Spaces of holomorphic maps from Stein manifolds to Oka manifolds

Finnur Lárusson

University of Adelaide

September 2013

Deforming continuous maps to holomorphic maps

Let X and Y be complex manifolds. Can every continuous map $X \to Y$ be deformed to a holomorphic map?

Deforming continuous maps to holomorphic maps

Let X and Y be complex manifolds. Can every continuous map $X \to Y$ be deformed to a holomorphic map?

- $\mathbb{C}^* \to \mathbb{D}^*$: every holomorphic map is constant (Liouville), so the only winding number realised by holomorphic maps is 0.

Deforming continuous maps to holomorphic maps

Let X and Y be complex manifolds. Can every continuous map $X \to Y$ be deformed to a holomorphic map?

- $\mathbb{C}^* \to \mathbb{D}^*$: every holomorphic map is constant (Liouville), so the only winding number realised by holomorphic maps is 0.
- $\mathbb{D}^* \to \mathbb{D}^*$: holomorphic maps only realise nonnegative winding numbers.

Deforming continuous maps to holomorphic maps

Let X and Y be complex manifolds. Can every continuous map $X \to Y$ be deformed to a holomorphic map?

- $\mathbb{C}^* \to \mathbb{D}^*$: every holomorphic map is constant (Liouville), so the only winding number realised by holomorphic maps is 0.
- $\mathbb{D}^* \to \mathbb{D}^*$: holomorphic maps only realise nonnegative winding numbers.
- $\mathbb{C} \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \to \mathbb{D}^*$: there are uncountably many homotopy classes of continuous maps, but only countably many classes of holomorphic maps.

Deforming continuous maps to holomorphic maps

Let X and Y be complex manifolds. Can every continuous map $X \to Y$ be deformed to a holomorphic map?

- $\mathbb{C}^* \to \mathbb{D}^*$: every holomorphic map is constant (Liouville), so the only winding number realised by holomorphic maps is 0.

- $\mathbb{D}^* \to \mathbb{D}^*$: holomorphic maps only realise nonnegative winding numbers.

- $\mathbb{C} \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \to \mathbb{D}^*$: there are uncountably many homotopy classes of continuous maps, but only countably many classes of holomorphic maps.

In all three examples, if the target $\mathbb{D}^*$ is replaced by $\mathbb{C}^*$, then every continuous map can be deformed to a holomorphic map.

Gromov's theorem

Theorem. If S is a Stein manifold and X is an Oka manifold, then every continuous map $S \to X$ can be deformed to a holomorphic map.

Gromov's theorem

Theorem. If S is a Stein manifold and X is an Oka manifold, then every continuous map $S \to X$ can be deformed to a holomorphic map.

A *Stein manifold* has many holomorphic maps **into** $\mathbb{C}$.
More precisely: A closed complex submanifold of $\mathbb{C}^n$ for some n.
There are many nontrivially equivalent characterisations.

Gromov's theorem

Theorem. If S is a Stein manifold and X is an Oka manifold, then every continuous map $S \to X$ can be deformed to a holomorphic map.

A *Stein manifold* has many holomorphic maps **into** $\mathbb{C}$.
More precisely: A closed complex submanifold of $\mathbb{C}^n$ for some n.
There are many nontrivially equivalent characterisations.

Every domain in $\mathbb{C}$ and every convex domain in $\mathbb{C}^n$ is Stein.

Gromov's theorem

Theorem. If S is a Stein manifold and X is an Oka manifold, then every continuous map $S \to X$ can be deformed to a holomorphic map.

A *Stein manifold* has many holomorphic maps **into** $\mathbb{C}$.
More precisely: A closed complex submanifold of $\mathbb{C}^n$ for some n.
There are many nontrivially equivalent characterisations.

Every domain in $\mathbb{C}$ and every convex domain in $\mathbb{C}^n$ is Stein.

An *Oka manifold* X has many holomorphic maps **from** $\mathbb{C}$.
A little more precisely: The Runge approximation theorem holds for holomorphic maps $\mathbb{C}^n \to X$.
There are even more nontrivially equivalent characterisations!

Gromov's theorem

Theorem. If S is a Stein manifold and X is an Oka manifold, then every continuous map $S \to X$ can be deformed to a holomorphic map.

A *Stein manifold* has many holomorphic maps **into** $\mathbb{C}$.
More precisely: A closed complex submanifold of $\mathbb{C}^n$ for some n.
There are many nontrivially equivalent characterisations.

Every domain in $\mathbb{C}$ and every convex domain in $\mathbb{C}^n$ is Stein.

An *Oka manifold* X has many holomorphic maps **from** $\mathbb{C}$.
A little more precisely: The Runge approximation theorem holds for holomorphic maps $\mathbb{C}^n \to X$.
There are even more nontrivially equivalent characterisations!

All complex Lie groups and their homogeneous spaces are Oka.

Gromov's theorem

Theorem. If S is a Stein manifold and X is an Oka manifold, then every continuous map $S \to X$ can be deformed to a holomorphic map.

A *Stein manifold* has many holomorphic maps **into** $\mathbb{C}$.
More precisely: A closed complex submanifold of $\mathbb{C}^n$ for some n.
There are many nontrivially equivalent characterisations.

Every domain in $\mathbb{C}$ and every convex domain in $\mathbb{C}^n$ is Stein.

An *Oka manifold* X has many holomorphic maps **from** $\mathbb{C}$.
A little more precisely: The Runge approximation theorem holds for holomorphic maps $\mathbb{C}^n \to X$.
There are even more nontrivially equivalent characterisations!

All complex Lie groups and their homogeneous spaces are Oka.
$\mathbb{C}^*$ is Oka but $\mathbb{D}^*$ is not.

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

Can it be done for all f at once, in a way that depends continuously on f and leaves f fixed if it is holomorphic to begin with?

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

Can it be done for all f at once, in a way that depends continuously on f and leaves f fixed if it is holomorphic to begin with?

In other words, $\boxed{\text{is } \mathscr{O}(S,X) \text{ a deformation retract of } \mathscr{C}(S,X)\text{?}}$

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

Can it be done for all f at once, in a way that depends continuously on f and leaves f fixed if it is holomorphic to begin with?

In other words, is $\mathscr{O}(S, X)$ a deformation retract of $\mathscr{C}(S, X)$?

Parametrised versions of Gromov's theorem are known for 'small' parameter spaces: Euclidean compacts, e.g. finite polyhedra.

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

Can it be done for all f at once, in a way that depends continuously on f and leaves f fixed if it is holomorphic to begin with?

In other words, $\boxed{\text{is } \mathscr{O}(S,X) \text{ a deformation retract of } \mathscr{C}(S,X)\text{?}}$

Parametrised versions of Gromov's theorem are known for 'small' parameter spaces: Euclidean compacts, e.g. finite polyhedra. Hence also for CW complexes (FL 2004).

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

Can it be done for all f at once, in a way that depends continuously on f and leaves f fixed if it is holomorphic to begin with?

In other words, **is $\mathscr{O}(S, X)$ a deformation retract of $\mathscr{C}(S, X)$?**

Parametrised versions of Gromov's theorem are known for 'small' parameter spaces: Euclidean compacts, e.g. finite polyhedra. Hence also for CW complexes (FL 2004).

But $\mathscr{C}$ and $\mathscr{O}$ are CW complexes only in trivial cases. They are metrisable, but a metrisable CW complex is locally compact.

The problem

Every continuous map f from a Stein manifold S to an Oka manifold X can be deformed to a holomorphic map.

Can it be done for all f at once, in a way that depends continuously on f and leaves f fixed if it is holomorphic to begin with?

In other words, $\boxed{\text{is } \mathscr{O}(S,X) \text{ a deformation retract of } \mathscr{C}(S,X)?}$

Parametrised versions of Gromov's theorem are known for 'small' parameter spaces: Euclidean compacts, e.g. finite polyhedra. Hence also for CW complexes (FL 2004).

But $\mathscr{C}$ and $\mathscr{O}$ are CW complexes only in trivial cases. They are metrisable, but a metrisable CW complex is locally compact.

Using homotopy theory and infinite-dimensional topology, we can solve the problem for reasonable S and arbitrary X.

Reformulate the problem

By basic algebraic topology, the following are equivalent.

(i) $\mathscr{O}(S,X)$ is a deformation retract of $\mathscr{C}(S,X)$.

(ii) The inclusion $\iota : \mathscr{O}(S,X) \hookrightarrow \mathscr{C}(S,X)$ is a homotopy equivalence and has the homotopy extension property.

Reformulate the problem

By basic algebraic topology, the following are equivalent.

(i) $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

(ii) The inclusion $\iota : \mathscr{O}(S, X) \hookrightarrow \mathscr{C}(S, X)$ is a homotopy equivalence and has the homotopy extension property.
That is, ι is an acyclic cofibration in the h-structure on **Top**.

Reformulate the problem

By basic algebraic topology, the following are equivalent.

(i) $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

(ii) The inclusion $\iota : \mathscr{O}(S, X) \hookrightarrow \mathscr{C}(S, X)$ is a homotopy equivalence and has the homotopy extension property. That is, ι is an acyclic cofibration in the h-structure on **Top**.

The h-structure (*h* for *Hurewicz*) is one of the two classical frameworks for standard homotopy theory. The other is the q-structure (*q* for *Quillen*).

Reformulate the problem

By basic algebraic topology, the following are equivalent.

(i) $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

(ii) The inclusion $\iota : \mathscr{O}(S, X) \hookrightarrow \mathscr{C}(S, X)$ is a homotopy equivalence and has the homotopy extension property. That is, ι is an acyclic cofibration in the h-structure on **Top**.

The h-structure (*h* for *Hurewicz*) is one of the two classical frameworks for standard homotopy theory. The other is the q-structure (*q* for *Quillen*).

A parametrised version of Gromov's theorem for finite polyhedra implies that ι is a weak homotopy equivalence.
How can we bridge the gap?

ANRs and the mixed structure on **Top**

Two main topological ingredients:

The brand new m-structure (*m* for *mixed*), due to Cole (2006): a third framework for standard homotopy theory.

The theory of ANRs (absolute neighbourhood retracts for metric spaces).

ANRs and the mixed structure on **Top**

Two main topological ingredients:

The brand new m-structure (*m* for *mixed*), due to Cole (2006): a third framework for standard homotopy theory.

The theory of ANRs (absolute neighbourhood retracts for metric spaces).

To cut a long story short:

Theorem (FL). Suppose $\mathscr{C}(S, X)$ is ANR. Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$ if and only if $\mathscr{O}(S, X)$ is ANR.

ANRs and the mixed structure on **Top**

Two main topological ingredients:

The brand new m-structure (*m* for *mixed*), due to Cole (2006): a third framework for standard homotopy theory.

The theory of ANRs (absolute neighbourhood retracts for metric spaces).

To cut a long story short:

Theorem (FL). Suppose $\mathscr{C}(S, X)$ is ANR. Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$ if and only if $\mathscr{O}(S, X)$ is ANR.

Theorem (Milnor 1959, Smrekar-Yamashita 2009). $\mathscr{C}(S, X)$ is ANR if S is *finitely dominated*.

ANRs and the mixed structure on **Top**

Two main topological ingredients:

The brand new m-structure (*m* for *mixed*), due to Cole (2006): a third framework for standard homotopy theory.

The theory of ANRs (absolute neighbourhood retracts for metric spaces).

To cut a long story short:

Theorem (FL). Suppose $\mathscr{C}(S, X)$ is ANR. Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$ if and only if $\mathscr{O}(S, X)$ is ANR.

Theorem (Milnor 1959, Smrekar-Yamashita 2009). $\mathscr{C}(S, X)$ is ANR if S is *finitely dominated*.

We need a good sufficient condition for $\mathscr{O}(S, X)$ to be ANR.

Absolute neighbourhood retracts

A metrisable space A is ANR if whenever A is embedded as a closed subspace of a metric space B, some neighbourhood of A in B retracts onto A.

Absolute neighbourhood retracts

A metrisable space A is ANR if whenever A is embedded as a closed subspace of a metric space B, some neighbourhood of A in B retracts onto A.

There are several other characterisations, including a 'combinatorial' one (Dugundji-Lefschetz) that we use.

Absolute neighbourhood retracts

A metrisable space A is ANR if whenever A is embedded as a closed subspace of a metric space B, some neighbourhood of A in B retracts onto A.

There are several other characterisations, including a 'combinatorial' one (Dugundji-Lefschetz) that we use.

ANRs have many nice properties. Being ANR is a local property. Every ANR is locally contractible (and conversely for finite-dimensional spaces).

Absolute neighbourhood retracts

A metrisable space A is ANR if whenever A is embedded as a closed subspace of a metric space B, some neighbourhood of A in B retracts onto A.

There are several other characterisations, including a 'combinatorial' one (Dugundji-Lefschetz) that we use.

ANRs have many nice properties. Being ANR is a local property. Every ANR is locally contractible (and conversely for finite-dimensional spaces).

A CW complex is ANR if and only if it is locally finite.
ANRs and CW complexes have the same homotopy types.

Absolute neighbourhood retracts

A metrisable space A is ANR if whenever A is embedded as a closed subspace of a metric space B, some neighbourhood of A in B retracts onto A.

There are several other characterisations, including a 'combinatorial' one (Dugundji-Lefschetz) that we use.

ANRs have many nice properties. Being ANR is a local property. Every ANR is locally contractible (and conversely for finite-dimensional spaces).

A CW complex is ANR if and only if it is locally finite.
ANRs and CW complexes have the same homotopy types.

A metrisable space is ANR if and only if every open subset has the homotopy type of a CW complex (Cauty 1994).

The main result

Theorem. Let X be an Oka manifold and let S be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points, e.g. an affine algebraic manifold.

Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

The main result

Theorem. Let X be an Oka manifold and let S be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points, e.g. an affine algebraic manifold.

Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

The hypothesis on S is not necessary, but I do not know whether it can be omitted. For example:

The main result

Theorem. Let X be an Oka manifold and let S be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points, e.g. an affine algebraic manifold.

Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

The hypothesis on S is not necessary, but I do not know whether it can be omitted. For example:

- For arbitrary S, $\mathscr{O}(S, \mathbb{C}^n)$ is a deformation retract of $\mathscr{C}(S, \mathbb{C}^n)$.

The main result

Theorem. Let X be an Oka manifold and let S be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points, e.g. an affine algebraic manifold.

Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

The hypothesis on S is not necessary, but I do not know whether it can be omitted. For example:

- For arbitrary S, $\mathscr{O}(S, \mathbb{C}^n)$ is a deformation retract of $\mathscr{C}(S, \mathbb{C}^n)$.
- $\mathscr{O}(\mathbb{C} \setminus \mathbb{N}, \mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(\mathbb{C} \setminus \mathbb{N}, \mathbb{C}^*)$.

Still, $\mathscr{C}$ and $\mathscr{O}$ are not ANR: they are not semilocally contractible, so they do not even have the homotopy type of an ANR (or of a CW complex).

The main result

Theorem. Let X be an Oka manifold and let S be a Stein manifold with a strictly plurisubharmonic Morse exhaustion with finitely many critical points, e.g. an affine algebraic manifold. Then $\mathscr{O}(S, X)$ is a deformation retract of $\mathscr{C}(S, X)$.

The hypothesis on S is not necessary, but I do not know whether it can be omitted. For example:

- For arbitrary S, $\mathscr{O}(S, \mathbb{C}^n)$ is a deformation retract of $\mathscr{C}(S, \mathbb{C}^n)$.
- $\mathscr{O}(\mathbb{C} \setminus \mathbb{N}, \mathbb{C}^*)$ is a deformation retract of $\mathscr{C}(\mathbb{C} \setminus \mathbb{N}, \mathbb{C}^*)$.

Still, $\mathscr{C}$ and $\mathscr{O}$ are not ANR: they are not semilocally contractible, so they do not even have the homotopy type of an ANR (or of a CW complex).

Paper on the arXiv and on my webpage.