The cogrowth series for BS(N,N) is D-finite

Murray Elder, (U Newcastle, Australia)
Andrew Rechnitzer, (UBC, Canada)
Buks van Rensburg, (York U, Canada)
Tom Wong, (UBC, Canada)

AustMS 2013, Group Actions special session

Cogrowth

(G,X) a group with finite generating set

c_n = number of words in $\left(\mathsf{X} \cup \mathsf{X}^{-1}\right)^n$ equal to the identity in G

$n \mapsto c_n$ is the *cogrowth function* for (G,X)

Cogrowth

(G,X) a group with finite generating set

c_n = number of words in $\left(X \cup X^{-1}\right)^n$ equal to the identity in G

$n \mapsto c_n$ is the *cogrowth function* for (G,X)

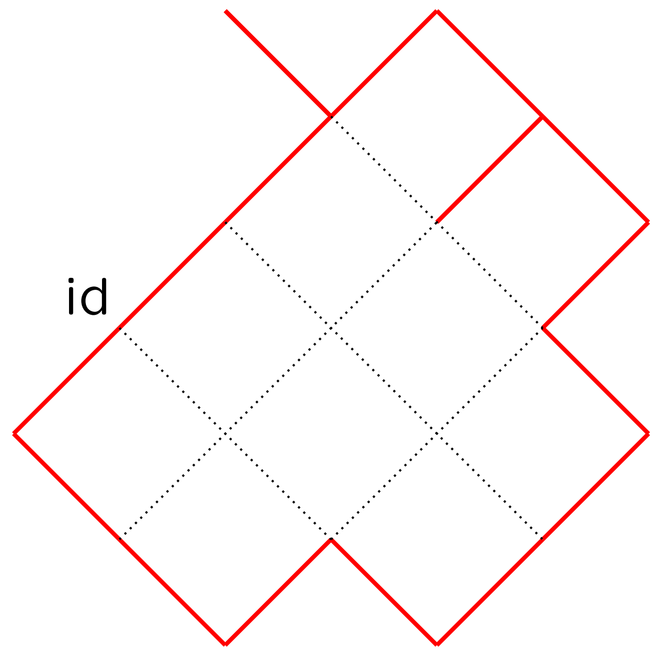
$c_n \leq (2|X|)^n$ so $\limsup c_n^{1/n} \leq 2|X|$

Thm(Grigorchuk/Cohen): G is amenable iff $\limsup c_n^{1/n} = 2|X|$

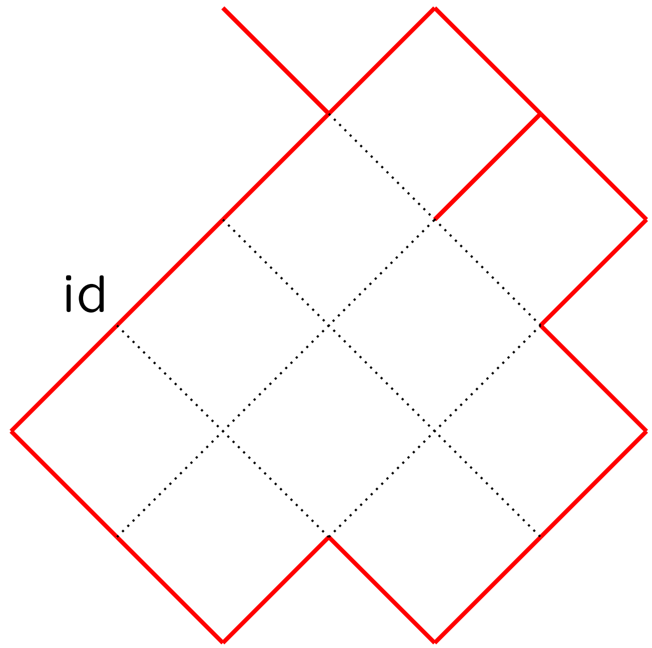
BS(N,M)

is the 1 relator group $\langle a, b \mid ba^{\mathsf{N}} = a^{\mathsf{M}}b \rangle$

BS(1,1) is just $\mathbb{Z}^2$



BS(1,1)

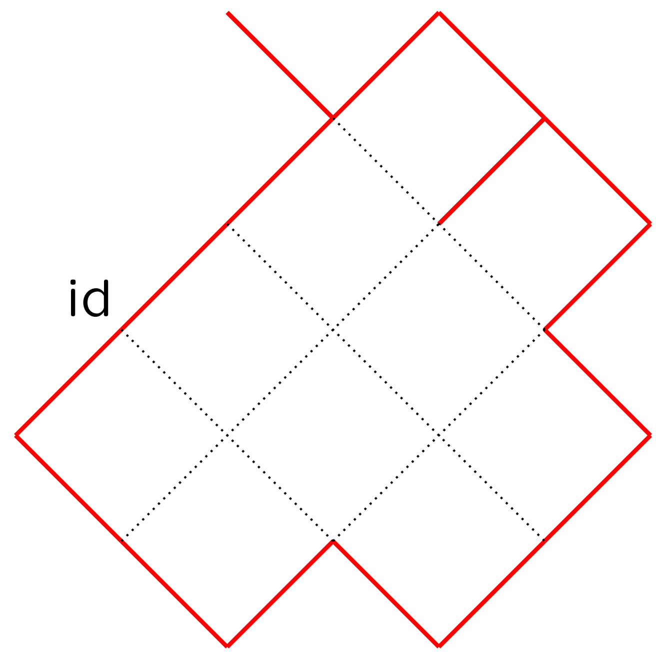


a	a	b^{-1}	b	a	b	a^{-1}	a	b	a^{-1}	b	a^{-1}	a^{-1}	b^{-1}	a^{-1}	b^{-1}	b^{-1}	a
$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
$+$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$	$+$

BS(1,1)

$$c_{2n} = \binom{2n}{n}\binom{2n}{n}$$

$$c_{2n+1} = 0$$

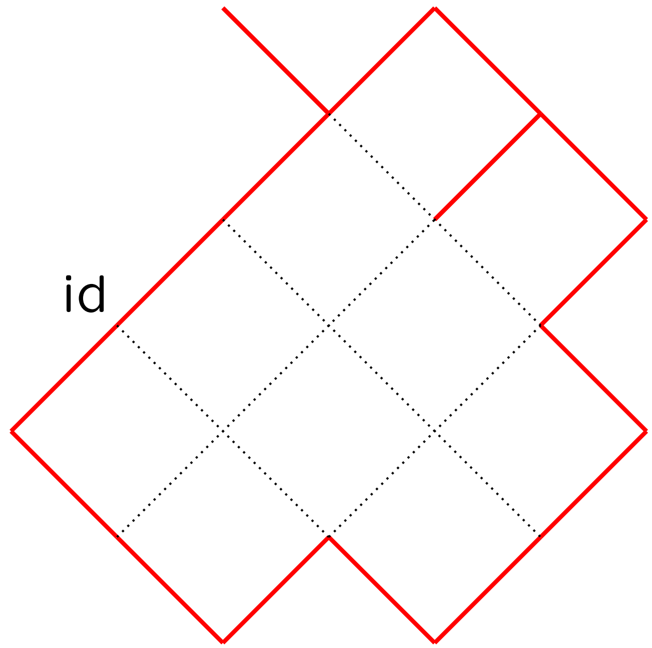


a	a	b^{-1}	b	a	b	a^{-1}	a	b	a^{-1}	b	a^{-1}	a^{-1}	b^{-1}	a^{-1}	b^{-1}	b^{-1}	a
$+$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
$+$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$	$+$

BS(1,1)

$c_{2n} = \binom{2n}{n}\binom{2n}{n}$

$c_{2n+1} = 0$



which satisfies $(n+1)^2 c_{2n+2} = 4(2n+1)^2 c_{2n}$

(sequence A002894 OEIS)

BS(1,1)

$\{c_n\}$ satisfies $(\frac{n}{2}+1)^2 c_{n+2} = 4(n+1)^2 c_n$

so the sequence $\{c_n\}$ is *P-recursive*

(satisfies a linear recurrence with polynomial coefficients)

BS(1,1)

$\{c_n\}$ satisfies $(\frac{n}{2}+1)^2 c_{n+2} = 4(n+1)^2 c_n$

so the sequence $\{c_n\}$ is *P-recursive*

(satisfies a linear recurrence with polynomial coefficients)

Thm(Stanley): $\{a_n\}$ is P-recursive iff $\sum_n a_n z^n$ is *D-finite*

(satisfies a linear differential equation with polynomial coefficients)

Why D-finite?

- closed under addition and multiplication
- includes rational and algebraic functions
- fast to compute terms of the sequence from the DEs
- can compute asymptotics of the sequence from the DEs

This project: understanding the cogrowth series $\sum_n c_n z^n$ for BS(N,N)

Kouksov
— cogrowth series is rational iff the group is finite

Not many explicit cogrowth series (closed form, etc) known
— free groups, abelian groups, some free products

Experimental work (ERvRW) to compute cogrowth rates for groups whose amenability is unknown
— need exact results for comparison/validation

Thm(ERvRW): cogrowth series $\sum_n c_n z^n$ is D-finite

Proof sketch: instead of counting just words = id, count more.

Let $g_{n,k}$ be the number of words of length n that evaluate to a^k in BS(N,N)

so $g_{n,0} = c_n$, but it is easier to count $g_{n,k}$ then *diagonalise* its generating function at $q = 0$

Define $G(z;q) = \sum_{n,k} g_{n,k} z^n q^k$

$$[q^0]G(z;q) = \sum_{n,k} g_{n,0} z^n$$

Thm(ERvRW): cogrowth series $\sum_n c_n z^n$ is D-finite

Proof sketch: instead of counting just words = id, count more.

Let $g_{n,k}$ be the number of words of length n that evaluate to a^k in BS(N,N)

so $g_{n,0} = c_n$, but it is easier to count $g_{n,k}$ then *diagonalise* its generating function at $q = 0$

Define $G(z;q) = \sum_{n,k} g_{n,k} z^n q^k$ $\qquad$ $[q^0]G(z;q) = \sum_{n,k} g_{n,0} z^n$

Thm(ERvRW): $G(z;q)$ **is algebraic**

Since the *diagonal* of an D-finite function is D-finite (Lipshitz), the result follows.

Details

Proving that $G(z;q)$ is algebraic is pretty cool, see

http://arxiv.org/abs/1309.4184

for details.

For the rest of the talk I will explain how we compute explicitly the cogrowth rate, which is the exponential growth rate of the cogrowth function, *i.e.* $\limsup c_n^{1/n}$

Lemma: $g_{n,k} = g_{n,-k}$

Proof: switch $a \longleftrightarrow a^{-1}$ in words counted by $g_{n,k}$

Eg in BS(10,10):

$$a^{13}ba^{-10}b^{-1}a^{2} \quad \longleftrightarrow \quad a^{-13}ba^{10}b^{-1}a^{-2}$$

Lemma: $g_{n,k} = 0$ for $|k| > n$

Proof: if w has length n, replace $a^{\pm \mathsf{N}} b^{\pm 1}$ by $b^{\pm 1} a^{\pm \mathsf{N}}$ and freely reduce. These moves do not increase length, and repeating them gives a word with no $a^{\pm \mathsf{N}}$ subwords except possibly on the right.

Eg in BS(10,10): $\quad a^{13} b a^{12} b \ldots \quad \longrightarrow \quad a^{3} b a^{2} b a^{20} \ldots$

Lemma: $g_{n,k} = 0$ for $|k| > n$

Proof: if w has length n, replace $a^{\pm \mathsf{N}} b^{\pm 1}$ by $b^{\pm 1} a^{\pm \mathsf{N}}$ and freely reduce. These moves do not increase length, and repeating them gives a word with no $a^{\pm \mathsf{N}}$ subwords except possibly on the right.

Eg in BS(10,10): $a^{13} b a^{12} b \ldots \quad \longrightarrow \quad a^{3} b a^{2} b a^{20} \ldots$

If w equals a power of a, there can be no $b^{\pm 1}$ letters in the resulting word (*Britton's lemma*)

So the resulting word a^k is no longer than n, so $|k| \leq n$.

Computing the cogrowth

The diagonal of $G(z;q) = \sum_{n,k} g_{n,k} z^n q^k$ is not so easy to work with.

Instead, consider the generating function with $q = 1$:

$$G(z;1) = \sum_n \left(\sum_k g_{n,k} \right) z^n$$

Computing the cogrowth

The diagonal of $G(z;q) = \sum_{n,k} g_{n,k} z^n q^k$ is not so easy to work with.

Instead, consider the generating function with $q = 1$:

$$G(z;1) = \sum_n g_n z^n \quad \left(\text{where} \quad g_n = \sum_k g_{n,k}\right).$$

Computing the cogrowth

The diagonal of $G(z;q) = \sum_{n,k} g_{n,k} z^n q^k$ is not so easy to work with.

Instead, consider the generating function with $q = 1$:

$$G(z;1) = \sum_n g_n z^n \left(\text{where} \quad g_n = \sum_k g_{n,k}\right).$$

Thm(ERvRW): $\limsup c_n^{1/n} = \limsup g_n^{1/n}$

So to compute cogrowth we find the asymptotic growth rate of a function that is counting more than just trivial words!

Thm(ERvRW): $\limsup c_n^{1/n} = \limsup g_n^{1/n}$

The proof makes use of a *"most popular"* argument that is popular in statistical physics.

Thm(ERvRW): $\limsup c_n^{1/n} = \limsup g_n^{1/n}$

The proof makes use of a *"most popular"* argument that is popular in statistical physics.

Proof: Let $\mu_{\text{all}} = \limsup g_n^{1/n}$ and $\mu_0 = \limsup c_n^{1/n} = \limsup g_{n,0}^{1/n}$

Since $g_{n,k}$ are nonnegative and $g_{n,0} \leq g_n$ we have $\mu_{\text{all}} \geq \mu_0$.

Proof continued:

Now we prove that $\mu_{\text{all}} \leq \mu_0$.

Recall that $g_{n,k} = 0$ when $|k| > n$

so there is some k^* so that g_{n,k^*} is maximised (k^* is *popular*)

So $g_{n,k^*} \leq \sum_k g_{n,k} = g_n \leq (2n+1) g_{n,k^*}$

so taking lim sup we get the same answer, so $\mu_{\text{all}} = \limsup g_{n,k^*}$

Proof continued:

Now consider words that equal a^{k^*}, and words that equal a^{-k^*}. Put them together and you get a word equal to a^0, so

$$(g_{n,k})^2 = g_{n,k^*} \cdot g_{n,-k^*} \leq g_{2n,0} \qquad (g_{n,k} = g_{n,-k})$$

Raise both sides to $1/2n$:

$$(g_{n,k^*})^{1/n} \leq (g_{2n,0})^{1/2n}$$

send $n \to \infty$:

$$\mu_{\text{all}} = \limsup (g_{n,k^*})^{1/n} \leq \limsup (g_{2n,0})^{1/2n} = \mu_0$$

Computing the cogrowth

The rate of growth of $G(z;1)$ (which is algebraic) is therefore the same as the cogrowth.

We can find it by taking the explicit polynomial equation satisfied by $G(z;1)$ and solving the discriminant *

N	μ_0
1	4
2	3.792765039
3	3.647639445
4	3.569497357

*David A. Klarner and Patricia Woodworth. Asymptotics for coefficients of algebraic functions. Aequationes Math. 23, 1981.

Computing the cogrowth

N	μ_0
1	4
2	3.792765039
3	3.647639445
4	3.569497357
5	3.525816111
6	3.500607636
7	3.485775158
8	3.476962757
9	3.471710431
10	3.468586539

The cogrowth rate $\mu_0 = \mu_{\text{all}}$ in BS(N,N) up to 10 (the polynomials and DEs up to 10 are online).

Note that the cogrowth rate for the 2-generator free group is $\sqrt{12} = 3.464101615$

Computing the cogrowth

N	μ_0
1	4
2	3.792765039
3	3.647639445
4	3.569497357
5	3.525816111
6	3.500607636
7	3.485775158
8	3.476962757
9	3.471710431
10	3.468586539

The cogrowth rate $\mu_0 = \mu_{\text{all}}$ in BS(N,N) up to 10 (the polynomials and DEs up to 10 are online).

Note that the cogrowth rate for the 2-generator free group is $\sqrt{12} = 3.464101615$

Thanks!
http://arxiv.org/abs/1309.4184