THE UNIVERSITY OF SYDNEY

MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

Semester2, 2003 — Lecturer: R. Howlett

Tutorial 3

1. Check that $\left\{ \begin{pmatrix} \frac{1}{\sqrt6} \\ \frac{1}{\sqrt6} \\ -\frac{2}{\sqrt6} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} \\ 0 \end{pmatrix} \right\}$ is an orthonormal set.

Solution.

Call the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ (respectively). We find that

$$\begin{aligned} \underset{\sim}{u}\cdot\underset{\sim}{u} &= (\frac{1}{\sqrt6})^2 + (\frac{1}{\sqrt6})^2 + (-\frac{2}{\sqrt6})^2 = \frac{1+1+4}{6} = 1 \\ \underset{\sim}{v}\cdot\underset{\sim}{v} &= (\frac{1}{\sqrt2})^2 + (-\frac{1}{\sqrt2})^2 + 0^2 = \frac12 + \frac12 = 1 \\ \underset{\sim}{u}\cdot\underset{\sim}{v} &= (\frac{1}{\sqrt6})(\frac{1}{\sqrt2}) + (\frac{1}{\sqrt6})(\frac{-1}{\sqrt2}) + (\frac{-2}{\sqrt6})0 = 0, \end{aligned}$$

as required.

2. Let $\underset{\sim}{a}_1 = (2,2,-1)^T$ and $\underset{\sim}{a}_2 = (-1,2,2)^T$.

(*i*) Check that $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ is an orthogonal set of vectors. Normalize $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ to produce an orthonormal set.

(*ii*) Let $\underset{\sim}{v} = (0,3,0)^T$. Find the projections of $\underset{\sim}{v}$ onto the one-dimensional spaces spanned by $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$.

(*iii*) Use Part (*ii*) to find the projection of $\underset{\sim}{v}$ onto the subspace W of $\mathbb{R}^3$ spanned by $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$.

(*iv*) Express $\underset{\sim}{v}$ as the sum of two vectors, one in W and the other orthogonal to W.

Solution.

(*i*) $\underset{\sim}{a}_1\cdot\underset{\sim}{a}_2 = -2 + 2^2 - 2 = 0$; so the vectors are orthogonal to each other, as required. Now $\|\underset{\sim}{a}_1\| = \sqrt{2^2+2^2+(-1)^2} = \sqrt9 = 3$, and similarly $\|\underset{\sim}{a}_2\| = 3$ also. To normalize you divide each vector by its length; you get the following orthonormal set:

$$\left\{ \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \right\}.$$

(*ii*) The projection of $\underset{\sim}{v}$ onto Span($\underset{\sim}{a}_1$) is

$$\underset{\sim}{p}_1 = \frac{\underset{\sim}{a}_1\cdot\underset{\sim}{v}}{\underset{\sim}{a}_1\cdot\underset{\sim}{a}_1}\underset{\sim}{a}_1 = \frac69\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -2/3 \end{pmatrix}$$

and the projection of $\underset{\sim}{v}$ onto Span($\underset{\sim}{a}_2$) is

$$\underset{\sim}{p}_2 = \frac{\underset{\sim}{a}_2\cdot\underset{\sim}{v}}{\underset{\sim}{a}_2\cdot\underset{\sim}{a}_2}\underset{\sim}{a}_2 = \frac69\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 4/3 \\ 4/3 \end{pmatrix}.$$

(*iii*) Since $\{\underset{\sim}{a}_1, \underset{\sim}{a}_2\}$ is an orthogonal set, the projection of any vector onto $W = \text{Span}(\underset{\sim}{a}_1, \underset{\sim}{a}_2)$ is the sum of its projections onto Span($\underset{\sim}{a}_1$) and Span($\underset{\sim}{a}_2$). So the projection $\underset{\sim}{p}$ of $\underset{\sim}{v}$ onto W is

$$\underset{\sim}{p} = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 = \begin{pmatrix} 4/3 \\ 4/3 \\ -2/3 \end{pmatrix} + \begin{pmatrix} -2/3 \\ 4/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 8/3 \\ 2/3 \end{pmatrix}.$$

(*iv*) By the definition of projection, $\underset{\sim}{v} - \underset{\sim}{p}$ is orthogonal W. So the required expression is $\underset{\sim}{v} = \underset{\sim}{p} + (\underset{\sim}{v} - \underset{\sim}{p}) = \frac23(1,4,1) + \frac13(-2,1,-2)$.

3. Apply the Gram-Schmidt process to the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Solution.

Call the given vectors $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$, $\underset{\sim}{a}_3$. Recall that the Gram-Schmidt formula is

$$\underset{\sim}{u}_i = \underset{\sim}{a}_i - \sum_{j=1}^{i-1} \frac{\underset{\sim}{a}_i\cdot\underset{\sim}{u}_j}{\underset{\sim}{u}_j\cdot\underset{\sim}{u}_j}\underset{\sim}{u}_j.$$

Firstly, $\underset{\sim}{u}_1 = \underset{\sim}{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Now since $\underset{\sim}{a}_2\cdot\underset{\sim}{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1+1+0 = 0$,

$$\underset{\sim}{u}_2 = \underset{\sim}{a}_2 - \frac{\underset{\sim}{a}_2\cdot\underset{\sim}{u}_1}{\underset{\sim}{u}_1\cdot\underset{\sim}{u}_1}\underset{\sim}{u}_1 = \underset{\sim}{a}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, since $\underset{\sim}{u}_1\cdot\underset{\sim}{u}_1 = 1^2+1^2+1^2 = 3$ and $\underset{\sim}{u}_2\cdot\underset{\sim}{u}_2 = (-1)^2+1^2+0^2 = 2$, as well as

$$\underset{\sim}{a}_3\cdot\underset{\sim}{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 4, \quad\text{and}\quad \underset{\sim}{a}_3\cdot\underset{\sim}{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1,$$

we obtain that

$$\begin{aligned} \underset{\sim}{u}_3 &= \underset{\sim}{a}_3 - \frac{\underset{\sim}{a}_3\cdot\underset{\sim}{u}_1}{\underset{\sim}{u}_1\cdot\underset{\sim}{u}_1}\underset{\sim}{u}_1 - \frac{\underset{\sim}{a}_3\cdot\underset{\sim}{u}_2}{\underset{\sim}{u}_2\cdot\underset{\sim}{u}_2}\underset{\sim}{u}_2 \\ &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac43\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac12\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac16\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \end{aligned}$$

4. Show that for any real number θ, the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Solution.

A matrix A is orthogonal if and only if it is square and satisfies $A^TA = I$. The given matrix A is certainly square (it is 2×2), and

$$\begin{aligned} A^TA &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} (\cos\theta)^2 + (\sin\theta)^2 & \cos\theta(-\sin\theta) + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & (-\sin\theta)^2 + (\cos\theta)^2 \end{pmatrix} = I, \end{aligned}$$

as required.

5. Show that if Q is symmetric and orthogonal, then $Q^2 = I$.

Solution.

Since Q is orthogonal, $Q^TQ = I$. Since Q is symmetric, $Q^T = Q$. Substituting the value for Q^T from the second equation into the first gives $Q^2 = I$.

6. (*i*) Let A be an $m \times n$ matrix and B a $p \times q$ matrix. What condition on the numbers n, m, p and q is necessary and sufficient for the product AB to exist? When this condition holds, what is the shape of AB?

(*ii*) Let $\underset{\sim}{u}$ be a (column) vector in $\mathbb{R}^n$. Using Part (*i*), show that $\underset{\sim}{u}^T\underset{\sim}{u}$ and $\underset{\sim}{u}\underset{\sim}{u}^T$ both exist, and determine their shapes. Show, furthermore, that $(\underset{\sim}{u}\underset{\sim}{u}^T)^2$ is a scalar multiple of $\underset{\sim}{u}\underset{\sim}{u}^T$, and show that the scalar involved equals $\|\underset{\sim}{u}\|^2$.

(*iii*) Let $\underset{\sim}{u}$ be as in Part (*ii*), and suppose in addition that $\underset{\sim}{u}$ has length 1. Show that the matrix $I - 2\underset{\sim}{u}\underset{\sim}{u}^T$ is both symmetric and orthogonal. (Here I is the $n \times n$ identity matrix).

Solution.

(*i*) AB exists if and only if $n = p$, and then AB is an $m \times q$ matrix. (The product of an $m \times n$ matrix by an $n \times q$ matrix gives an $m \times q$ matrix.)

(*ii*) $\underset{\sim}{u}$ is $n \times 1$ and $\underset{\sim}{u}^T$ is $1 \times n$. So, by Part (*i*), $\underset{\sim}{u}\underset{\sim}{u}^T$ is $n \times n$ and $\underset{\sim}{u}^T\underset{\sim}{u}$ is 1×1. Note that a 1×1 matrix is just a scalar. Thus $\underset{\sim}{u}^T\underset{\sim}{u} = k$, some real number. In fact, $\underset{\sim}{u}^T\underset{\sim}{u} = \underset{\sim}{u} \cdot \underset{\sim}{u} = \|\underset{\sim}{u}\|^2$; so $k = \|\underset{\sim}{u}\|^2$. Now

$$(\underset{\sim}{u}\underset{\sim}{u}^T)^2 = \underset{\sim}{u}\underset{\sim}{u}^T\underset{\sim}{u}\underset{\sim}{u}^T = \underset{\sim}{u}(\underset{\sim}{u}^T\underset{\sim}{u})\underset{\sim}{u}^T = \underset{\sim}{u}\,k\,\underset{\sim}{u}^T = k(\underset{\sim}{u}\underset{\sim}{u}^T),$$

since the multiplication of scalars by vectors is commutative. So $(\underset{\sim}{u}\underset{\sim}{u}^T)^2$ is a scalar multiple of $\underset{\sim}{u}\underset{\sim}{u}^T$, and the scalar is $k = \|\underset{\sim}{u}\|^2$, as required.

(*iii*) Let $M = I - 2\underset{\sim}{u}\underset{\sim}{u}^T$. Then

$$M^T = (I - 2\underset{\sim}{u}\underset{\sim}{u}^T)^T = I^T - 2(\underset{\sim}{u}\underset{\sim}{u}^T)^T = I - 2(\underset{\sim}{u}^T)^T\underset{\sim}{u}^T = I - 2\underset{\sim}{u}\underset{\sim}{u}^T,$$

since I is symmetric and transposing reverses multiplication. So M^T equals M; that is, M is symmetric. To check that M is orthogonal we must show that $M^TM = I$, but since $M^T = M$ this is just $M^2 = I$.

Observe that $M = I - 2N$, where $N = \underset{\sim}{u}\underset{\sim}{u}^T$, the matrix we considered in Part (*ii*). There we showed that $N^2 = kN$, where $k = \|\underset{\sim}{u}\|^2$. Since we are assuming now that $\underset{\sim}{u}$ has length 1, we have $k = 1$, and $N^2 = N$. So

$$M^2 = (I - 2N)^2 = I - 4N + 4N^2 = I - 4N + 4N = I,$$

as required.