Proposition. *If $\phi\colon FG \to Mat_d(F)$ is a representation of a group algebra FG then the restriction of ϕ to the basis G of FG gives a matrix representation of the group G. Conversely, if $\psi\colon G \to \mathrm{GL}(d, F)$ is a matrix representation of G then we can obtain a matrix representation of FG by extending the domain of definition of ψ to the whole of FG by the formula*

$$\psi(\sum_{g\in G} \lambda_g g) = \sum_{g\in G} \lambda_g(\psi g).$$

Thus, representations of G are essentially the same as representations of FG.

Proof. Let $\phi\colon FG \to \mathrm{Mat}_d(F)$ be a representation of FG, and let ψ be the restriction of ϕ to G. (In other words, for each $g \in G$ we define $\psi g = \phi g$. This makes sense since G is a subset of FG.) Since $(\phi\alpha)(\phi\beta) = \phi(\alpha\beta)$ holds for all α, $\beta \in FG$, we certainly have $(\psi g)(\psi h) = \psi(gh)$ for all g, $h \in G$. So to prove that ψ is a representation of G it remains to show that ψg is invertible for all g (so that ψ can be interpreted as a map from G to $\mathrm{GL}(d, F)$ instead of a map from G to $\mathrm{Mat}_d(F)$). But $(\psi g)(\psi g^{-1}) = \psi(gg^{-1}) = \psi 1_G = I$ (since part of the definition of a representation of an algebra is that the identity element must be mapped to the identity matrix). So ψg has an inverse (namely, $\psi(g^{-1})$), as required.

Conversely, let $\psi\colon G \to \mathrm{GL}(d, F)$ be a representation of G, and define $\phi\colon FG \to \mathrm{Mat}_d(F)$ by

$$\phi(\sum_{g\in G} \lambda_g g) = \sum_{g\in G} \lambda_g(\psi g).$$

This yields a well defined function on FG since each element of FG is uniquely expressible in the form $\sum_{g\in G} \lambda_g g$ (where the coefficients λ_g are elements of F). Then for all choices of scalars λ_g, μ_g we have

$$\begin{aligned} \phi\Big(\sum_{g\in G} \lambda_g g\Big)\phi\Big(\sum_{g\in G} \mu_g g\Big) &= \Big(\sum_g \lambda_g(\psi g)\Big)\Big(\sum_h \mu_h(\psi h)\Big) = \sum_g\sum_h \lambda_g\mu_h\psi(gh) \\ = \sum_{k\in G}\Big(\sum_{\{g,h\mid gh=k\}} \lambda_g\mu_h\Big)\psi k &= \phi\Big(\sum_{k\in G}\Big(\sum_{\{g,h\mid gh=k\}} \lambda_g\mu_h\Big)k\Big) = \phi\Big(\Big(\sum_g \lambda_g g\Big)\Big(\sum_h \mu_h h\Big)\Big), \end{aligned}$$

and hence ϕ preserves multiplication. It remains to prove that ψ is linear and that $\psi 1_G = I$. This is left to the reader. □

Let $R^{(1)}$, $R^{(2)}$, ..., $R^{(s)}$ be a full set of irreducible unitary representations of G. Let d_i be the degree of $R^{(i)}$. In accordance with the proposition above, we can make each $R^{(i)}$ into a representation of $\mathbb{C}G$ so that $R^{(i)}(\sum_g \lambda_g g) = \sum_g \lambda_g(R^{(i)}g)$. Now define A to be the $\mathbb{C}$-algebra which is the direct sum of the full matrix algebras $\mathrm{Mat}_{d_1}(\mathbb{C})$, $\mathrm{Mat}_{d_2}(\mathbb{C})$, ..., $\mathrm{Mat}_{d_s}(\mathbb{C})$. Thus each element of A is an ordered s-tuple of matrices $(M_1, M_2, \ldots, M_s)$, where M_i is a $d_i \times d_i$ matrix, and the operations of addition, multiplication and scalar multiplication for A are all defined componentwise. Define a function $\phi\colon \mathbb{C}G \to A$ by

$$\phi\alpha = (R^{(1)}\alpha, R^{(2)}\alpha, \ldots, R^{(s)}\alpha)$$

for all $\alpha \in FG$. We shall prove that the function ϕ is an isomorphism of $\mathbb{C}$-algebras. Thus we obtain the following theorem.

Wedderburn's Theorem. *The complex group algebra of a finite group G is isomorphic to a direct sum of full matrix algebras.*

Wedderburn's Theorem can reasonably be called the main theorem in the study of complex representations of finite groups. It is, as we shall see, easy to deduce from the orthogonality relations; so one could also argue that the main theorem is really the orthogonality of coordinate functions. But Wedderburn's Theorem gives the results a structural flavour which is in the spirit of modern algebra.

We need only to prove that the map ϕ defined above is an isomorphism. It is obviously preserves addition, multiplication and scalar multiplication, since each $R^{(i)}$ preserves all of these. In other words, ϕ is an algebra homomorphism. Now the vector space dimension of $\mathrm{Mat}_d(\mathbb{C})$ is d^2; so the dimension of A is $\sum_i d_i^2$, which, as we know, equals $|G|$. This is also the dimension of FG. So if ϕ is surjective it will also have to be injective.

Choose any $k \in \{1, 2, \ldots, s\}$ and $i, j \in \{1, 2, \ldots, d_k\}$, and let $\alpha = \frac{1}{|G|} \sum_{g \in G} \overline{(R_{ij}^{(k)} g)} g$ (where $R_{ij}^{(k)}$ is the (i, j) coordinate function of $R^{(k)}$. The (p, q)-entry of $R^{(l)}\alpha$ is

$$\frac{1}{|G|} \sum_{g \in G} \overline{(R_{ij}^{(k)} g)} (R_{pq}^{(l)} g),$$

which (by the orthogonality of coordinate functions) is zero unless $l = k$ and $(p, q) = (i, j)$, in which case it is nonzero. So

$$\phi\alpha = (0, 0, \ldots, \Delta_{ij}, \ldots, 0)$$

where the only nonzero component of the right hand side is the kth component, and this component (denoted here as Δ_{ij}) has nonzero (i, j)-entry and is zero elsewhere. The elements of A that we obtain in this way as we vary k, i and j clearly form a basis for A, and since all these elements are in the image of ϕ it follows that the image of ϕ is the whole of A, as required.