Virtual neighbourhood techniques for pseudo-holomorphic spheres

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Introduction

The Gromov-Witten invariants "*count*" stable pseudo-holomorphic curves of genus g and n-marked points in a symplectic manifold (X, ω, J).

This requires some good understanding of the moduli space of stable pseudo-holomorphic curves.

Earlier foundational works were done in 1990s by

1. K. Fukaya and K. Ono (general symplectic manifolds)
2. J.Li and G. Tian (smooth projective varieties)
3. G. Liu and G. Tian

using the notion of virtual fundamental cycle, and its variants.

Raun also proposed virtual neighbourhood technique as a dual approach.

More recently, there are some renewed interests in the refined structure for the underlying moduli spaces of stable maps.

Joint with Bohui Chen and Anmin Li, we investigated some interesting issues arising from the infinite dimensional analysis in the Gromov-Witten theory, in particular,

- non-smooth action of certain reparametrization groups,
- "gluing" local perturbations together,
- how to define an invariant.

Reference:

B. Chen, A. Li and B. Wang, *Virtual neighborhood technique for pseudo-holomorphic spheres*, arXiv: 1306.3276.

Orbifold Fredholm system

Definition

An orbifold Fredhom system is a triple $\begin{array}{c} \mathcal{E} \\ \downarrow \uparrow S \\ \mathcal{B} \end{array}$ consisting of

- a Banach orbifold $\mathcal{B}$,
- a Banach orbifold bundle $\mathcal{E}$ over $\mathcal{B}$, and
- a Fredholm section $S : \mathcal{B} \to \mathcal{E}$.

The central object is the moduli space $\mathcal{M} = S^{-1}(0)$.

In practice, $\mathcal{B}$ is only a topological orbifold or even (topological) orbifold stratified spaces, and $\mathcal{E}$ is only a topological Banach orbifold bundle. For examples, this happens when we consider the (compactified) moduli space of pseudo-holomorphic spheres.

Example: pseudo-holomorphic spheres

Consider the Fredholm system $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{E}}, \bar{\partial}_J)$,

- $\widetilde{\mathcal{B}} = \{u \in W^{1,p}(\Sigma, X) | f_*([\Sigma]) = A\}$, the space of $W^{1,p}$-maps $\Sigma = S^2 \to (X, \omega, J)$ with a fixed homology class $A \in H_2(X, \mathbb{Z})$.
- $\widetilde{\mathcal{E}}$ be the Banach bundle over $\widetilde{\mathcal{B}}$ whose fiber at $u : \Sigma \to X$ is

$$L^p(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} u^*TX),$$

  the space of L^p-section of the bundle $\Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} u^*TX$.
- The Cauchy-Riemann operator defines a smooth Fredholm section $\bar{\partial}_J : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{E}}$.

The zero set of this section $\widetilde{\mathcal{M}} = \bar{\partial}_J^{-1}(0) \subset \widetilde{\mathcal{B}}$ is the moduli space of **parametrized** pseudo-holomorphic spheres in (X, ω, J) .

Non-smooth actions of $PSL(2,\mathbb{C})$

The actual moduli space is the quotient of $\widetilde{\mathcal{M}}$ under the action of the reparametrization group $PSL(2,\mathbb{C})$.

The action of $G = PSL(2,\mathbb{C})$ on $\widetilde{\mathcal{B}}$ is given by the reparametrization of the domain Σ:

$$(g,u) \quad \mapsto \quad g \cdot u = u \circ g^{-1}. \tag{1}$$

Consider the quotient

$$(\mathcal{B} = \widetilde{\mathcal{B}}/G, \mathcal{E} = \widetilde{\mathcal{E}}/G, S).$$

The main technical issue is that this action (1) is only continuous. So $\mathcal{B}$ is a only topological orbifold and $\mathcal{E}$ is a topological orbifold vector bundle with a continuous section $S : \mathcal{B} \to \mathcal{E}$.

We need to find a way to get the slice and neighbourhood theorem for this non-smooth action.

Key steps in dealing with non-smooth actions

1. Each point $u \in \widetilde{\mathcal{M}}$ is a smooth map, and the G-orbit $\mathcal{O}_u$ is a smooth submanfiold of $\widetilde{\mathcal{B}}$.
   $\Longrightarrow$ The tangent bundle $T\mathcal{O}_u$ is a smooth sub-bundle of $T\widetilde{\mathcal{B}}|_{\mathcal{O}_u}$.
2. Recall that there is an L^2-inner product $\Omega_{u_0} : T_{u_0}\widetilde{\mathcal{B}} \times T_{u_0}\widetilde{\mathcal{B}} \to \mathbb{R}$ that is given by

$$\Omega_{u_0}(v, w) = \int_\Sigma \langle v(x), w(x)\rangle_{u_0(x)} dvol_\Sigma(x). \tag{2}$$

Lemma

The G-invariant metric defined by Ω_{u_0} is a smooth metric on $T\widetilde{\mathcal{B}}|_{\mathcal{O}_{u_0}}$.

Proof

$$\begin{aligned}\Omega_{g\cdot u_0}(v, w) &= \int_\Sigma h_{u_0(x)}(v(g(x)), w(g(x)))dvol_\Sigma(x)\\ &= \int_\Sigma h_{u_0(g^{-1}(x))}(v(y), w(y))Jac_g^{-1}dvol_\Sigma(y).\end{aligned}$$

Define $\mathcal{N}_{\mathcal{O}_u} = (T\mathcal{O}_u)^{\perp}$ with respect to the L^2-metric. Then $\mathcal{N}_{\mathcal{O}_u}$ is smooth sub-bundle $T\widetilde{\mathcal{B}}|_{\mathcal{O}_u}$.
Therefore,

- There exists a small $\epsilon > 0$ such that the exponential map

$$\exp_u : \mathcal{N}^{\epsilon}_{\mathcal{O}_u} \longrightarrow \widetilde{\mathcal{B}}$$

  provides a G-invariant **tubular neighbourhood** $\mathcal{T}^{\epsilon}_u$ of $\mathcal{O}_u$ in $\widetilde{\mathcal{B}}$.
- The image of the fiber at u gives a G-slice $\mathcal{S}^{\epsilon}_u$ for the G-action at u.

These slices only define a topological Banach (orbifold!) coordinate charts

$$\{\mathbf{U}^{\epsilon}_u = (\mathcal{S}^{\epsilon}_u, G_u)\}_{[u]\in\mathcal{M}}$$

for a neighbourhood $\mathcal{M} = \widetilde{\mathcal{M}}/G$ in $\mathcal{B} = \widetilde{\mathcal{B}}/G$.

Assume that $\mathcal{M}$ is compact, there exist finitely many points

$$\{[u_1],[u_2],\cdots,[u_n]\}$$

in $\mathcal{M}$ with their smooth representatives

$$u_1,u_2,\cdots,u_n \tag{3}$$

in $\widetilde{\mathcal{B}}$ whose slices $\{\mathbf{U}^{\epsilon_i}_{u_i}\}_{i=1}^n$, prescribed as in the previous slide, define a system of Banach orbifold charts for $\mathcal{U}=\bigcup_{i=1}^n \mathbf{U}^{\epsilon_i}_{u_i}$.

Given a smooth function f on $\mathbf{U}^{\epsilon_i}_{u_i}$, then, under the coordinate change Φ_{u_i,u_j}, $f\circ\Phi_{u_i,u_j}$ might not be a smooth function. A lemma comes to rescue.

Lemma

If f has a G-invariant smooth extension $\hat{f}:\mathcal{T}^{\epsilon_1}_{u_i}\to\mathbb{R}$, then

$$f\circ\Phi_{u_i,u_j}=\hat{f}|_{\mathcal{T}^{\epsilon_1}_{u_i}\cap\mathbf{U}^{\epsilon_j}_{u_j}}$$

is actually a smooth function.

Virtual manifolds/orbifolds

In order to deal with second issue of "gluing local perturbations", we need to review the notion of virtual manifolds proposed by Bohui Chen and Gang Tian.

Definition

A *virtual manifold* is a collection of smooth manifolds $\{X_I\}_{I\in\mathcal{I}}$ indexed by a partially ordered set $(\mathcal{I} = 2^{\{1,2,\cdots,n\}}, \subset)$, together with patching data

$$\{(\Phi_{I,J}, \phi_{I,J}) | I, J \in \mathcal{I}, I \subset J\},$$

where $\Phi_{I,J} : X_{J,I} \to X_{I,J}$ is a vector bundle with the zero section $\phi_{I,J} : X_{I,J} \to X_{J,I}$ for open sub-manifolds $X_{I,J}$ and $X_{J,I}$ of X_I and X_J respectively, whenever $I \subset J$. The patching datum $(\Phi_{I,J}, \phi_{I,J})$ for $I \subset J$ satisfying some coherent conditions.

Virtual orbifolds can be defined using the language of proper étale groupoids.

Virtual vector bundles

A **virtual vector bundle** over a virtual manifold $\{X_I\}$ is a virtual manifold

$$\mathbf{E} = \{E_I \to X_I\}$$

such that E_I is a vector bundle over X_I for each I, and for any ordered pair $I \subset J$, $E_{I,J} = E_I|_{X_{I,J}}$, $E_{J,i} = E_J|_{X_{J,I}}$ and

$$E_J|_{X_{J,I}} \cong \Phi^*_{I,J}(X_{J,I} \oplus E_I|_{X_{I,J}}) \tag{4}$$

as vector bundles over $X_{J,I}$.
A section of a virtual vector bundle $\mathbf{E}$ over $\{X_I\}$ is a collection of sections $\{\sigma_I : X_I \longrightarrow E_I,\}$ called a **virtual section** of $\mathbf{E}$ if any ordered pair $I \subset J$,

$$\sigma_J|_{X_{J,I}} = (s_{X_{J,I}}, \sigma_I|_{X_{I,J}} \circ \Phi_{I,J}), \tag{5}$$

where $s_{X_{J,I}}$ is the canonical section of the bundle $\Phi^*_{I,J}(X_{J,I})$ over $X_{J,I}$.

Theorem (Chen-Li-W.)

Given an orbifold Fredholm system $(\mathcal{B}, \mathcal{E}, S)$ such that $\mathcal{M} = S^{-1}(0)$ is compact, then there exists a virtual orbifold system, that is, a collection of triples

$$\{(\mathcal{V}_I, \mathbf{E}_I, \sigma_I) | I \subset \{1, 2, \cdots, n\}\},$$

where

1. $\mathcal{V} = \{\mathcal{V}_I\}$ is a finite dimensional virtual orbifold,
2. $\mathbf{E} = \{\mathbf{E}_I\}$ is a finite rank virtual orbifold vector bundle over $\mathcal{V}$ with a virtual section $\sigma = \{\sigma_I\}$ such that the zero sets $\{\sigma_I^{-1}(0)\}$ form a cover of $\mathcal{M}$.

Moreover, under certain assumptions, the virtual integration

$$\int_{\mathcal{V}}^{vir} \alpha$$

is well-defined for any virtual differential form $\alpha = \{\alpha_I\}$ on $\mathcal{V}$.

Theorem (Chen-Li-W.)

Let $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{E}}, \bar{\partial}_J)$ be the Fredholm system associated to pseudo-holomorphic spheres with the action of the reparametrization group $G = PSL(2, \mathbb{C})$. Then there exists a virtual orbifold system such that

1. the zero sets $\{\sigma_I^{-1}(0)\}_{I \subset \{1,2,\cdots,n\}}$ form a cover of $\mathcal{M}$;
2. the genus zero Gromov-Witten invariant is defined by the the virtual integration.

- For the general Gromov-Witten invariant, we need to develop a general gluing principle for virtual orbifold stratified spaces.
- This virtual neighbourhood technique is very useful for the development of quantum K-theory.