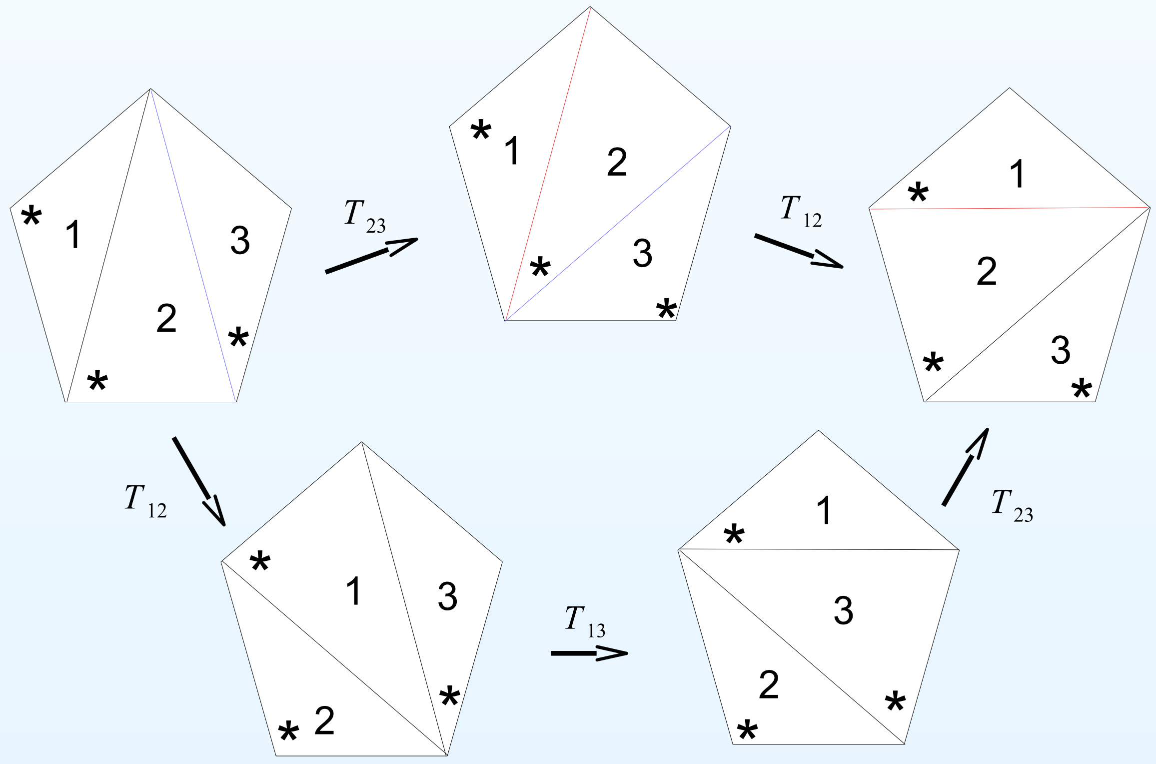
Using Hopf algebra techniques and quantum groups representations



$p_v\,,\, q_v$ for triangle v.

$$T_{vw} = g_b(e^{2\pi b(q_v + p_w - q_w)})e^{-2\pi i p_v q_w},$$

where g_b is Faddeev quantum dilogarithm function



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[Kashaev]

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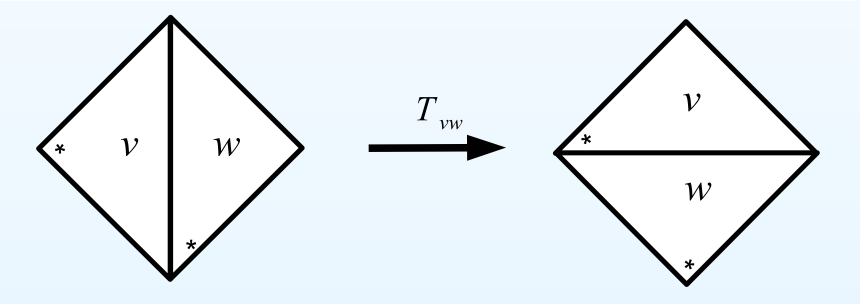

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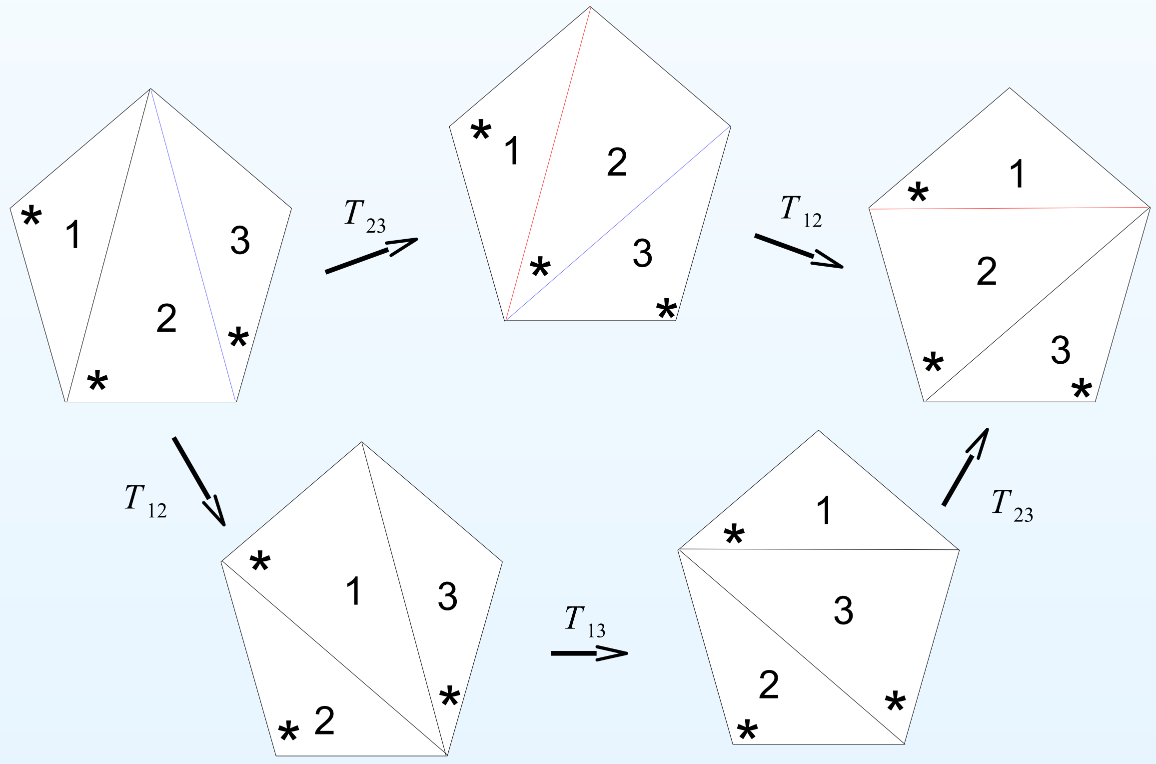

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Quantum Teichmüller spaces

[Kashaev]

Quantum Dilogarithm Function

It is a special function. It has relation with double gamma function.

$$e_b(x) = g_b(\frac{1}{2\pi b}) = exp[\int_{\mathbb{R}+i0} \frac{dw}{w} \frac{e^{-2ixw}}{4\sinh(wb)\sinh(wb^{-1})}]$$

[Faddaeev]

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[Faddaeev]

- It has the following shift and reflection relations:

$$e_b(x - \frac{ib^{\pm 1}}{2}) = (1 + e^{2\pi b^{\pm 1} x}) e_b(x + \frac{ib^{\pm 1}}{2})$$

$$e_b(x)e_b(-x) = e^{-i\pi(1-Q^2/2)/6} e^{i\pi x^2}.$$

$Q = b + b^{-1}$

- It has the asymptotic behaviour :

$e_b(x) = 1 \qquad x \to -\infty$

$e_b(x) = e^{-i\pi(1-Q^2/2)/6} e^{i\pi x^2} \qquad x \to +\infty$

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[Faddaeev]

- we have the following variant of the pentagon relation :

$$[p, x] = \frac{1}{2\pi i}$$

$$e_b(p)e_b(x) = e_b(x)e_b(x+p)e_b(p)$$

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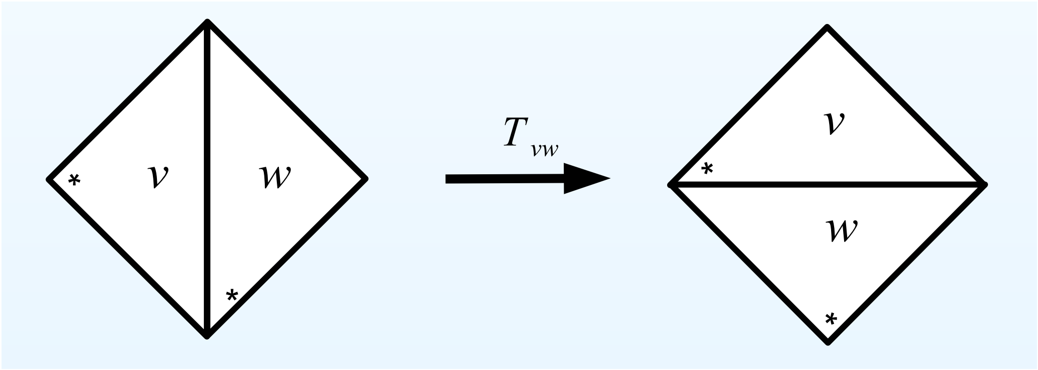
- Also, for non-commutative variables U,V such that $UV = q^2 VU$, $q = e^{i\pi b^2}$ satisfies the pentagon relation

$$g_b(U)g_b(V) = g_b(V)g_b(q^{-1}UV)g_b(U).$$

Goal of the Game



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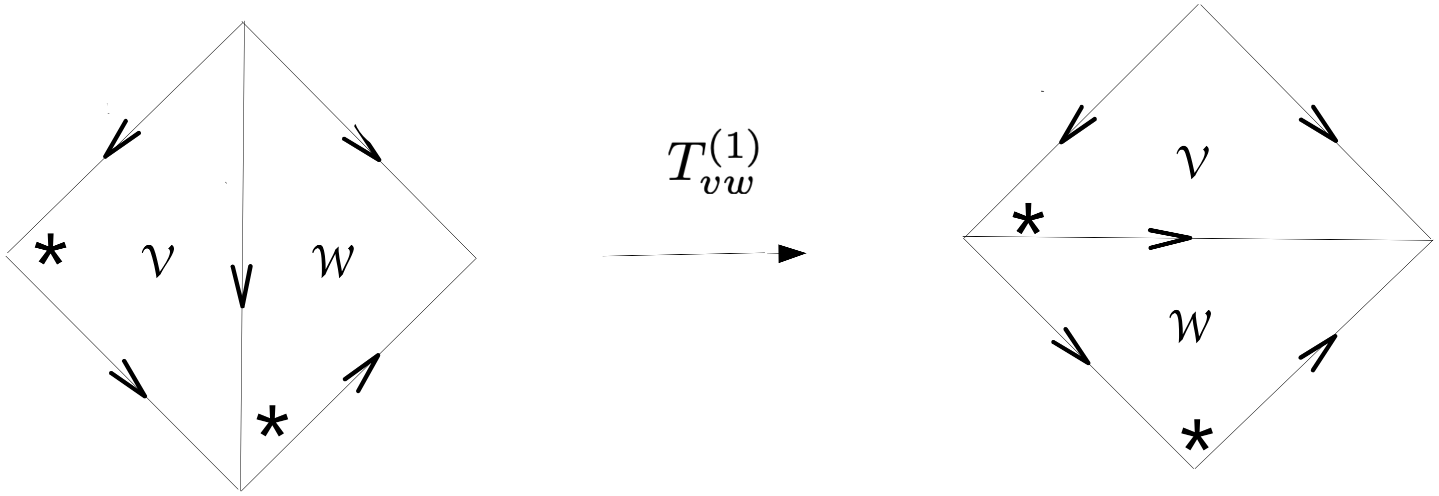

$$T^{(1)}_{vw} = ?$$

Physics Motivation

II) AGT (4d-2d) Correspondence

SUSY Gauge theory

- Gauge theory describes the interactions that bind quarks into Hadrons.
- Understanding strong coupling behaviour remains as an challenge (confinement).
- Adding Supersymmetry: Simplify computational difficulties, add more mathematical structure.

Physics Motivation

4d, $\mathcal{N} = 2$
Gauge theory S^4

Physics Motivation

Alday, Gaiotto, and Tachikawa '09:

made the observation that the four-sphere partition function theories can be expressed in terms of the correlation functions of Liouville conformal field theory on Riemann surface .

4d, $\mathcal{N}=2$
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4d-2d correspondence (AGT)

[Alday, Gaiotto, Tachikawa.' 09]

2d CFT
(Liouville)

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[Alday, Gaiotto, Tachikawa.' 09]

2d CFT
(Liouville)

4D gauge theory	2D CFT
instanton partition function	conformal block
perturbative part	3-point function (Liouville)
coupling constants	cross ratios
masses	external momenta
Coulomb moduli	internal momenta
generalized S-duality	crossing symmetry
Omega background $\epsilon_1 = b,\ \epsilon_2 = b^{-1}$	Coupling constant/central charge

Physics Motivation

Alday, Gaiotto, and Tachikawa ’09:

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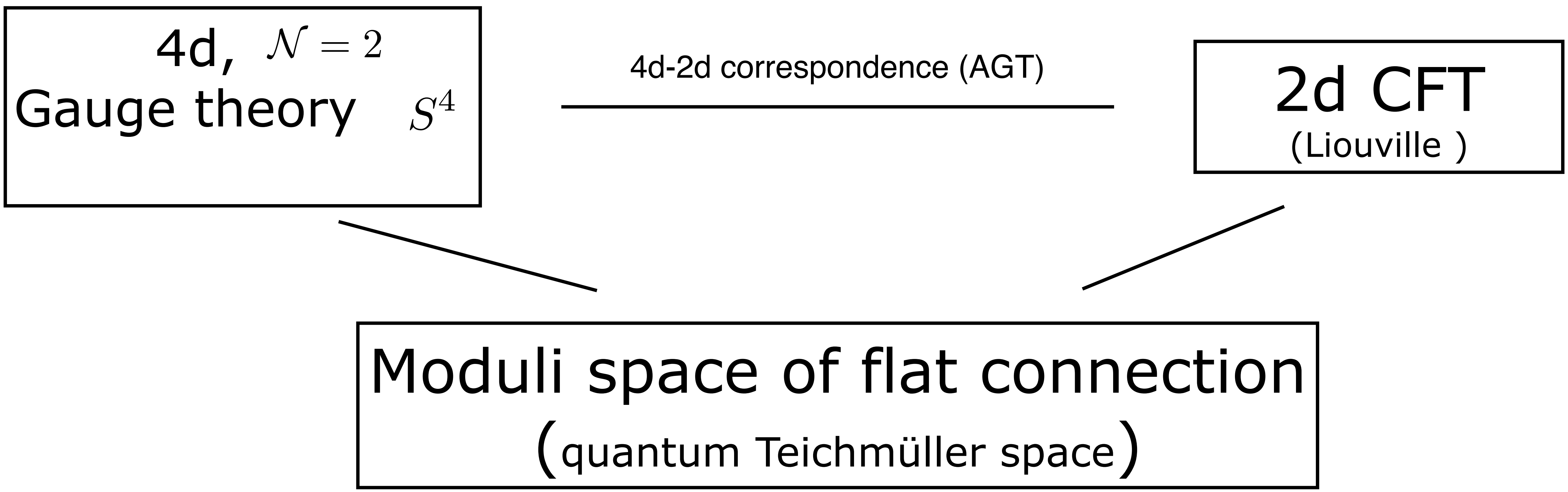


One can show that this moduli space contains a component which can be identified with Teichmüller space.

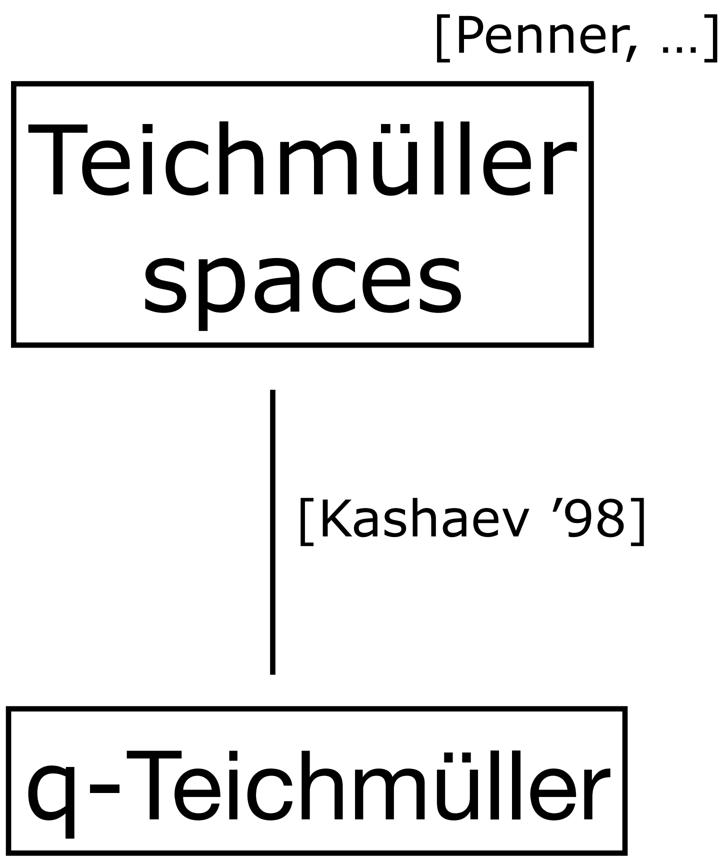
Motivations

TQFT

AGT Correspondence

Math: Quantise the Classical Teichmüller space

Mathematics Motivation



Mathematics Motivation

[Penner, ...]

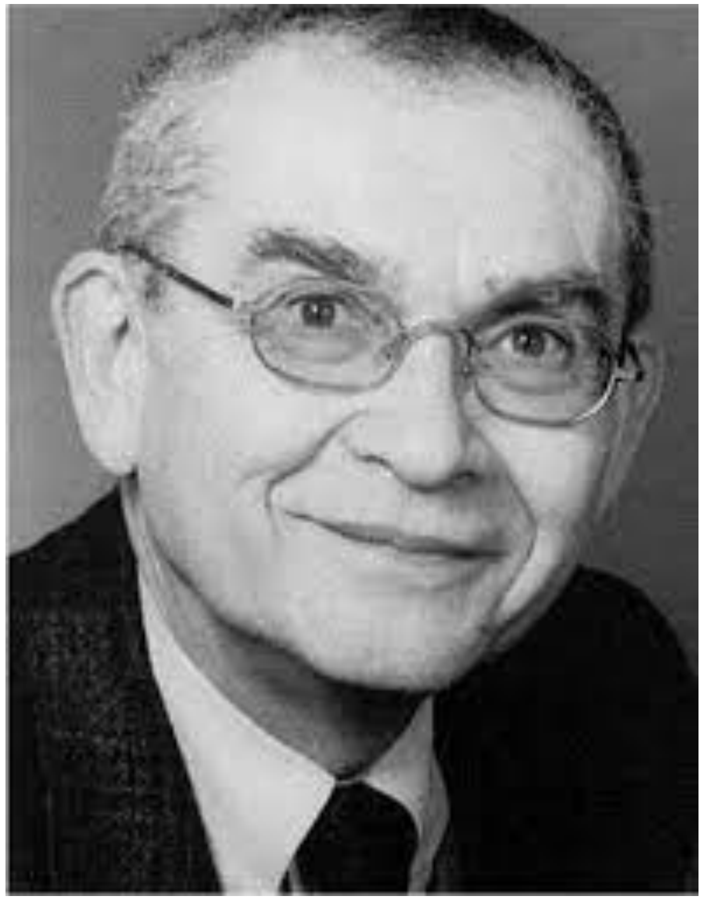
Teichmüller spaces

[Fock-Buchbacher '13/ Penner-Zeitlin '15]

Super-Teichmüller

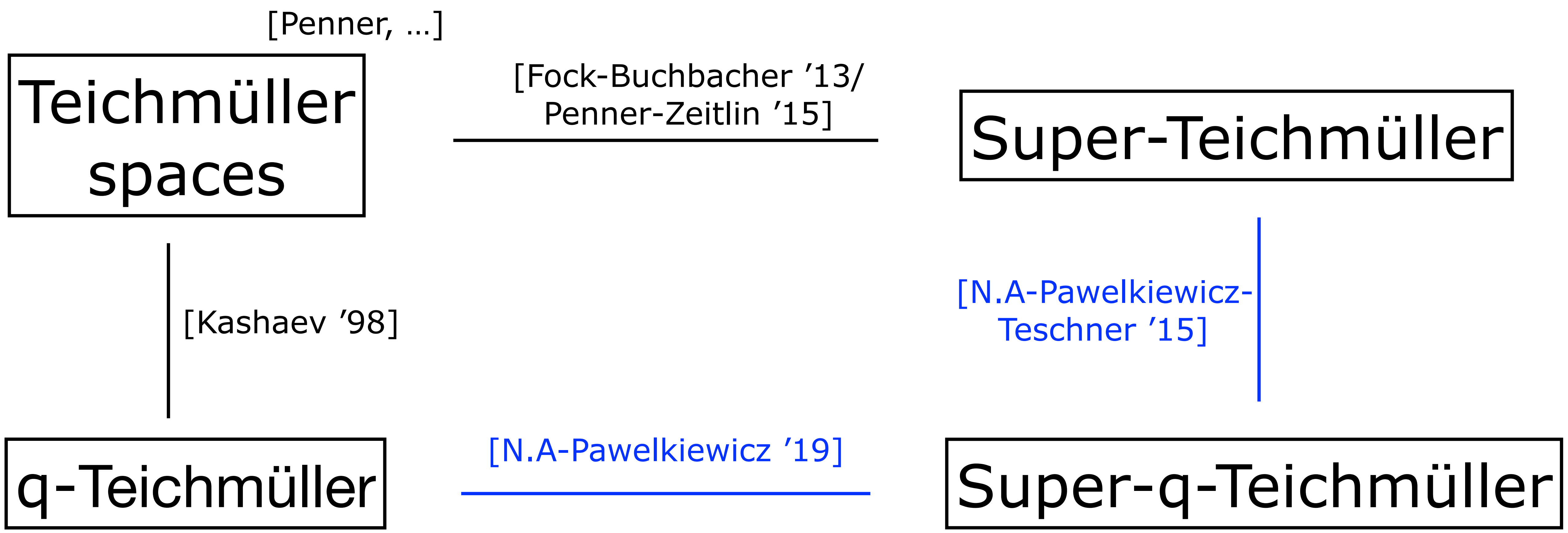
[Kashaev '98]

q-Teichmüller



In the late 80s the problem of construction of Penner's coordinates on Super Teichmüller was introduced in Yu.I. Manin's Moscow seminar.

Mathematics Motivation



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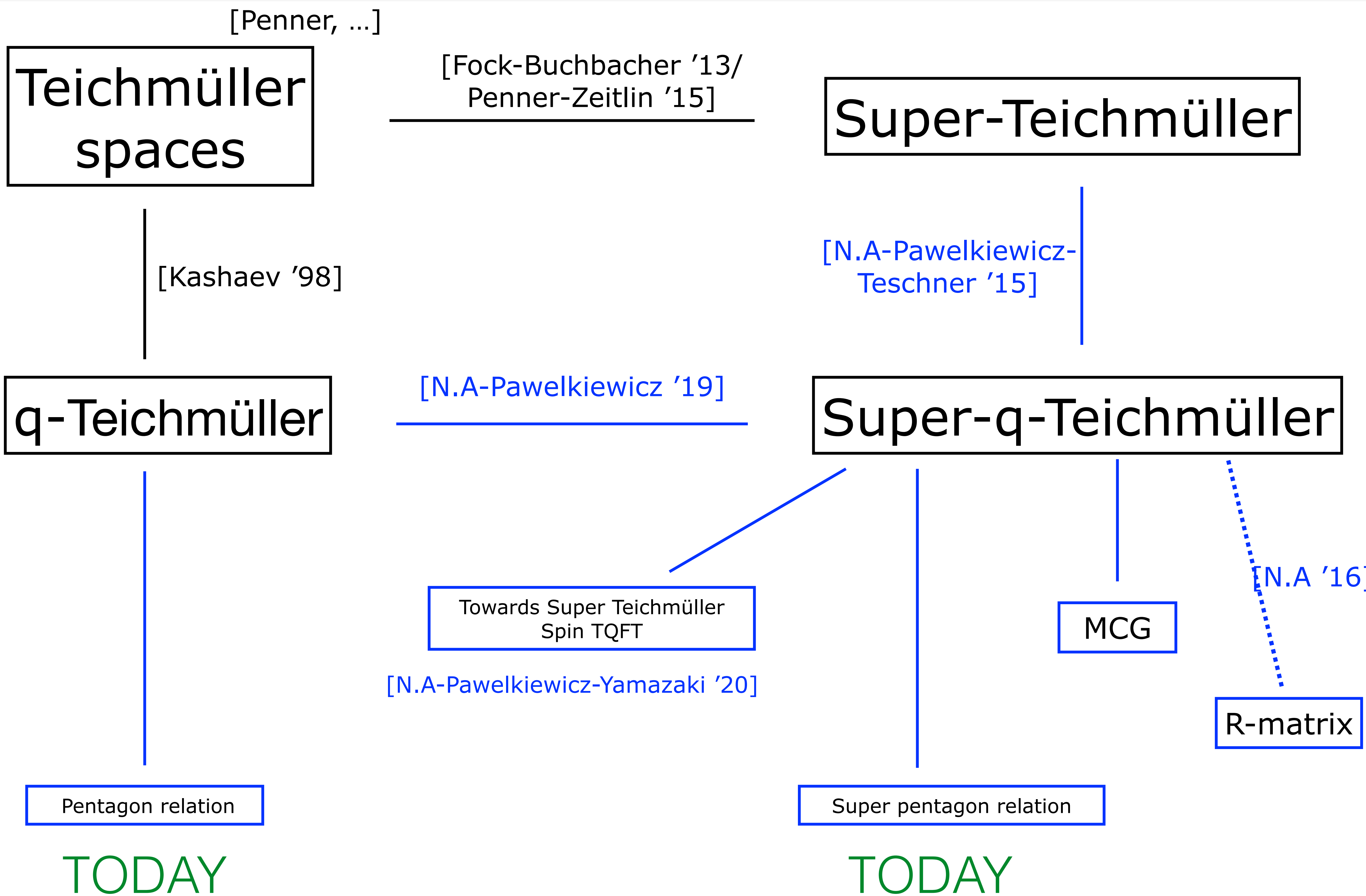
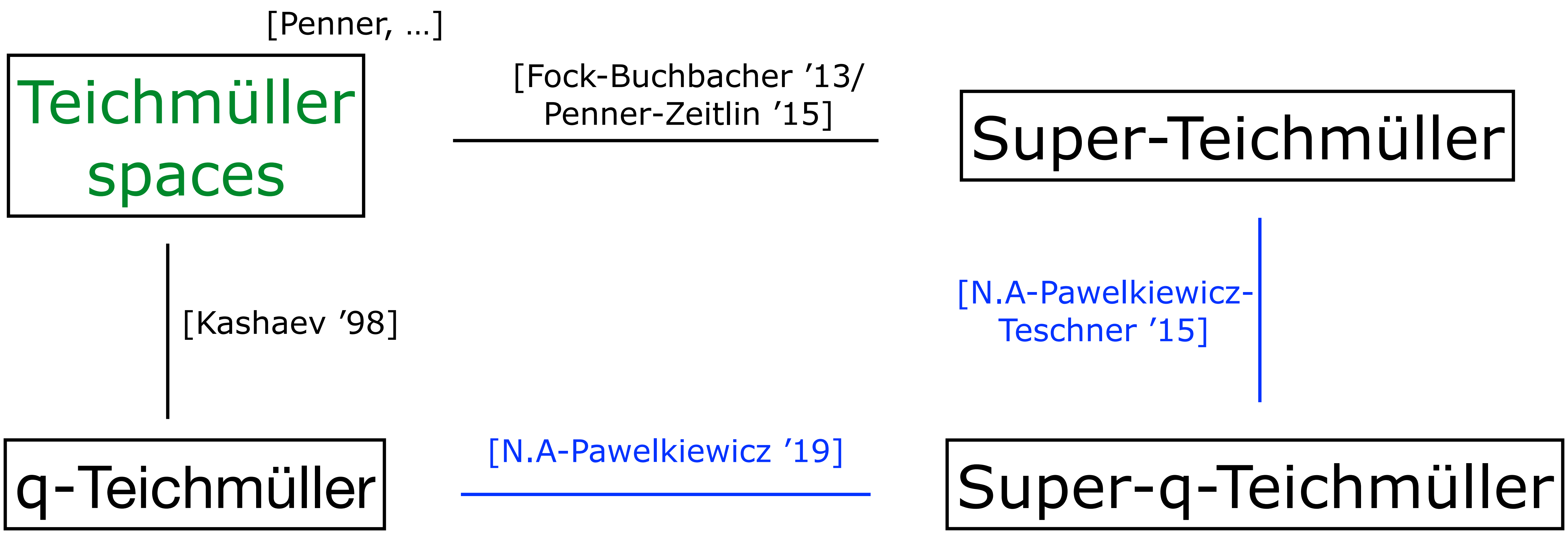


Plan for the rest of the talk



I) Super Teichmüller space and Super Penner coordinate on that.

II) Super Kashaev coordinate to be able to quantise.

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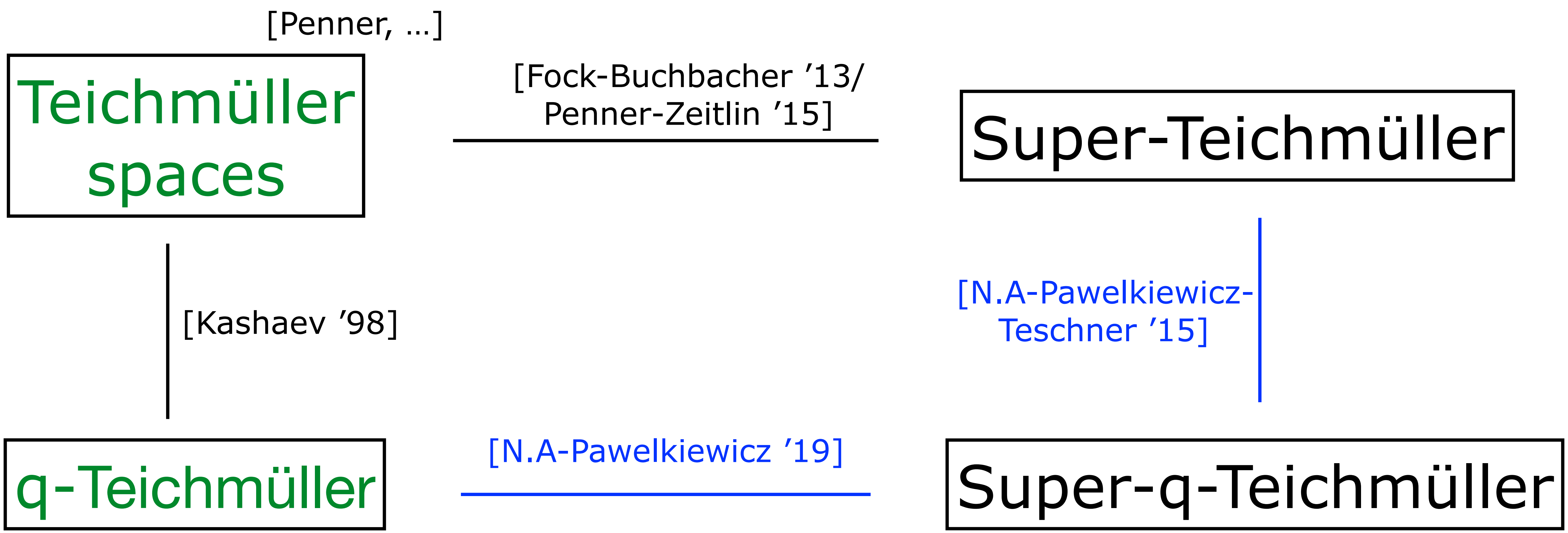

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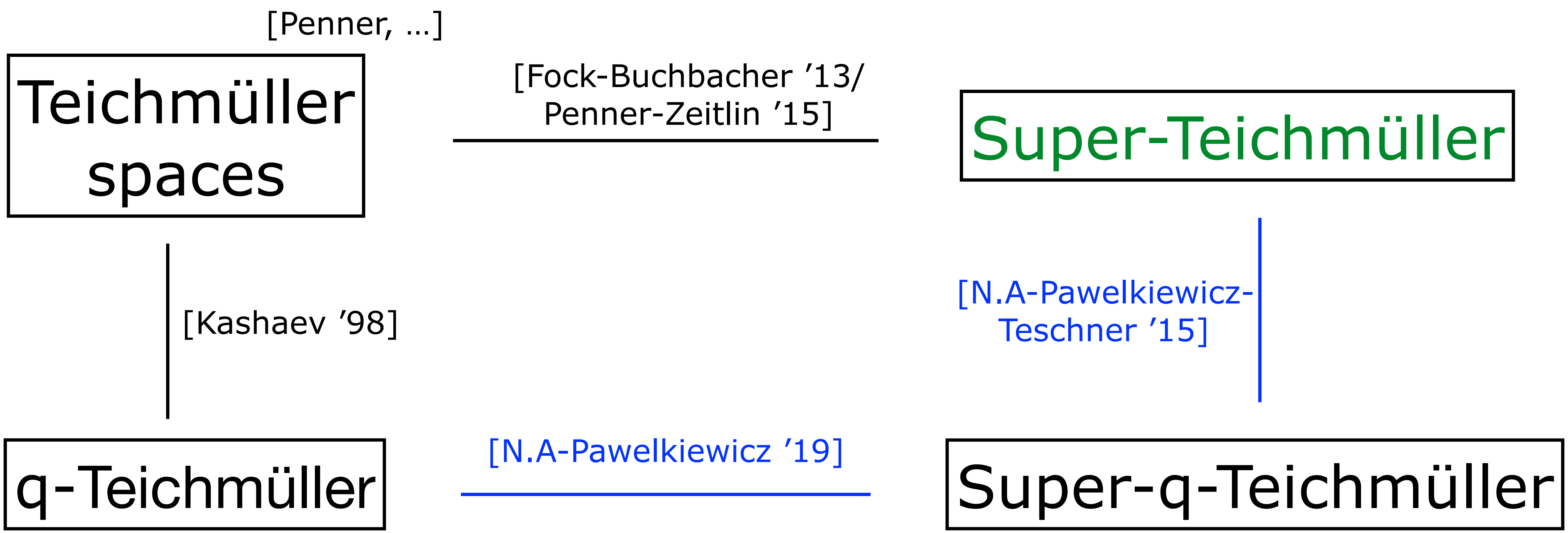

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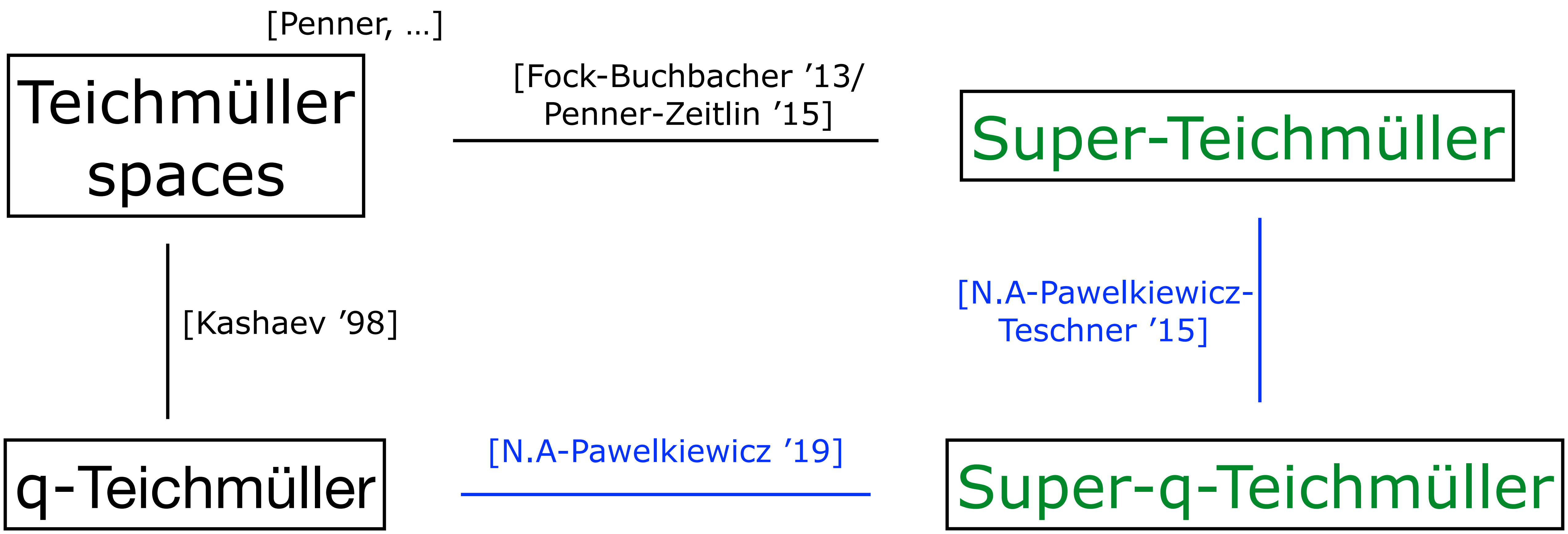

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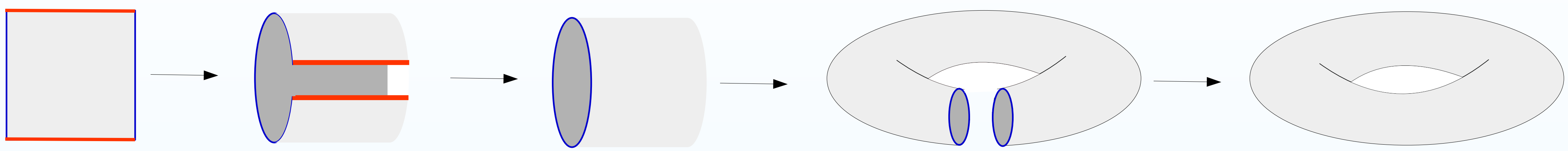
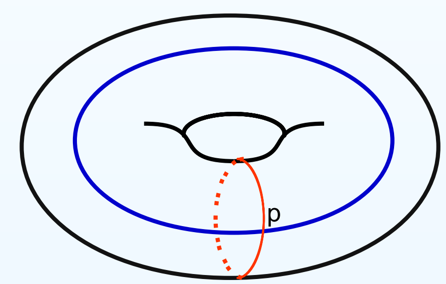
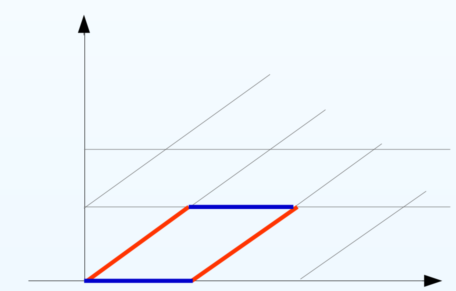
Teichmüller space





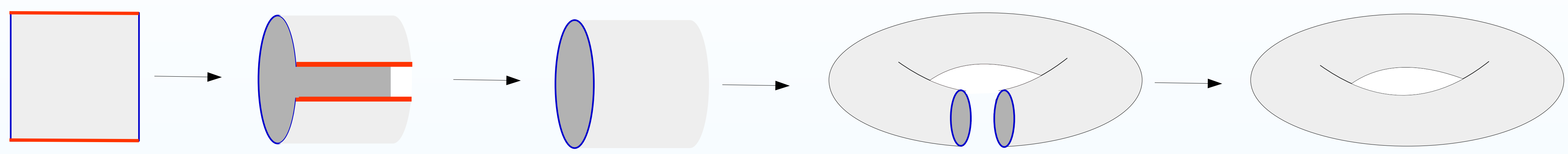


Example: Punctured Torus $F_{1,1}$

$$F_{1,1} \sim \mathbb{C}/\Lambda$$

The torus can be written as complex plane modulo lattice Λ. Different Λ will give different complex structures.

Teichmüller space



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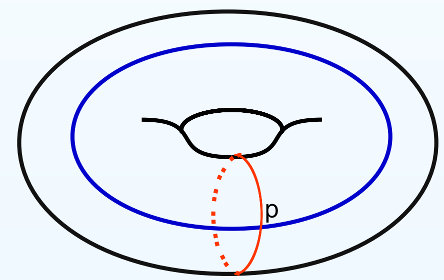
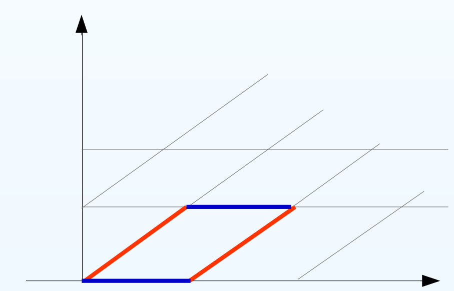




The torus can be written as complex plane modulo lattice Λ. Different Λ will give different complex structures.

$$F_{1,1} \sim \mathbb{C}/\Lambda$$

◀ How many inequivalent complex structures are there?

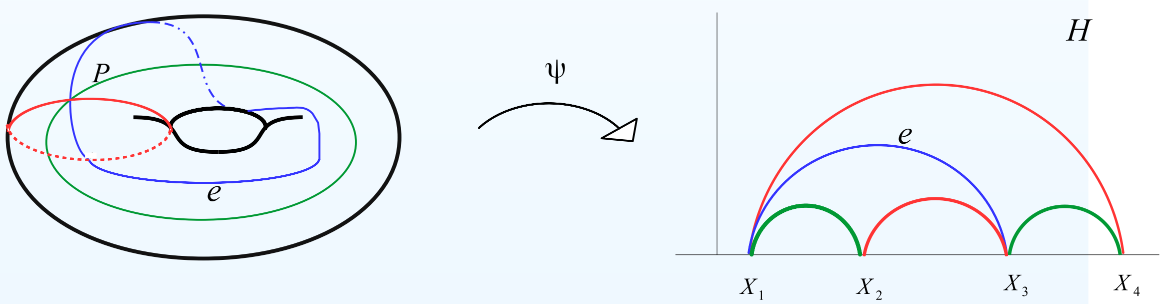
Teichmüller space of Riemann surfaces $\mathcal{T}_{g,n}$:

Space of deformations of complex structures.

Teichmüller space

- Riemann surface $F_{g,n}$ with genus $g \geq 0$ and $n \geq 1$ boundary components.
- Complex manifold with conformal structure, constant negative curvature.
- Boundary components being punctures, i.e. holes of zero length.

- Any Riemann surface represented as a quotient of upper half plane by a discrete group of isometries Γ.
  $F \equiv H/\Gamma$



ψ is an uniformisation map

Definition:

The space of deformations of the metrics of constant negative curvature

$$\mathcal{T}_{g,n} = Hom(\pi_1(F_{g,n}) -> PSL(2,\mathbb{R})/PSL(2,\mathbb{R}))$$

$PSL(2,\mathbb{R})$ acts by conjugation.

System of Coordinates

Goal: Find a system of coordinates on $\mathcal{T}_{g,n}$,
so that the action of MC(F) is realised in the simplest possible way.

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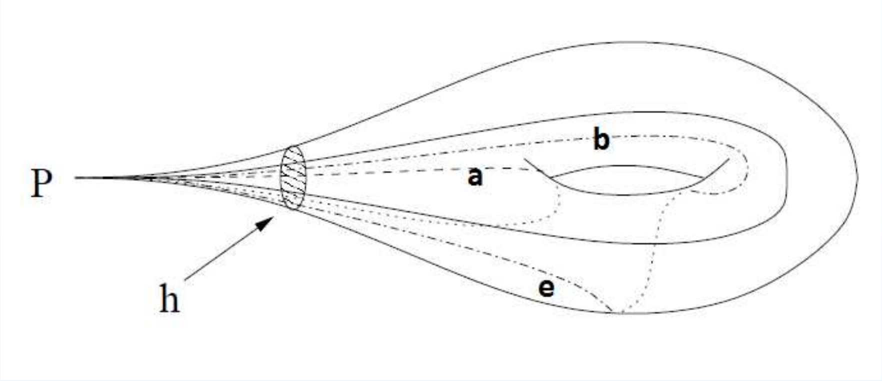
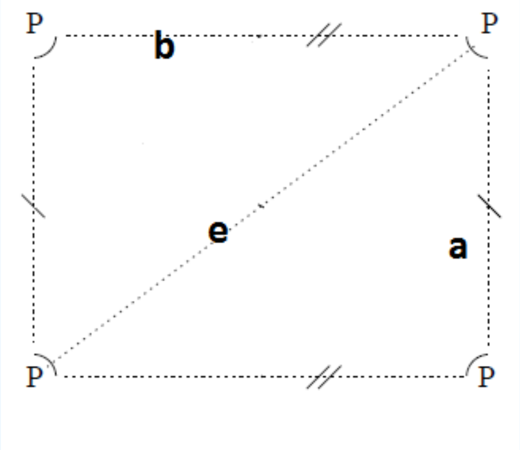
An ideal triangulation of $F_{g,n}$: Geodesics that start and end at the punctures.





Remember our game!

System of Coordinates

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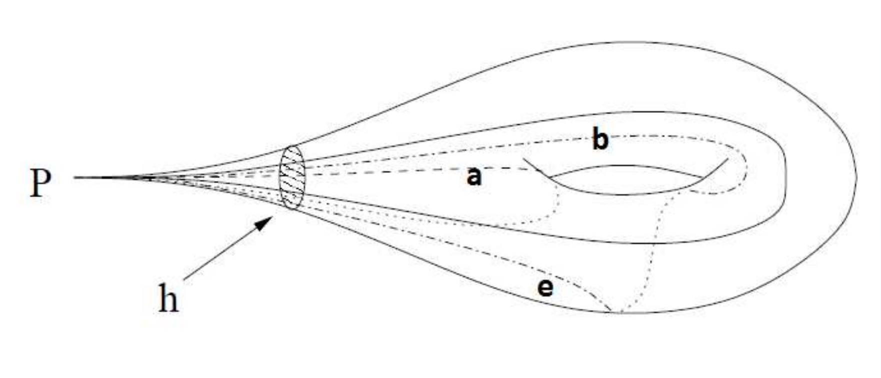

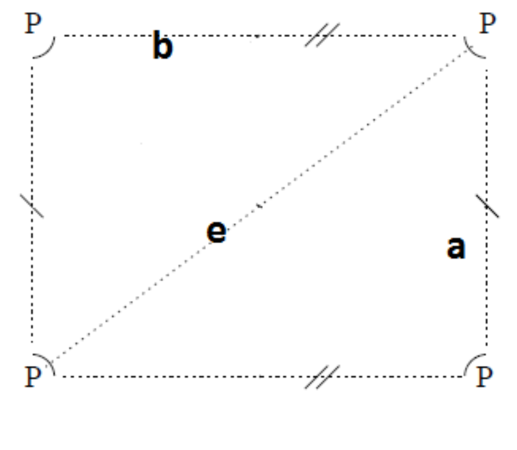

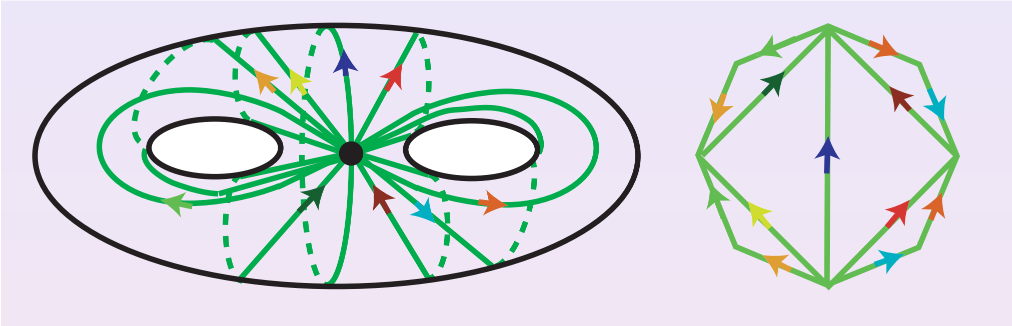

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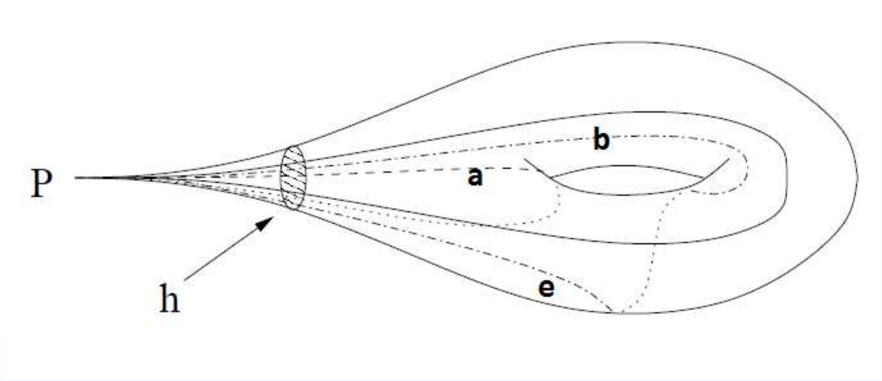
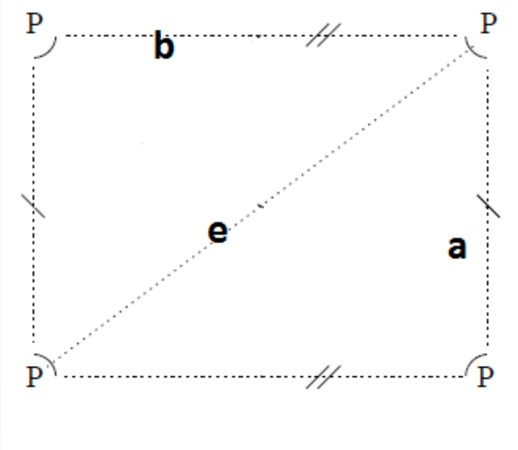
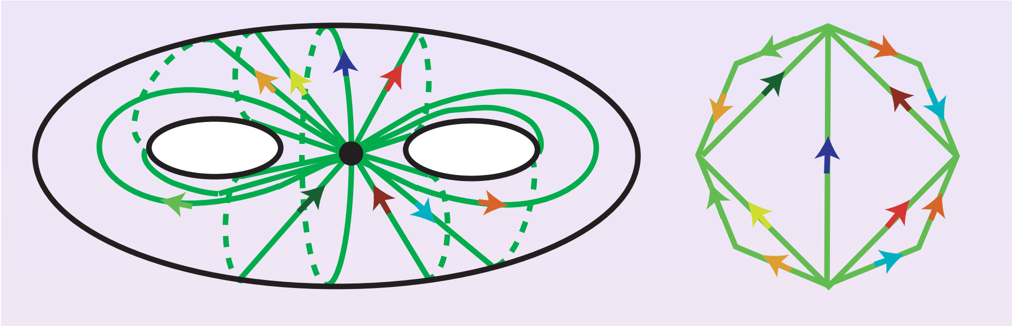






[Penner's 1980s]

A construction of coordinates associated to the

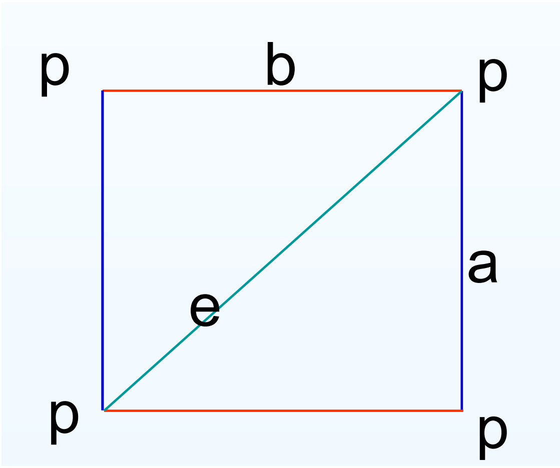
ideal triangulation of F , so one assigns one positive number for every edge.

System of Coordinates

- In order to obtain coordinates on $\mathcal{T}_{g,n}$

one can consider penner/shear coordinates z_e to the edge e

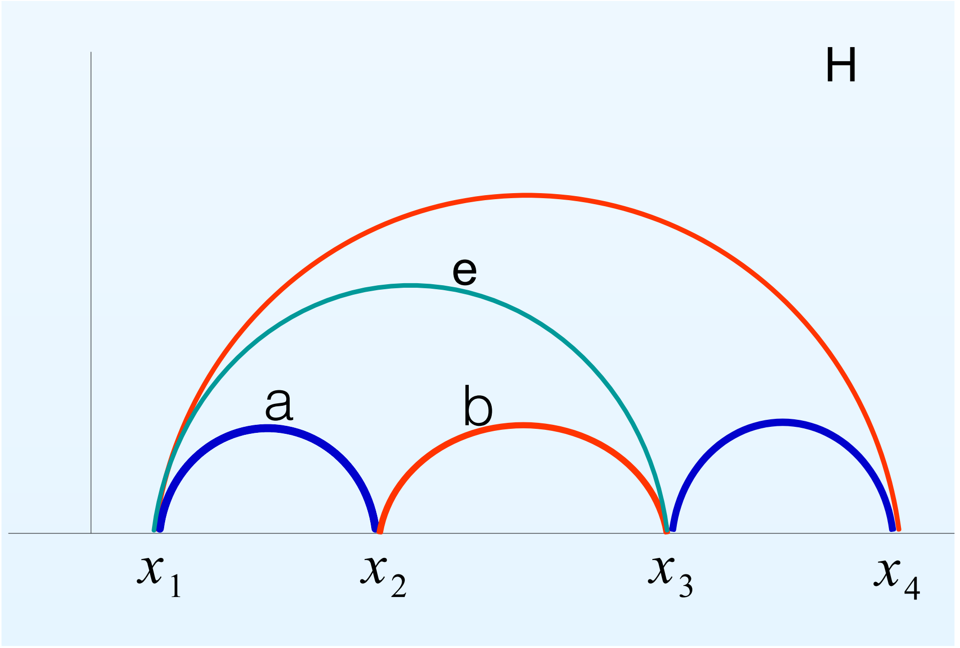
$$e^{z_e} = \frac{ac}{bd}$$



- Conformal cross ration is:

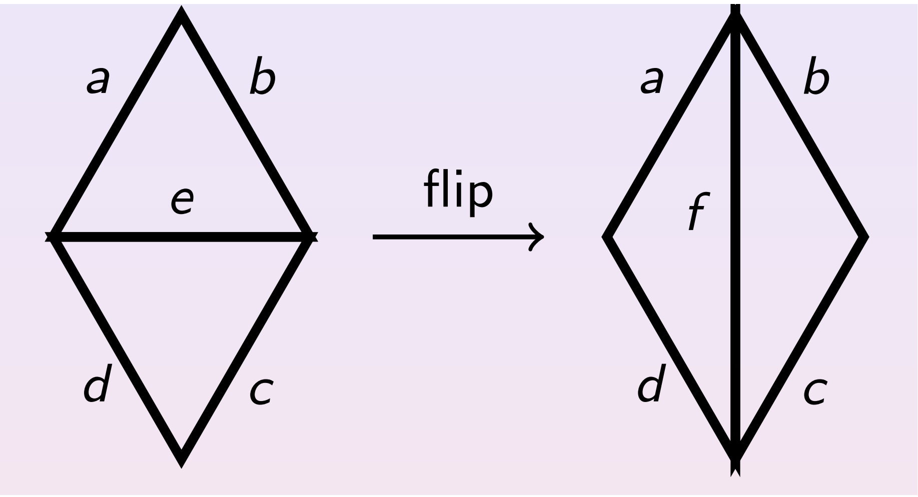
$$e^z = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}$$

$$a = x_1 - x_2$$



Teichmüller space

The theory should be independent of the choice of triangulations.



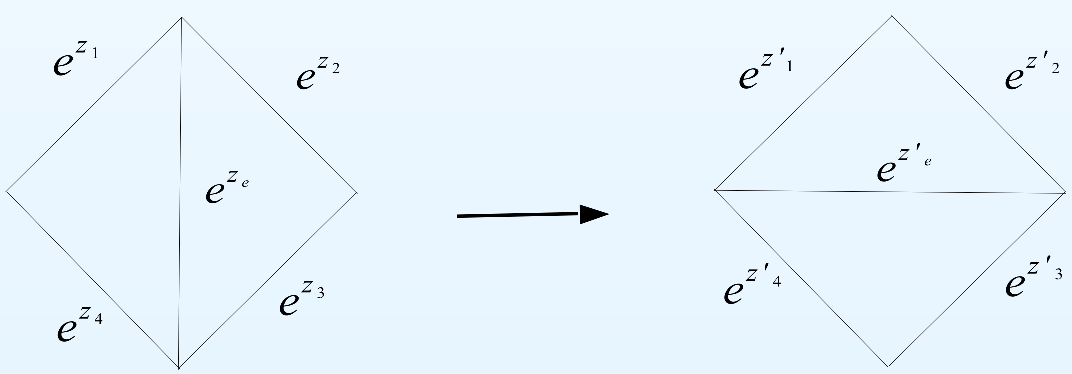
The action of MC(F) can be described combinatorially using elementary transformations called flips:

$$ef = ac + bd$$

Ptolemy relation

Teichmüller space

Change of shear coordinates under the change of triangulation



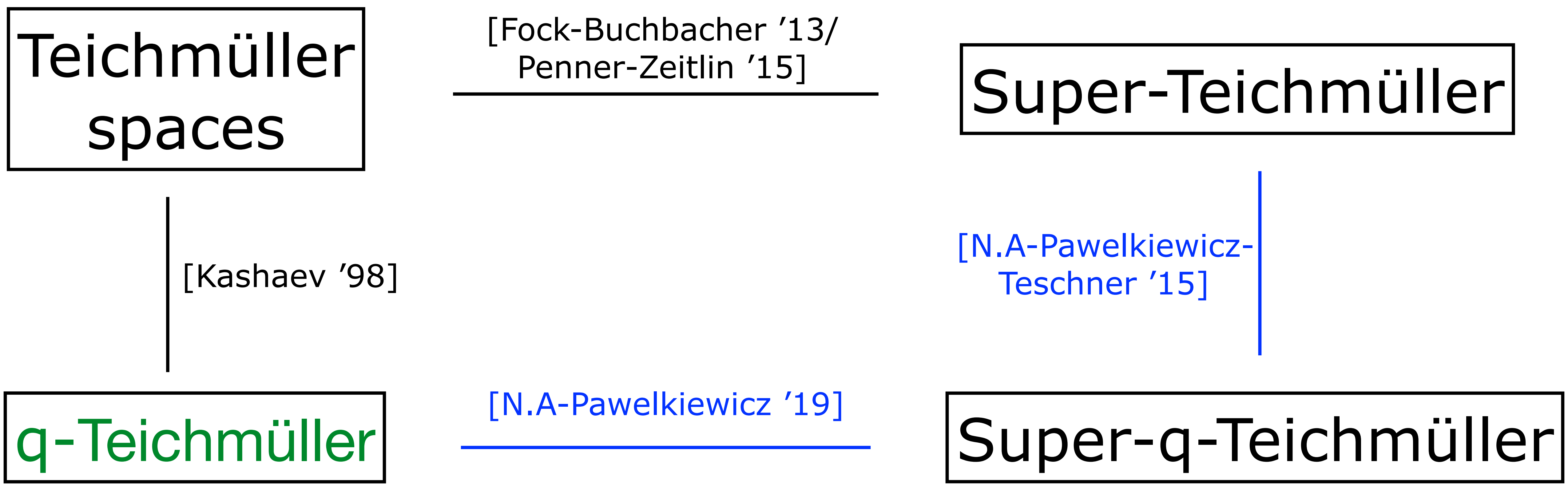
$$\begin{aligned}
e^{z'_e} &= e^{-z_e},\\
e^{z'_1} &= e^{\frac{z_1}{2}}(1+e^{z_e})e^{\frac{z_1}{2}},\\
e^{z'_2} &= e^{\frac{z_2}{2}}(1+e^{-z_e})^{-1}e^{\frac{z_2}{2}},\\
e^{z'_3} &= e^{\frac{z_3}{2}}(1+e^{z_e})e^{\frac{z_3}{2}},\\
e^{z'_4} &= e^{\frac{z_4}{2}}(1+e^{-z_e})^{-1}e^{\frac{z_4}{2}}.
\end{aligned}$$

Transformation of coordinates via the change of triangulation is therefore governed by Ptolemy relations.

This leads to the prominent geometric example of cluster algebra.

[S. Fomin and A. Zelevinsky 2000]

Plan for the rest of the talk



Teichmüller space has symplectic structure.

Remark: Given by the Weil-Petersson form.

Quantisation

Kashaev coordinates: [Kashave/ Fock-Chekhov]

- Pair of variables (p_v, q_v) for each triangle v.
- Can be written in term of Penner/shear coordinates.

$$\{p_v, p_w\} = 0,$$
$$\{q_v, q_w\} = 0,$$
$$\{p_v, q_w\} = \delta_{v,w},$$

$$[p_v, p_w] = 0,$$
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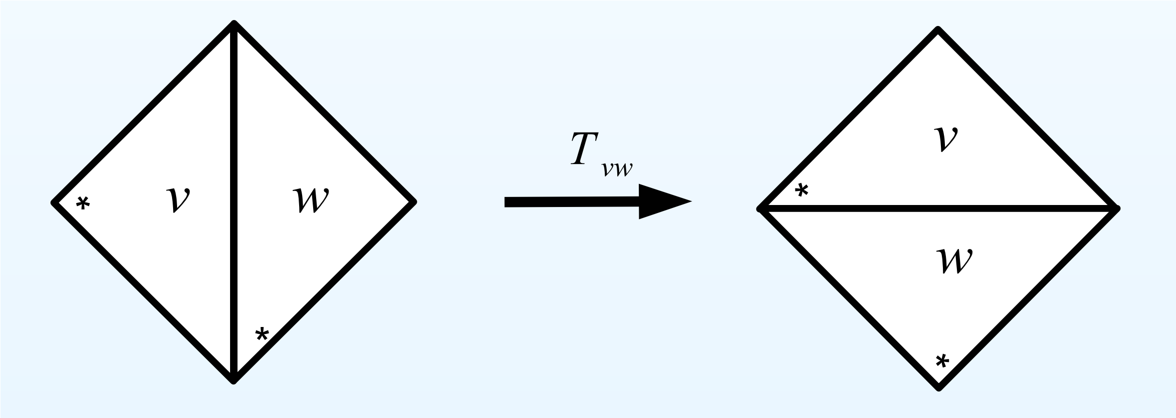
$$[p_v, q_w] = \frac{1}{2\pi i}\delta_{vw},$$



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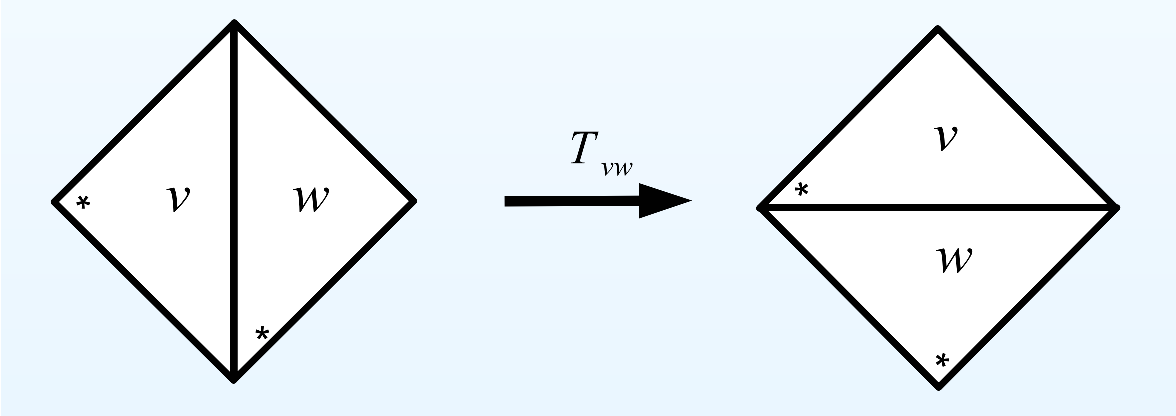


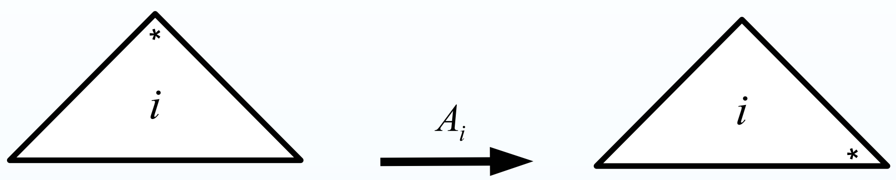
$$T_{vw} : \mathcal{H}_v \otimes \mathcal{H}_w \to H_v \otimes \mathcal{H}_w$$

$$U = e^{2\pi b q}$$
$$V = e^{2\pi b p}$$

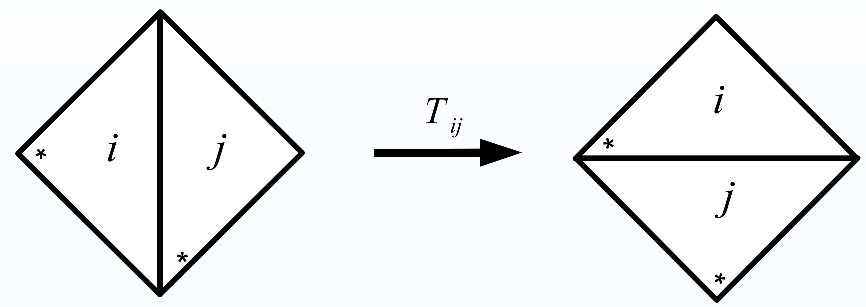
The pentagon equation for T is equivalent to the pentagon equation for special function $g_b(x)$ for the case $UV = q^2 VU$

$$g_b(U) g_b(V) = g_b(V) g_b(q^{-1} UV) g_b(U).$$

Quantum Teichmüller



$$A : \mathcal{H}_v \to \mathcal{H}_v$$

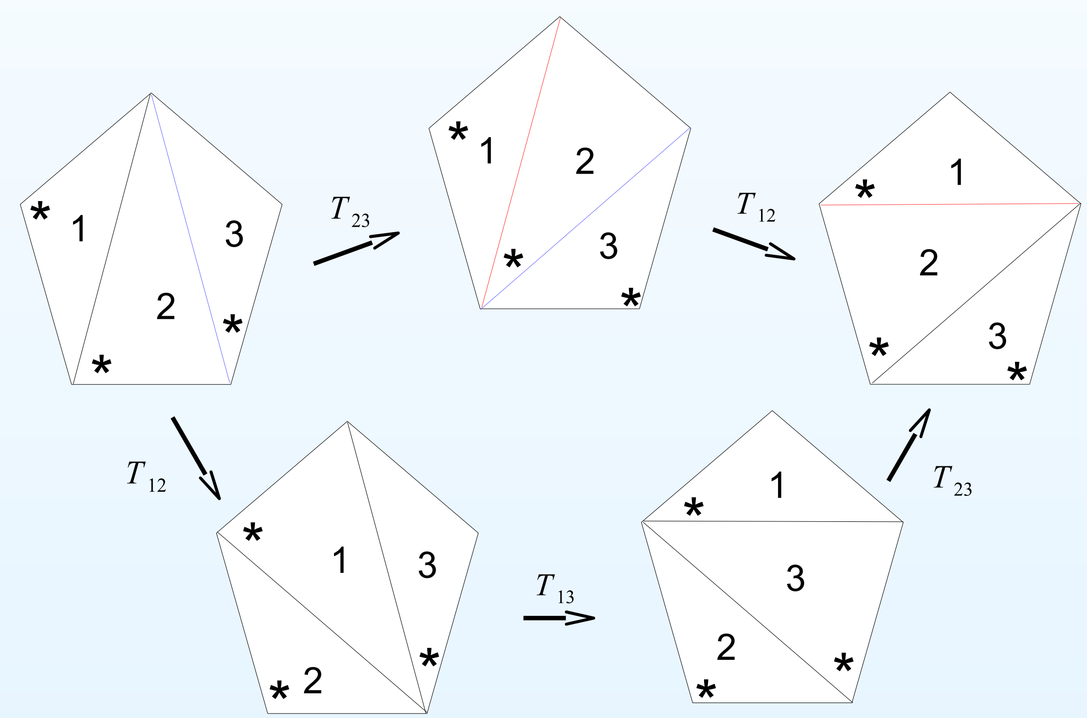


$$T : \mathcal{H}_v \otimes \mathcal{H}_w \to \mathcal{H}_v \otimes \mathcal{H}_w$$

$$A_v = e^{-i\pi/3} e^{i3\pi q_v^2} e^{i\pi(p_v+q_v)^2}$$

$$T_{vw} = g_b(e^{2\pi b(q_v+p_w-q_w)})e^{-2\pi i p_v q_w},$$

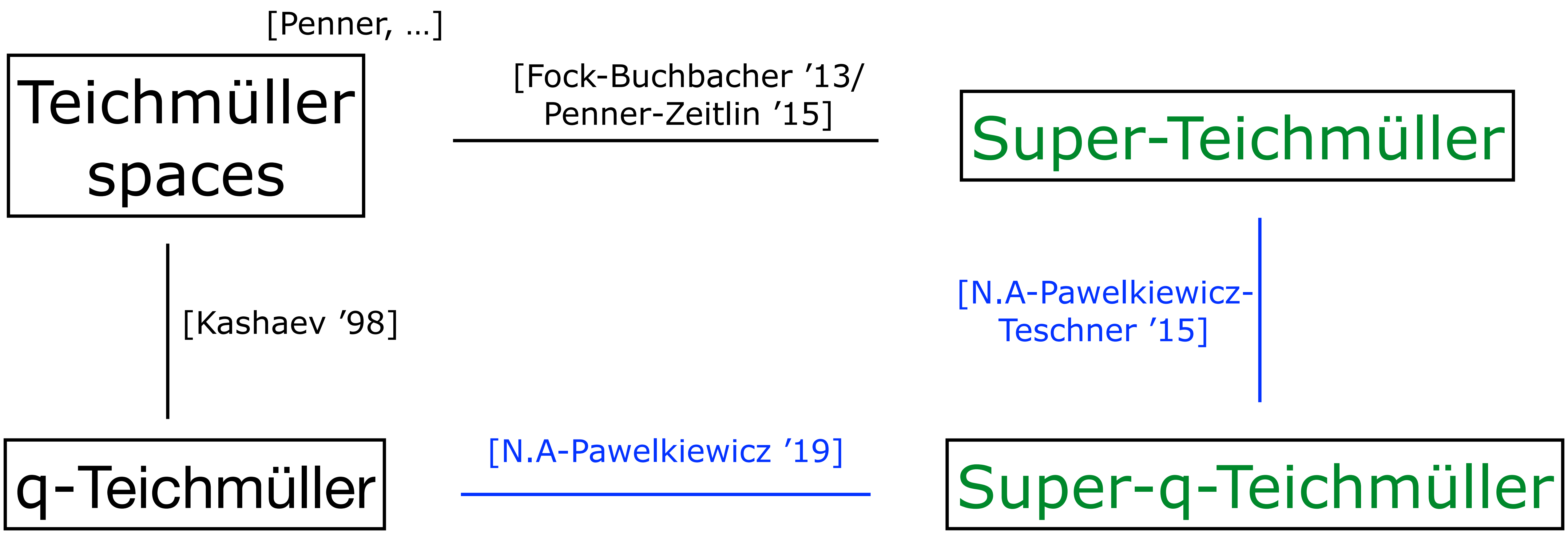
where g_b is Faddeev quantum dilogarithm function



$$T_{23}T_{13}T_{12} = T_{12}T_{23}$$

- Exist also relations between A and T

Plan for the rest of the talk



I') Super Teichmüller space and Super Penner coordinate on that.

II') Super Kashaev coordinate to be able to quantise.

Super Generalisation

- A super Riemann surfaces is a complex super manifold of dimension 1|1 obeys subtle additional condition.

- Define Super Riemann surfaces in terms of $\mathbb{Z}_2$-graded algebraic geometry. we replace commutative algebra by $\mathbb{Z}_2$-graded super commutative algebra.

- A typical $\mathbb{Z}_2$-graded ring has even z_i and odd generators θ_i with:

$$z_i z_j = z_j z_i \qquad z_i \theta_j = \theta_j z_i \qquad \theta_i \theta_j = -\theta_j \theta_i$$

- One can define $(n|m)$ supermanifold based on superspace.

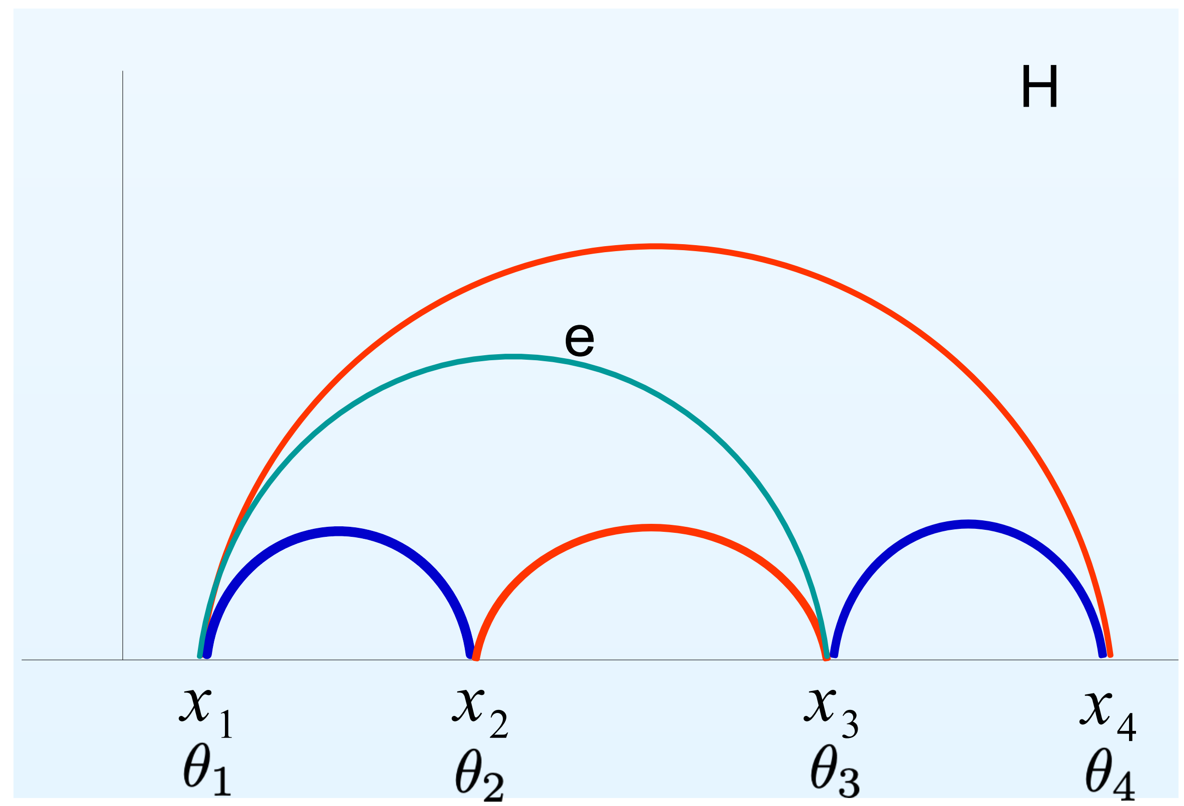
Super Generalisation

Step1) Super-Teichmüller

Uniformisation theory for super Riemann surfaces, as quotient of super upper half plane by discrete subgroup $OSp(1|2)$.

[P. Bryant, L. Hodgkin, L. Crane, J. Rabin,]

Points on upper half plane have coordinates (x, θ)



Super Generalisation

Step1) Super-Teichmüller

Uniformisation theory for Super Riemann surfaces, as quotient of super upper half plane by discrete subgroup

From now on let

$$ST(F) = Hom\,(\pi_1(F), OSp(1|2))/(OSp(1|2))$$

Super-Fuchsian representations comprising Hom are defined to be those whose are Fuchsian group, corresponding to F

$$\pi_1 \to OSp(1|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

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Lie super algebra:

$OSp(1|2)$ acts on $H^+, \partial H^+ = \mathbb{R}^{1|1}$ by superconformal fractional-linear transformations

$$z \to \frac{az+b}{z+dc} + \eta\frac{\gamma z+\delta}{(cz+d)^2}$$

$$\eta \to \frac{\gamma z+\delta}{cz+d} + \eta\frac{1+\frac{1}{2}\delta\gamma}{cz+d}$$

Factor H^+/Γ where Γ is a discrete group of $OSp(1|2)$ such that its projection is a Fuchsian group, are called super-Riemann surfaces.

Super Generalisation

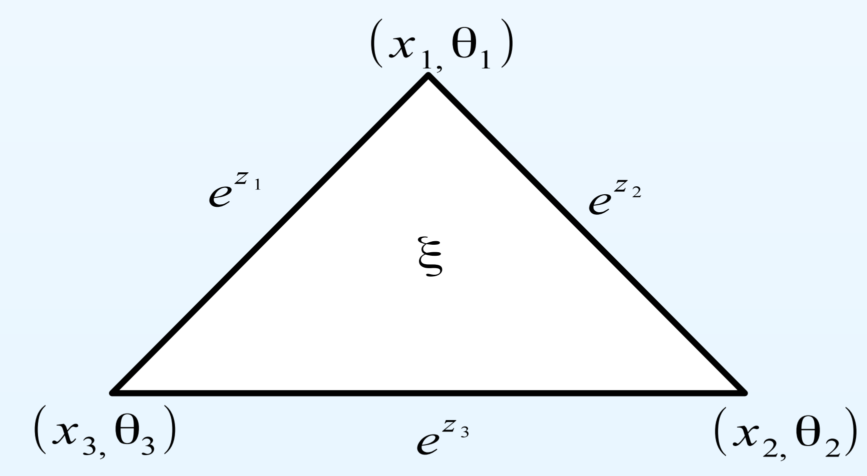
Step2) Shear/Penner Coordinates

Then there are global coordinates:

- one even coordinate called a λ-length for each edge.
- one odd coordinate called a ξ invariant for each face.

The above λ-lengths and ξ-invariants establish a real-analytic homeomorphism

$$C \to \mathbb{R}_+^{6g-6+3n|4g-4+2n}/\mathbb{Z}_2$$



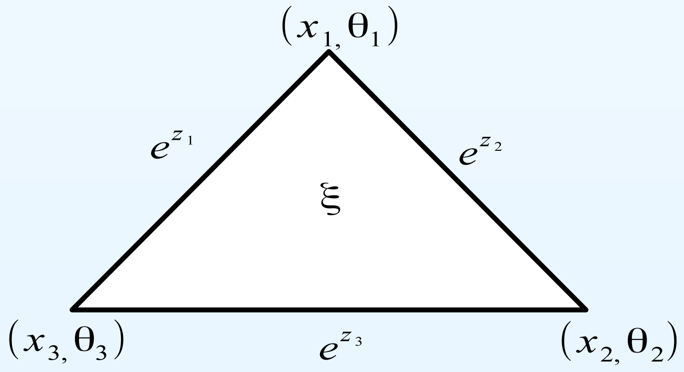
Super Generalisation

Step2) Shear/Penner Coordinates

Superconformal invariants x even , θ odd variable

even: $e^{z_e} = \frac{X_{12}X_{34}}{X_{14}X_{23}}, \quad X_{ij} = x_i - x_j - \theta_i\theta_j$,

odd: $\xi = \pm i \frac{x_{23}\theta_1 + x_{31}\theta_2 + x_{12}\theta_3 - \frac{1}{2}\theta_1\theta_2\theta_3}{(X_{12}X_{23}X_{31})^{\frac{1}{2}}}$,



Extra fermionic variables ξ attached to the face of triangle

Super Generalisation

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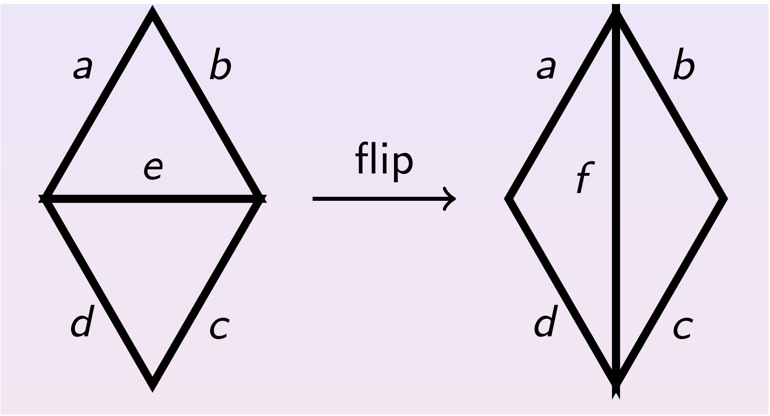
$$ef = ac + bd$$ Ptolemy relation



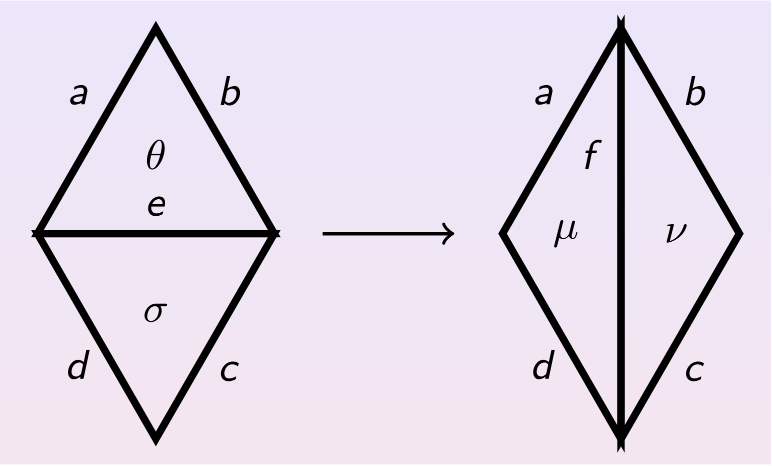
Super generalisation:

$$ef = (ac + bd)(1 + \frac{\sigma\theta\chi}{1 + \sqrt{\chi}})$$

$$\chi = \frac{ac}{bd}$$

$$\mu = \frac{\sigma - \theta\sqrt{\chi}}{1 + \chi}$$

$$\nu = \frac{\sigma + \theta\sqrt{\chi}}{1 + \chi}$$



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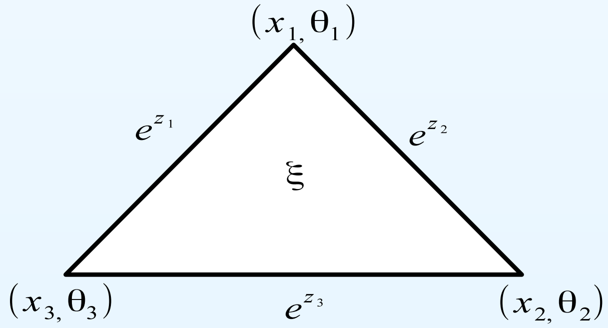


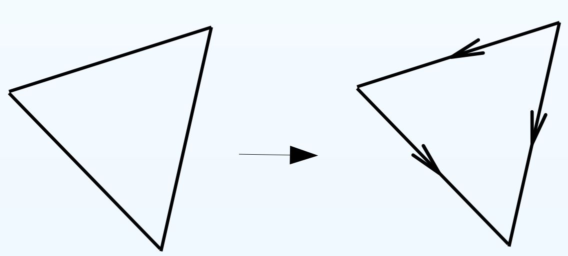
Extra fermionic variables ξ attached to the face of triangle

Spin structure

Odd variable defined up to a sign, to fix it $\rightarrow$ define the spin structure.

Spin structure can be encoded by Kasteleyn orientation.

[Cimasoni, Reshetikhin'07]



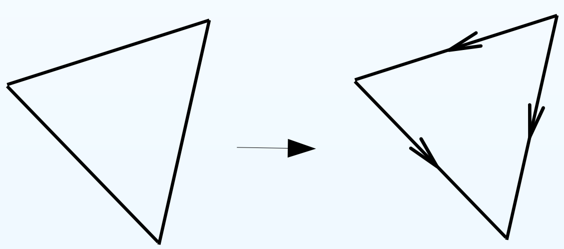
Remark: Over a Riemann surface spin structure correspond to square roots of canonical bundle.

Summary of what I have to do

Odd variable defined up to a sign, to fix it $\longrightarrow$ define the spin structure.

Spin structure can be encoded by Kasteleyn orientation.

[Cimasoni, Reshetikhin'07]

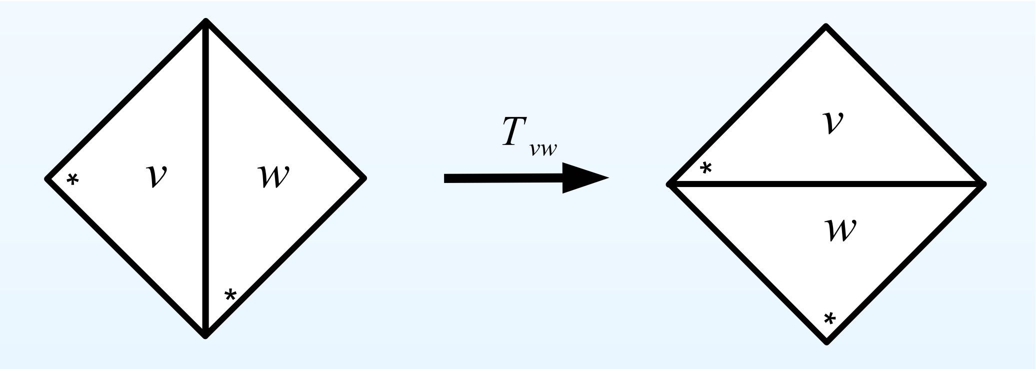


Step3) Kashaev Coordinates

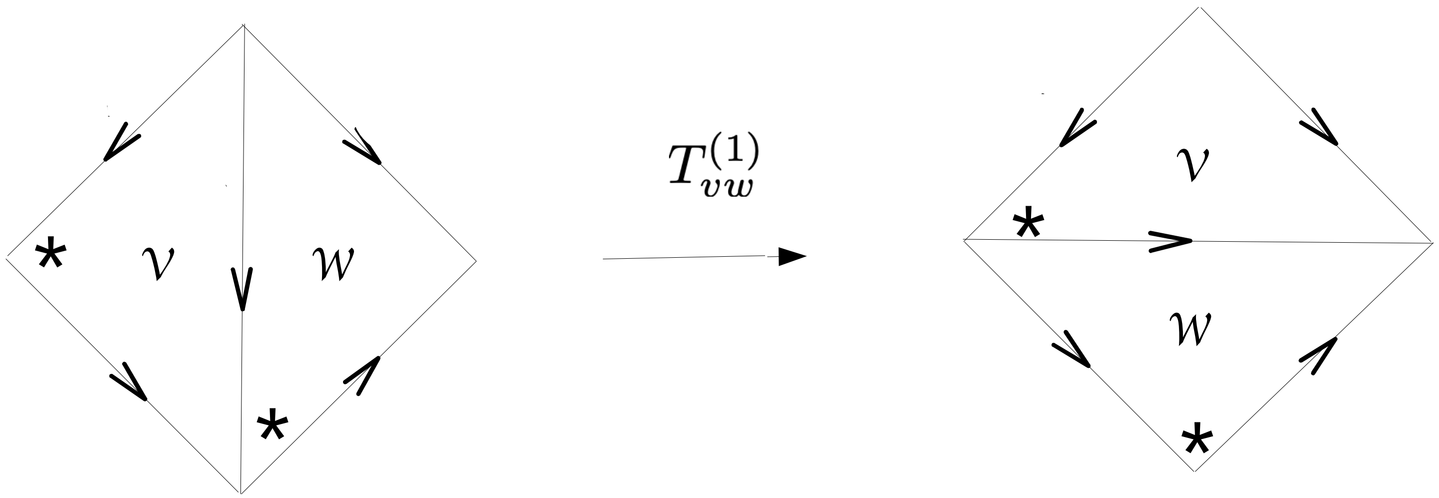
Identify the representation space of quantum group with Hilbert space

- Hilbert space associated to the face of the triangle.
- Identify (p_v, q_v, ξ_v) with super Kashaev for any triangle v .

Goal of the Game



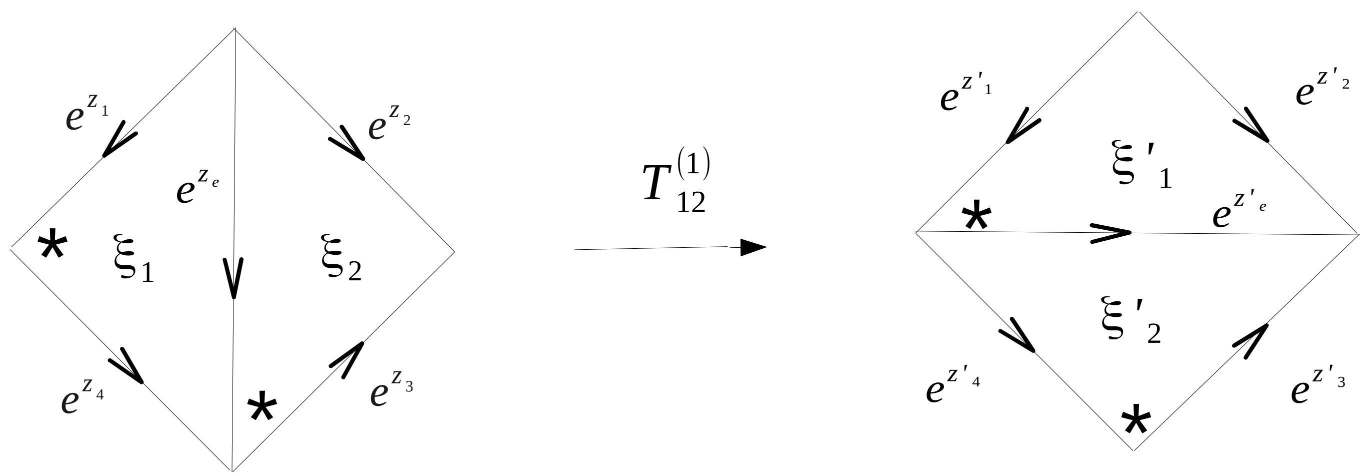
$$T_{vw} = g_b(e^{2\pi b(q_v + p_w - q_w)})e^{-2\pi i p_v q_w}$$



$$T^{(1)}_{vw} = ?$$

Super-q-Teichmüller

[N.A-Pawelkiewicz-Teschner ’15]



$$T_{vw} = g_b(e^{2\pi b(q_v + p_w - q_w)})e^{-2\pi i p_v q_w}$$

$$T_{12} = \frac{1}{2}[f_+(\alpha)\mathbb{I}\otimes\mathbb{I} - i f_-(\alpha)\xi\otimes\xi]e^{-i\pi p_1 q_2}$$

where $\alpha = q_1 + p_2 - q_2$, $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\xi\otimes\xi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$f_+ = e_R + e_{NS}$, $f_- = e_{NS} - e_R$, where, e_R, e_{NS} are connected to the supersymmetric analogues of double sine function.

super e_b function

The supersymmetric version of Fadeev's quantum dilogarithm function is

$$e_b(x) = g_b e^x$$

$$e_R = e_b(\frac{x + i(b - b^{-1}/2)}{2}) e_b(\frac{x - i(b - b^{-1}/2)}{2})$$
$$e_{NS} = e_b(\frac{x + i(b + b^{-1}/2)}{2}) e_b(\frac{x - i(b + b^{-1}/2)}{2})$$

$$T_{12} = \frac{1}{2}[f_+(\alpha)\mathbb{I} \otimes \mathbb{I} - i f_-(\alpha) \xi \otimes \xi] e^{-i\pi p_1 q_2}$$

[N.A-Pawelkiewicz-Teschner '15]

where $\alpha = q_1 + p_2 - q_2$, $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

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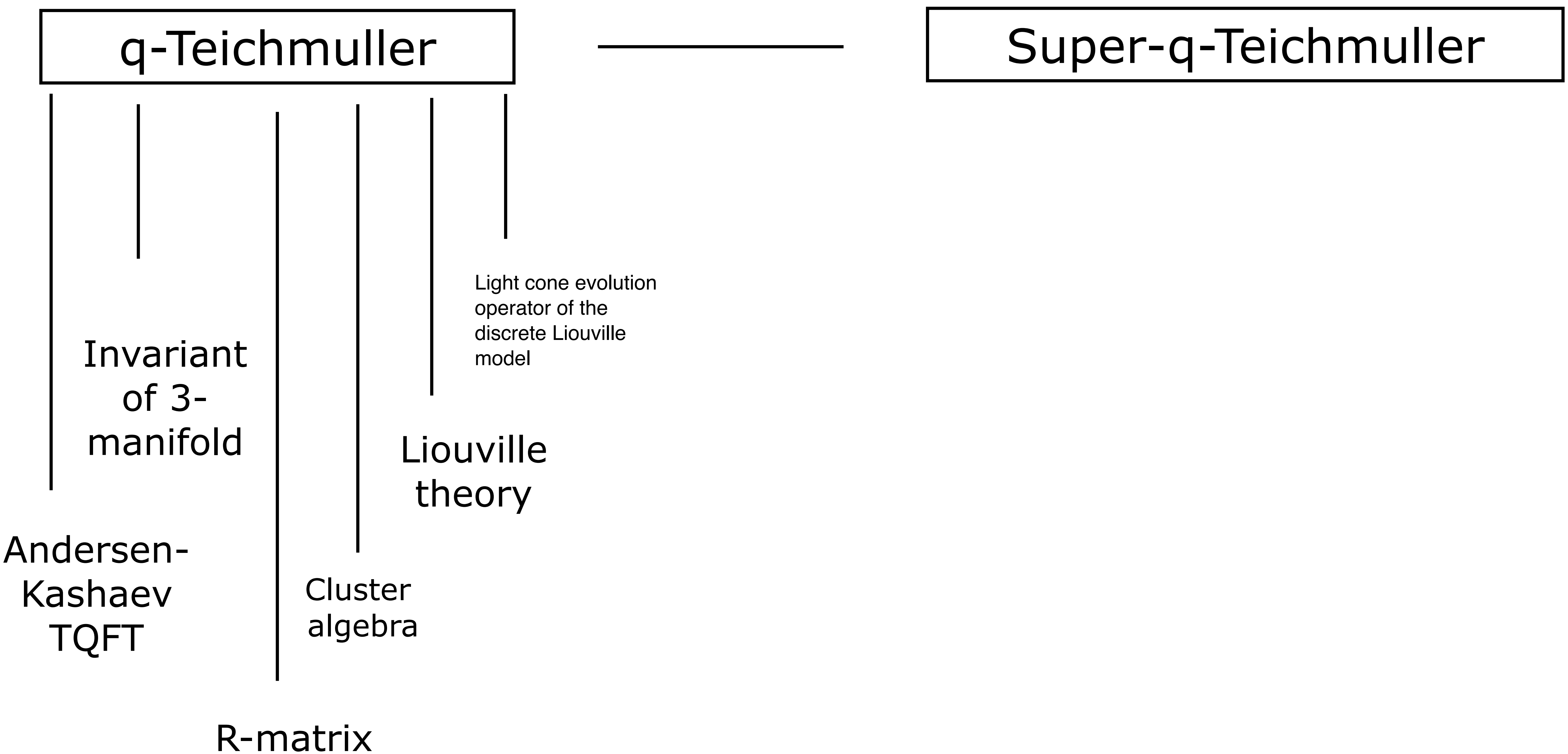
$f_+ = e_R + e_{NS}$, $f_- = e_{NS} - e_R$, where, e_R, e_{NS} are connected to the supersymmetric analogues of double sine function.

Outlook I

Higher Teichmüller spaces: $PSL(2,\mathbb{R})$ is replaced by some reductive group G. [V.Fock and A. Goncharov (2003)]



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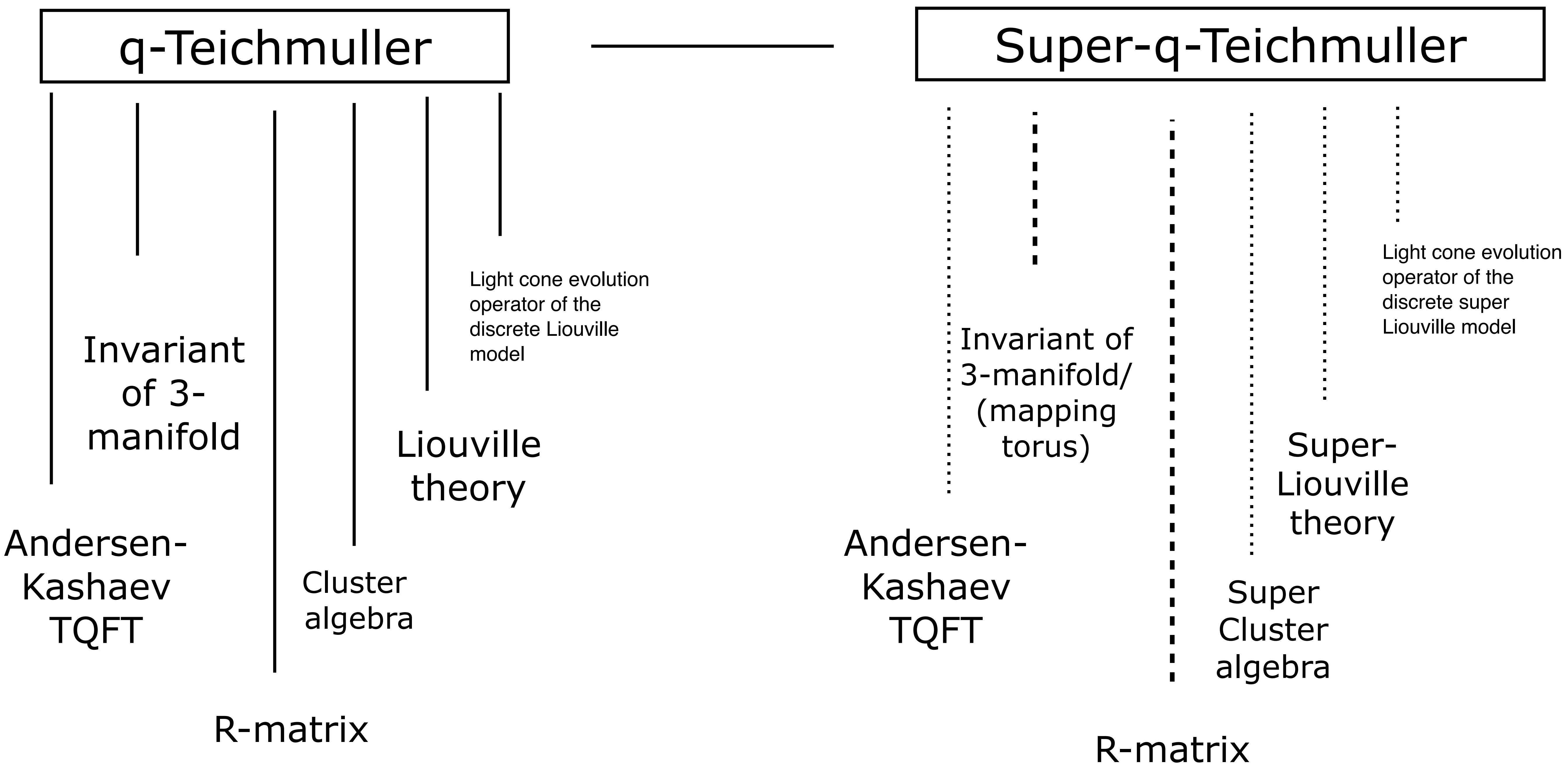
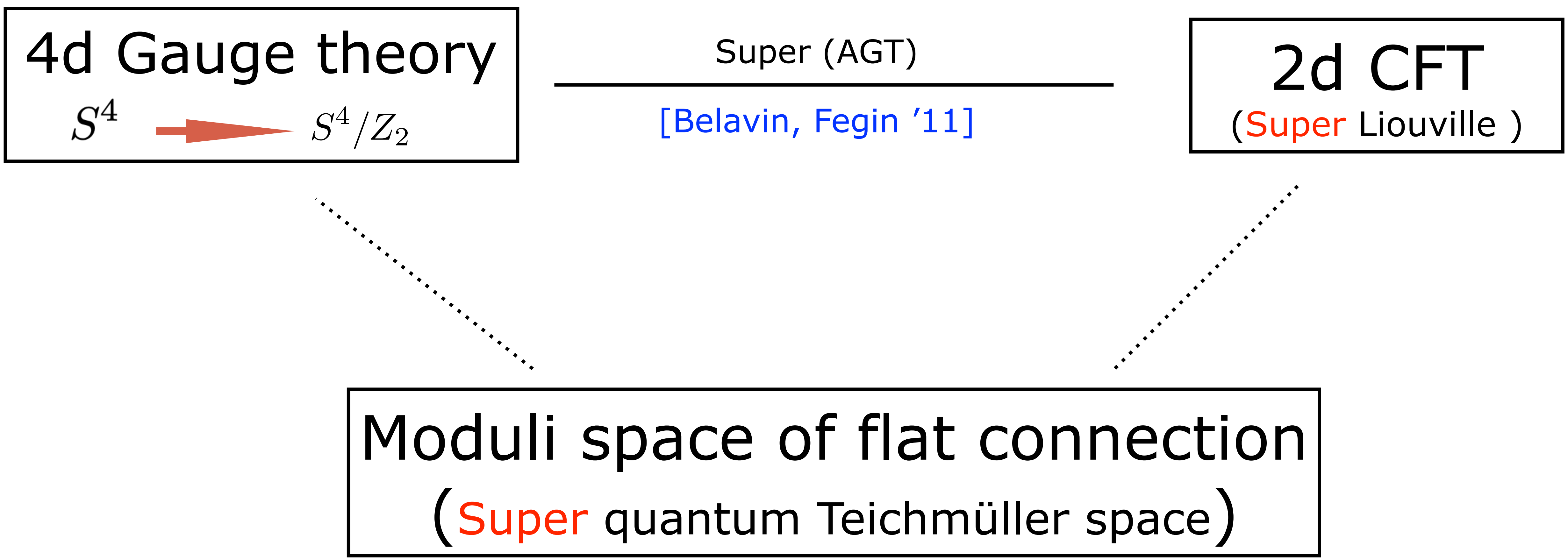


Outlook II


4d Gauge theory
S^4 → S^4/Z_2
Super (AGT)
[Belavin, Fegin '11]
2d CFT
(Super Liouville)
Moduli space of flat connection
(Super quantum Teichmüller space)

Thanks:)