Constant Mean Curvature Tori in $\mathbb{R}^3$ and S^3

Emma Carberry

University of Sydney

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Bonnet Theorem in terms of H

Compact constant mean curvature surfaces (soap bubbles) are critical points for the area functional under variations which preserve the enclosed volume.

Given a conformal immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$, with metric $4e^{2u}dzd\bar{z}$, the first and second fundamental forms can be written as

$$I = \begin{pmatrix} 4e^{2u} & 0 \\ 0 & 4e^{2u} \end{pmatrix}, \qquad II = \begin{pmatrix} 4He^{2u} + Q + \bar{Q} & i(Q - \bar{Q}) \\ i(Q - \bar{Q}) & 4He^{2u} - (Q + \bar{Q}) \end{pmatrix}$$

where H denotes the mean curvature and $Q = \langle f_{zz}, N \rangle$. The quadratic differential Qdz^2 is called the Hopf differential.

Theorem (Bonnet)

Given $4e^{2u}dzd\bar{z}$, Qdz^2 and H on $\mathbb{R}^2$ satisfying the Gauss-Codazzi equations, there is a conformal immersion $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that these are the metric, Hopf differential and mean curvature. This immersion is unique up to Euclidean motions.

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When H is constant, the Gauss-Codazzi equations are unchanged by $Q \mapsto \lambda Q$ for $\lambda \in S^1$, giving a one-parameter family of CMC surfaces.

We can extend the parameter $\lambda \in S^1$ to $\lambda \in \mathbb{C}^*$ and obtain that f has constant mean curvature if and only if a certain family of $sl(2,\mathbb{C})$-valued 1 forms ϕ_λ satisfy

$$d\phi_\lambda + [\phi_\lambda, \phi_\lambda] = 0 \quad \forall \lambda \in \mathbb{C} - \{0\} \text{ (Maurer-Cartan equation).}$$

The Maurer-Cartan equation states that the connections $d_\lambda = d + \phi_\lambda$ (in the trivial bundle) are all **flat**.

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Identifying $\mathbb{R}^3$ with $SU(2)$ via

$$e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

we obtain a moving frame $F : \mathbb{R}^2 \to SU(2)$ for f by requiring

$$\mathrm{Ad}_F e_1 = \frac{f_x}{|f_x|}, \qquad \mathrm{Ad}_F e_2 = \frac{f_y}{|f_y|}, \qquad \mathrm{Ad}_F e_3 = N.$$

Explicitly, one has

$$\phi_1 = F^{-1}dF = \frac{1}{2}\begin{pmatrix} u_z & -2He^u \\ Qe^{-u} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u} \\ 2He^u & u_{\bar{z}} \end{pmatrix} d\bar{z}$$

and defines

$$\phi_\lambda = \frac{1}{2}\begin{pmatrix} u_z & -2He^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u}\lambda \\ 2He^u\lambda & u_{\bar{z}} \end{pmatrix} d\bar{z}.$$

We have described a CMC immersion in terms of a family of flat $SL(2,\mathbb{C})$ connections $d+\phi_\lambda$.
The condition for $A_\lambda(z)$ to be parallel is then a Lax equation

$$dA_\lambda(z) = [A_\lambda(z), \phi_\lambda(z)]$$

which forces the characteristic polynomial

$$\det(A_\lambda(z) - yI) = 0$$

to be independent of z.
We say that the CMC immersion f is of finite-type if we may find an A which is

- "polynomial": $A(\lambda) = \sum_{k=-d}^{d} A_k \lambda^k$
- the connection forms ϕ_λ can be recovered from the coefficients of A.

We take such A of minimal degree and call the resulting algebraic curve $\det(A_\lambda(z) - yI) = 0$ the spectral curve X of f.

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Theorem (Hitchin, Pinkall-Sterling)

All constant mean curvature tori are of finite-type.

For each $z \in \mathbb{R}^2$ there is a line bundle E_z on X given by the eigenlines of $A_\lambda(z)$ and the map

$$z \mapsto E_z \otimes E_0^{-1} : T^2 \to \mathrm{Jac}(X)$$

is linear. The real 2-plane in the Jacobian given by the image of this map is intrinsically determined from the spectral curve X.

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Amongst the CMC immersions $\mathbb{R}^2 \to \mathbb{R}^3$ are those of finite-type.

These are in one-to-one correspondence with spectral curve data, consisting of

- a hyperelliptic curve X
- marked points $P_0, P_\infty \in X$
- a line bundle E_0 on X of degree $g+1$

all satisfying certain symmetries (Hitchin, Pinkall-Sterling, Bobenko).

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There is very similar spectral curve correspondence for CMC immersions $\mathbb{R}^2 \to S^3$ of finite-type.

Again all the CMC tori are of finite-type and they are characterised by their spectral data satisfying periodicity conditions.

We would like to understand the moduli spaces of CMC tori in $\mathbb{R}^3$ and S^3, in particular:

Question

1. *Can one deform these tori, at least infinitesimally, and if so what is the dimension of the space of deformations?*
2. *How common are the CMC tori amongst CMC planes of finite-type?*

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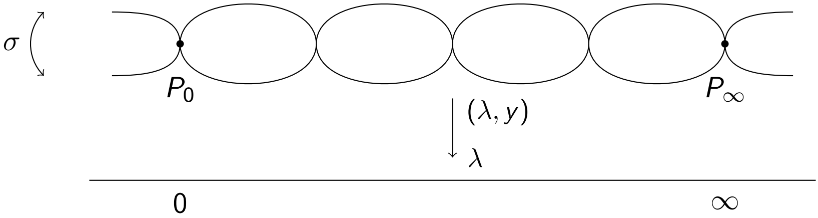
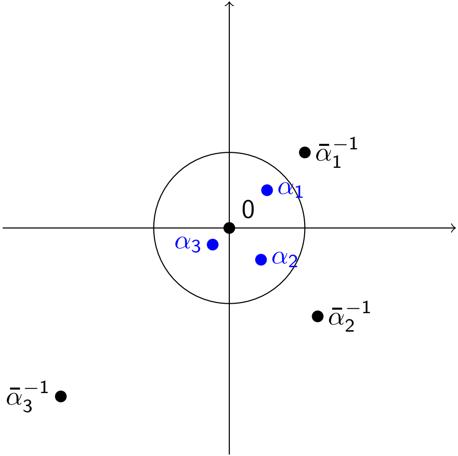
Writing the hyperelliptic curve X_a as $y^2 = \lambda a(\lambda)$, we have

- the hyperelliptic involution $\sigma : (\lambda, y) \mapsto (\lambda, -y)$



- an anti-holomorphic involution ρ without fixed points covering $\lambda \mapsto \bar{\lambda}^{-1}$

$$\rho : (\lambda, y) \mapsto (\bar{\lambda}^{-1}, \bar{y}\bar{\lambda}^{-g-1}).$$



Writing $\rho^* a$ to mean $\lambda^{\deg a} a(\bar{\lambda}^{-1})$,

$$\overline{\rho^* a} = a \qquad \text{reality condition.}$$

We consider the space $\mathcal{H}^g$ of smooth spectral curve data (X_a, λ) of genus g as an open subset of $\mathbb{R}^{2g}$, given by $(\alpha_1, \ldots, \alpha_g)$, where

$$X_a : \quad y^2 = \lambda a(\lambda) = \lambda \prod_{i=1}^{g} \frac{(\lambda - \alpha_i)(1 - \overline{\alpha_i}\lambda)}{|\alpha_i|}.$$

Let

$$\mathcal{B}_a = \left\{ \begin{array}{c} \text{meromorphic differentials } \Theta \text{ with} \\ \text{no residues, double poles at } P_0, P_\infty, \\ \sigma^*\Theta = -\Theta,\ \rho^*\Theta = -\overline{\Theta} \text{ and having} \\ \text{purely imaginary periods} \end{array} \right\}.$$

$\mathcal{B}_a$ is a real 2-plane.

We obtain a real-analytic rank two vector bundle

$$\mathcal{B} \to \mathcal{H}^g$$

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Periodicity Conditions for CMC $T^2 \to \mathbb{R}^3$

$X_a \in \mathcal{H}^g$ gives a doubly-periodic CMC immersion into $\mathbb{R}^3$

$\Leftrightarrow$

there exists a frame (Θ_1, Θ_2) of $\mathcal{B}_a$ such that

1. their periods lie in $2\pi\sqrt{-1}\mathbb{Z}$ ($\Leftrightarrow \Theta_1 = d\log\mu_1, \Theta_2 = d\log\mu_2$)
2. for some $\lambda_0 \in S^1$, called the Sym point
   (a) for γ a curve in X connecting the two points in $\lambda^{-1}(\lambda_0)$,

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Deformations of CMC Tori

For spectral curves of CMC tori in $\mathbb{R}^3$,

$$\# \text{ free parameters} = \# \text{ periodicity conditions.}$$

A CMC torus has $\mathbb{R}^g$ deformations, all isospectral.

A CMC torus in S^3 has $\mathbb{R}^g$ isospectral deformations.

For CMC tori in S^3, have an extra real parameter: the ratio $\frac{\lambda_1}{\lambda_2}$.

Theorem (Kilian, Schmidt, Schmitt)

An immersed CMC torus in S^3 has additionally a real 1-dimensional space of non-isospectral deformations, provided Θ_1, Θ_2 have no common zeros.
The moduli space of equivariant CMC tori in S^3 is a connected graph, with edges parameterised by the mean curvature. (A surface is equivalent if it is preserved set-wise by a 1-parameter family of isometries.)

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CMC Tori in S^3

For $\lambda_1 \neq \lambda_2 \in S^1$ define $\mathcal{P}^g(\lambda_1, \lambda_2) \subset \mathcal{H}^g$ to be the set of spectral curves of CMC tori with Sym points λ_1, λ_2.

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For each $\lambda_1 \neq \lambda_2 \in S^1$, $\mathcal{P}^g(\lambda_1, \lambda_2)$ is dense in $\mathcal{H}^g$.
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CMC tori in $\mathbb{R}^3$

Theorem (Ercolani–Knörrer–Trubowitz '93, Jaggy '94)

For every $g > 0$, there exist CMC tori of spectral genus g. There are at least countably many spectral curves of genus g satisfying the periodicity conditions.

In the Euclidean case, CMC tori are not dense amongst CMC planes of finite type.

Writing

$$\mathcal{P}_{\lambda_0} = \{X_a \in \mathcal{H} \mid X_a \text{ is a spectral curve of a CMC torus with Sym point } \lambda_0\},$$

the closure of $\mathcal{P}_{\lambda_0}$ is contained in the real subvariety

$$\mathcal{S}_{\lambda_0} = \{X_a \in \mathcal{H} \mid \text{ all } \Theta \in \mathcal{B}_a \text{ vanish at } \lambda_0\},$$

which has codimension at least 2.

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$$= \{X_a \in \mathcal{H} \mid \text{ all } \Theta \in \mathcal{B}_a \text{ have a common root on } S^1\},$$

which contains the closure of spectral curves of CMC tori, is in general not a subvariety.

However it is contained in the subvariety

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Intuitive Picture

Recall that for real varieties we may have smooth points of different dimension within the same irreducible component

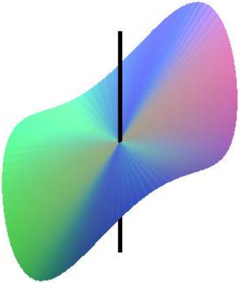

Figure : Cartan's Umbrella: $z(x^2 + y^2) = x^3$

The intuition is that for $g > 2$, $\mathcal{R}$ is as above and we conjecture that the points of $\mathcal{S}$ comprise the "cloth" of the umbrella.

Theorem (——, Schmidt)

For $X_a \in \mathcal{R}$ the following statements are equivalent:

1. $dim_a \mathcal{R} = 2g - 1$ *(i.e. codimension 1 in $\mathcal{H}$)*
2. X_a *belongs to the closure of* $\{X_{\tilde{a}} \in \mathcal{H} \mid \deg(\gcd(\mathcal{B}_{\tilde{a}})) = 1\}$.

Furthermore if one of these equivalent conditions is satisfied then X_a belongs to the closure of the spectral curves of constant mean curvature tori in $\mathbb{R}^3$.

This closure is contained in $\mathcal{S}^g$.

Theorem (——, Schmidt)

For $X_a \in \mathcal{R}$ the following statements are equivalent:

1. $\mathit{dim}_a \mathcal{R} = 2g - 1$ *(i.e. codimension 1 in $\mathcal{H}$)*
2. X_a *belongs to the closure of* $\{X_{\tilde{a}} \in \mathcal{H} \mid \deg(\gcd(\mathcal{B}_{\tilde{a}})) = 1\}$.

Furthermore if one of these equivalent conditions is satisfied then X_a belongs to the closure of the spectral curves of constant mean curvature tori in $\mathbb{R}^3$.

This closure is contained in $\mathcal{S}^g$.