The local motivic monodromy conjecture holds generically

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Joint work with Matt Larson and Sam Payne

Conjecture (first approximation)

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Local monodromy conjectures

If $\alpha \in \mathbb{Q}$ is a pole of a rational function associated to f, then $\exp(2\pi i \alpha)$ is an eigenvalue of a matrix associated to f.

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For $p \notin \{2, 3\}$,

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poles at $s = -1$ and $s = -5/6$

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For all but finitely many primes p, if $\alpha \in \mathbb{Q}$ is a pole of $Z_p(s)$, then $\exp(2\pi i \alpha)$ is an eigenvalue of monodromy for $H^*(\mathcal{F}; \mathbb{C})$.

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Example: $f(x, y) = y^2 - x^3$.
$Z_p(s)$ has poles: $\alpha \in \{-1, -5/6\}$
eigenvalues of monodromy: $\exp(2\pi i \alpha)$ for $\alpha \in \{-1, -5/6, -1/6\}$.

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3. By miracle, can compute by reducing to the monomial case using a resolution of singularities and change of variables formula.
   ($Z_p(s)$ Igusa '75, eigenvalues of monodromy A'Campo '75)

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The *local motivic zeta function* $Z_{\mathrm{mot}}(T)$ is a universal such invariant (Denef-Loeser '98). It is a rational function in variable T with coefficients in an appropriate Grothendieck ring of varieties with a group action, defined using motivic integration.

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Example: $f(x,y) = y^2 - x^3$.
The local motivic zeta function $Z_{\mathrm{mot}}(T)$ is

$$\frac{(\mathbb{L}-1)\left(\left(\frac{(\mathbb{L}-1)\mathbb{L}^{-1}T}{1-\mathbb{L}^{-1}T} + [Y_F(1)]\right)\mathbb{L}^{-5}T^6 + [\mu_3]\mathbb{L}^{-2}T^3(1+\mathbb{L}^{-3}T^3) + [\mu_2]\mathbb{L}^{-1}T^2(1+\mathbb{L}^{-2}T^2+\mathbb{L}^{-4}T^{-4})\right)}{1-\mathbb{L}^{-5}T^6},$$

where $\mathbb{L} = [\mathbb{A}^1]$, $Y_F(1)$ is an elliptic curve minus 6 points, with a free μ_6-action, and $Y_F(1)/\mu_6$ is isomorphic to $\mathbb{P}^1$ minus 3 points.

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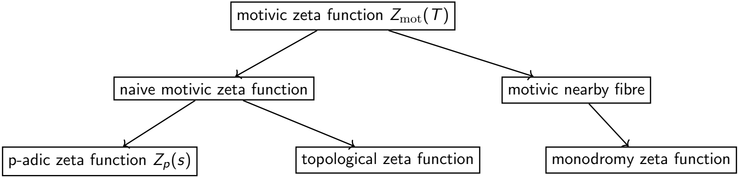

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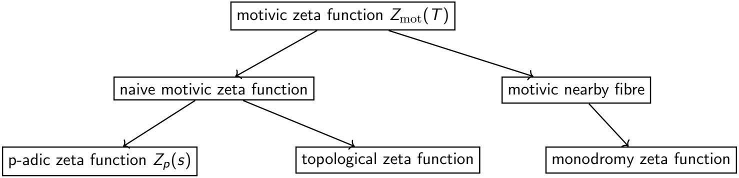

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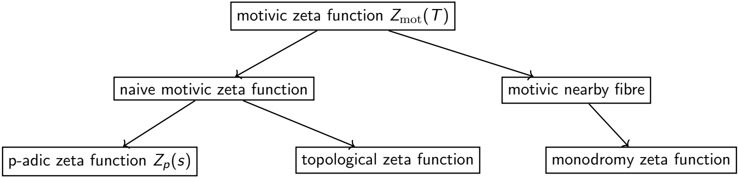

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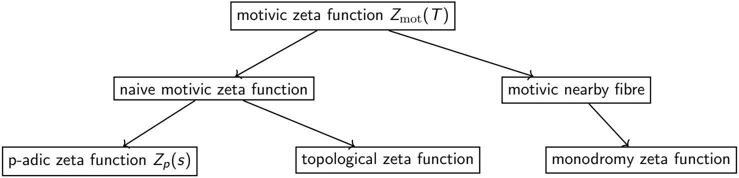

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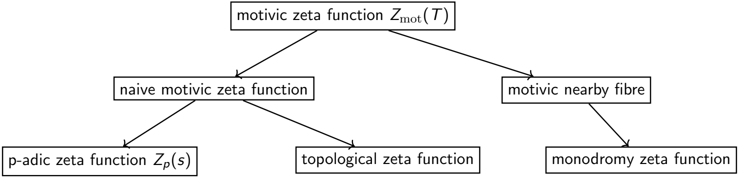

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vast literature; when $n = 3$, special cases due to Artalo Bartolo, Cassou-Nogues, Luengo, Melle Hernandez, Lemahieu, Veys, ...

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  $Z_{\mathrm{mot}}(T)$ is intrinsically defined. Any resolution of singularities gives set of "candidate poles", most not poles. Can you decide which "candidate poles" are poles? Similarly, which "candidate eigenvalues" are eigenvalues.

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- semi-algebraic geometry (Nicaise, Sebag '08)
  give Milnor fiber structure of smooth rigid variety, *analytic Milnor fiber*, realise $Z_{\mathrm{mot}}(T)$ is a Weil-type invariant.

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"combinatorial" formulas for zeta functions (Denef-Loeser '92, Guibert '02, Bories-Veys '16, Bultot-Nicaise '20) and eigenvalues of monodromy (Varchenko '76).

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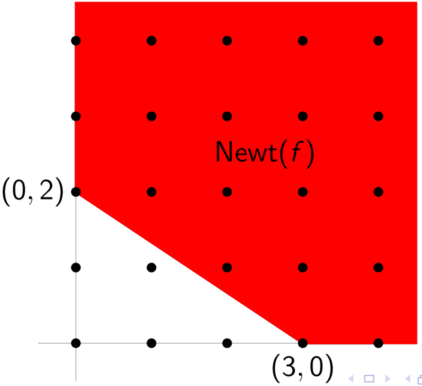

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Fundamental problems:

- formulas for "candidate poles" and "candidate eigenvalues" involve lots of cancellation
- combinatorics of polytopes often difficult in dimension 4 and above

A new frontier

The local motivic monodromy conjecture

If $\alpha \in \mathbb{Q}$ is a pole of $Z_{\mathrm{mot}}(T)$, then $\exp(2\pi i\alpha)$ is an eigenvalue of monodromy for $H^*(\mathcal{F};\mathbb{C})$.

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Theorem (Larson-Payne-S. '22)

The local motivic monodromy conjecture holds generically.

"generically" $=$ nondegenerate $+$ technical condition

e.g. nondegenerate and $n \leq 3$,

e.g. nondegenerate and simplicial Newton polyhedron

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- Formula involves terms of two types: from Ehrhart theory and from triangulations of simplices.

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- Let $\mathcal{S}$ be a triangulation of a simplex Δ.
  The *local h-polynomial* $\ell(\mathcal{S}; t)$ of $\mathcal{S}$ was introduced by Stanley '92 (we actually use generalization Athanasiadis, Nill '12).

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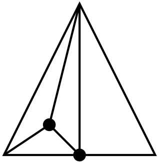

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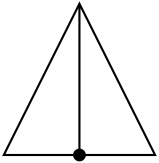

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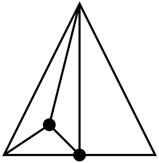

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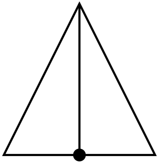

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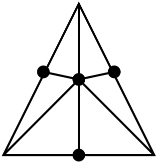

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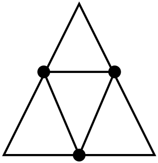

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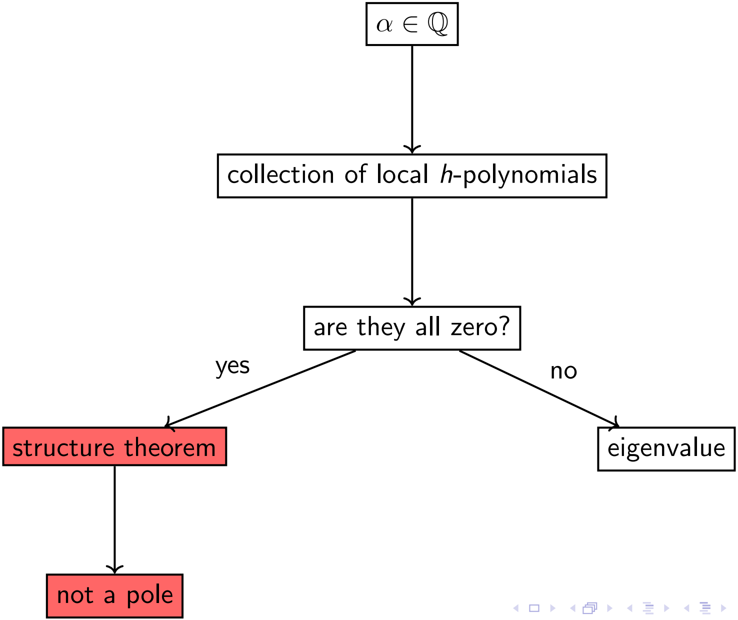

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Ex: if F is not a pyramid, then a full partition is a decomposition $F = F_1 \sqcup F_2$, with F_1, F_2 interior faces.

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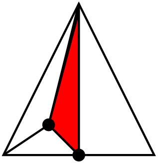

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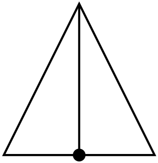

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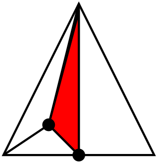

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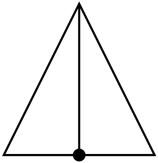

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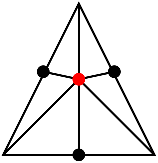

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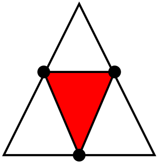

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