Part II

§1 Recap and direction

- Again k alg. closed of char. $p>0$, G connected, simply connected, semisimple$/k$, $T\subseteq B\subseteq G$.

  Associated root system $(\Phi\subseteq X, \Phi^\vee\subseteq X^\vee)$, Weyl group W.
  Φ^+ positive roots.

- Three types of multiplicities:

  (1) G_1: Vermas $Z_w = Z_{w\cdot 0} = U(\mathfrak{g})\otimes_{U(\mathfrak{b})} k_{w\cdot 0}$
  $\twoheadrightarrow$ simples L_w $\rightsquigarrow$ $[Z_{w'} : L_w] = d_k(w',w)$.

  (2) G: Weyl modules $V(\mu) = H^0(-w_0\mu)^*$, $\mu\in X^+$,
  and simples $L(\lambda)$, $\lambda\in X^+$ $\rightsquigarrow$ $[V(\mu):L(\lambda)] = b_k(\mu,\lambda)$.
  and in part. $b_k(w',w)$.

  (3) G_1T: Vermas $\hat{Z}(\lambda)$ $\twoheadrightarrow$ $\hat{L}(\lambda)$ simples
  $\rightsquigarrow$ $d_k'(w',w) = [\hat{Z}(w'\cdot 0), \hat{L}(w\cdot 0)]$.

- <u>Theorem (AJS, 94)</u>: [Independence of p]: For all $w, w'\in W$ such that the relevant quantities make sense, there exist integers $d(w',w)$, $b(w',w)$, $d'(w',w)$ for which

$$d(w',w) = d_k(w',w),\quad b(w',w) = b_k(w',w),\quad d'(w',w) = d_k'(w',w)$$

whenever $p\gg 0$.

- <u>Strategy of proof</u>:

$$\boxed{\text{Thm A}} \xRightarrow[\text{facts on idempotents}]{\text{alcove geom.}} \text{Result in (3)} \Rightarrow \text{Result in (1)}, \quad \text{Result in (3)} \Rightarrow \text{Result in (2)}$$

Thm A: Structure result on endomorphism algebra of projectives obtained by wall-crossing functors in deformation category of G_1T-mod.

(obtained by uniform base change from $\mathbb{Z}$)

- Key construction: A functor $\mathcal{U}_\Omega$ from (part of) the deformation category into a "combinatorial" category $K(\Omega)$, where $\Omega = W_p \cdot O$.

- Def: For $\alpha \in \Delta^+$, $n \in \mathbb{Z}$, define

$$H^+_{\alpha,n} \text{ (resp. } H^-_{\alpha,n}) = \{v \in V^* : \langle \alpha, v \rangle > n \text{ (resp. } < n)\}.$$

Then $V^* = H^-_{\alpha,n} \sqcup H_{\alpha,n} \sqcup H^+_{\alpha,n}$. Let P be the coarsest partition refining these.

$$\mathcal{F} = \{\text{components of } P\} = \{\text{facets}\}$$
$$\supseteq \mathcal{A} = \{\text{open facets}\} = \{\text{alcoves}\},$$
$$\mathcal{W} = \{\text{facets} \subseteq \text{some } H_{\alpha,n}\} = \{\text{walls}\}.$$

- Have $W_{aff} \curvearrowright \mathcal{F}$ stabilising $\mathcal{A}, \mathcal{W}$. Fix fund. alcove A_0 (in dom. chamber) and let $A_{0,s} \in \mathcal{W}$ be its wall fixed by $s \in S_{aff}$. Then:

$$\begin{tikzcd} W_{aff} \arrow[r, "\cong"', "\text{orbit}"] \arrow[d] & \mathcal{A} \arrow[d] \\ W_{aff}/\langle s \rangle \arrow[r, "\cong", "\text{orbit}"'] & \mathcal{A}^s \end{tikzcd} = \text{orbit of } A_{0,s}$$

$$\begin{tikzcd} A \arrow[d] \\ \bar{A}, \end{tikzcd} \qquad \begin{tikzcd} B^- & & B^+ \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

sections

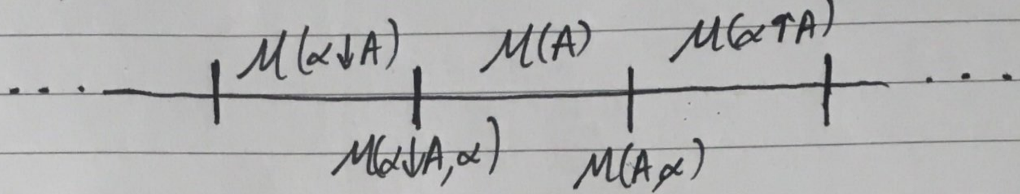
- Rings: Let $S = S(X \otimes_{\mathbb{Z}} k)$ and

$$S^\beta = S[\alpha^{-1} : \beta \neq \alpha \in \Delta^+] \subseteq S^\emptyset = S[\alpha^{-1} : \alpha \in \Delta^+].$$

M an S-module $\rightsquigarrow$ $M^\beta = S^\beta \otimes_S M$, $M^\emptyset = S^\emptyset \otimes_S M$.

- Soergel (95): Suppose $\mathcal{O} \subseteq \mathcal{F}$ is a W_{aff}-orbit. Then $K_k(\mathcal{O}) = K(\mathcal{O})$ is the category of objects $\mathcal{M}$ consisting of the following data:
  (1) An $S^\emptyset$-module $\mathcal{M}(F)$ for each facet $F \in \mathcal{F}$
  (2) An S^β-submodule $\mathcal{M}(F, \beta)$ of $\mathcal{M}(F) \oplus \mathcal{M}(\beta \uparrow F)$ (resp. of $\mathcal{M}(F)$) if $\beta \uparrow F \neq F$ (resp. if $\beta \uparrow F = F$).

- <u>Example:</u> For $G = SL_2$, have $\mathcal{A} = \{\pm\alpha\}$; can represent elements of $K(\mathcal{A})$ by pictures



- <u>Combinatorial flavour:</u> We have translation functors

$$K = K(\mathcal{A}) \underset{T^s_{out}}{\overset{T^s_{on}}{\rightleftarrows}} K^s = K(\mathcal{A}^s), \quad s \in S_{aff},$$

given by

$$T^s_{on} M(B) = M(B_-) \oplus M(B_+), \quad T^s_{on} M(B, \alpha) = \begin{cases} M(B_-, \alpha) & \text{if } \alpha \uparrow B = B \\ M(B_-, \alpha) \oplus M(B_+, \alpha), & \text{else,} \end{cases}$$

and

$$T^s_{out} N(A) = N(\bar{A}), \quad T^s_{out} N(A, \alpha) = \begin{cases} \{(\alpha x + y, y) : x, y \in N(\bar{A}, \alpha)\} & \text{if } \alpha \uparrow \bar{A} = \bar{A},\ A = \bar{A}_-, \\ \alpha N(\bar{A}, \alpha) \oplus N(\alpha \uparrow \bar{A}, \alpha) & \text{if } \alpha \uparrow \bar{A} = \bar{A},\ A = \bar{A}_+, \\ N(\bar{A}, \alpha), & \text{else.} \end{cases}$$

Let $\Theta_s = T^s_{out} T^s_{on}$ = wall crossing.

- <u>Goal today:</u> See an action of Soergel bimodules on a subcat. of K which:
  - categorifies a known submodule $P^\circ \subseteq P$ = periodic Hecke
  - is equivalent to $\mathrm{Proj}(\mathrm{Rep}_0(G, T))$.

- <u>Riche – Williamson (18):</u> Conjectured existence of action on $\mathrm{Rep}(G)$, categorifying AS quotient in tilting subcat:
  $\rightsquigarrow$ character formula for tilting modules, in terms of p-Kazhdan – Lusztig polynomials.

Conjecture on action known for $G = GL_n$ (RW18) and on characters for $p > h$ (AMRW19).

§2 Soergel Bimodules

- Geometric representations: (W,S) Coxeter system
  $\rightsquigarrow V = \mathbb{R}\langle \alpha_s : s \in S \rangle$ with

$$(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right), \qquad m_{st} = |st|,$$

and $W \curvearrowright V$ (faithful) by $s(\lambda) = \lambda - 2(\lambda, \alpha_s)\alpha_s$, $s \in S$.

- Bott-Samelson bimodules: $R = \mathrm{Sym}(V)$, graded w/ V in deg. 2, $W \curvearrowright R$.
  $\longrightarrow$ invariant subrings $R^I \subseteq R$ for $I \subseteq W$.

  Take $B_s = R \otimes_{R^s} R(1)$, graded R-bimodule.
  Then for any expression $\underline{w} = (s_1, \dots, s_n)$, $s_i \in S$:

$$BS(\underline{w}) := B_{s_1} \cdots B_{s_n} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R(\ell(\underline{w}))$$

- Soergel bimodules: A summand of a finite direct sum of $BS(\underline{w})$'s, possibly with grading shifts, is a Soergel bimodule.
  $\rightsquigarrow$ SBimod $\subseteq$ Rgbim $= \{$graded R-bimodules f.g. on left and right$\}$

- Examples: (1) $W = S_2 = \langle s \rangle \rightsquigarrow$ indecomp. SBimods $\{R, B_s\}$

  Have
$$\begin{aligned} B_s B_s &\cong R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} R(2) \oplus R \otimes_{R^s} R = B_s(1) \oplus B_s(-1) \end{aligned}$$

(2) $W = S_3 = \langle s,t \rangle \rightsquigarrow$ indecomp. SBimods
$\{R, B_s, B_t, B_{st} = B_s B_t, B_{ts} = B_t B_s, B_{sts} \mid B_s B_t B_s\}$.

- <u>Soergel's categorification theorem</u> (SCT): Let $H \doteq H(W,S)$ be the $\mathbb{Z}[v^{\pm 1}]$-algebra gen. by δ_s, $s \in S$, under

$$\delta_s^2 = (v^{-1}-v)\delta_s + 1, \qquad \underbrace{\delta_s\delta_t\delta_s \cdots}_{m_{st}} = \underbrace{\delta_t\delta_s\delta_t \cdots}_{m_{st}},$$

<u>Then</u>: (1) There is a $\mathbb{Z}[v^{\pm 1}]$-algebra isomorphism,

$$H \underset{ch}{\overset{c}{\rightleftarrows}} [SBim]_\oplus, \quad b_s \mapsto [B_s],$$

where $\{b_x\}_{x \in W}$ is the <u>KL basis</u> and ch is explicit.

(2) There is a bijection,

$$W \xrightarrow{\cong} \{\text{indecomp. obj. in } SBim\}/\text{Shift}, \cong$$

$$w \longleftrightarrow B_w, \quad \text{with } B_w \mid BS(\underline{w}),$$

$\underline{w}$ reduced.

(3) The graded hom $\mathrm{Hom}^\bullet_{SBim}(B, B')$ is free on left and right of graded rank $(ch(B), ch(B'))$.

standard form on H

- <u>Generalising</u>: Can replace our $V = V_{geom}$ by other reps V of W, over other base fields k, char $k \neq 2$, $|k| = \infty$.

SCT holds if V is <u>reflection faithful</u>:

- faithful,
- for all $x \in W$, codim $\mathrm{Fix}(x) = 1 \Leftrightarrow x$ a reflection.

This often fails (e.g. char $k > 0$, e.g. affine Weyl group), which motivates the diagrammatic Hecke category $\mathcal{D}$:

- coincides with SBim (up to equiv.) for char. 0
- replaces them with better behaviour (e.g. SCT) otherwise

[<u>Example</u>: $\nexists$ faithful f.d. representation of an infinite W over $\mathbb{F}$ or even $\overline{\mathbb{F}}$ if $|\mathbb{F}| < \infty$.]

§ Another realisation

- Assume V is a f.d. $\mathbb{R}$-rep. of W such that, for certain $\alpha_s \in V$, $\alpha_s^\vee \in V^*$, $s \in S$, it holds that

  (1) $\langle \alpha_s^\vee, \alpha_s \rangle = 2$
  (2) $s(v) = v - \langle \alpha_s^\vee, v \rangle \alpha_s$
  (3) $\alpha_s \neq 0 \neq \alpha_s^\vee$
  (4) For all s,t with $m_{st} < \infty$, $\langle s,t \rangle \curvearrowright V$ is reflection faithful.

- <u>Notation:</u> $R = S(V)$, $Q = \mathrm{Frac}(R)$.
  $\deg(V) = 2$

  $W \curvearrowright V \rightsquigarrow W \curvearrowright R.$

- <u>First approx:</u> The category $\mathcal{C}$ consists of graded R-bimodules M which are f.g. and flat on the left, such that there is a (Q,R)-bimodule decomp.

$$Q \otimes_R M = \bigoplus_{w \in W} M_w^Q = M^Q$$

s.t. almost all $M_w^Q = 0$ and $mf = w(f)m$, $m \in M_w^Q, f \in R$.

- <u>Facts:</u> If $M \in \mathcal{C}$, then M is torsionfree on the right$/R$ and $M \hookrightarrow M^Q$ by flatness. Further, M is f.g. on the left and right$/R$.

~~[illegible]~~ • <u>Notation:</u> (1) Write $\mathrm{Supp}_W(M) = \{w \in W : M_w^Q \neq 0\}$
(2) If $I \subseteq W$, then have

$$M_I = f^{-1}\left(\bigoplus_{w \in I} M_w^Q\right) \subseteq M^I = f(M) \cap \bigoplus_{w \in I} M_w^Q.$$

~~[illegible]~~

- <u>Tensor products:</u> $M, N \in \mathcal{C} \rightsquigarrow M \otimes N = M \otimes_R N$ as R-bimodules,

$$(M \otimes N)^Q_w = \bigoplus_{xy=w} M^Q_x \otimes_Q N^Q_y.$$

$$\Rightarrow \operatorname{supp}(M \otimes N) = \operatorname{supp}(M)\operatorname{supp}(N)$$

- <u>Key objects:</u> (1) Let $w \in W$ and define

$R_w = R$ as left R-module, with right action $mf = w(f)m$, $m \in R_w$, $f \in R$.
Then $\operatorname{supp}(R_w) = \{w\}$ with

$$(R_w)^Q_x = \begin{cases} Q, & x = w, \\ 0, & x \neq w. \end{cases}$$

(2) For $s \in S$ consider $B_s = R \otimes_{R^s} R(1)$ as before.
Then $\operatorname{supp}(B_s) = \{1, s\}$, with

$$(B_s)^Q_1 = Q(\delta_s \otimes 1 - 1 \otimes s(\delta_s)),$$
$$(B_s)^Q_s = Q(\delta_s \otimes 1 - 1 \otimes \delta_s).$$

$\boxed{\delta_s \in V \text{ s.t. } \langle \alpha_s^\vee, \delta_s \rangle = 1}$

Interesting: $\operatorname{Hom}_{\mathcal{C}}(B_s \otimes -, -) \cong \operatorname{Hom}_{\mathcal{C}}(-, B_s \otimes -)$.

- <u>Soergel bimodules:</u> $\mathcal{BS} \underset{\text{full}}{\subseteq} \mathcal{C}$ consists of objects of the form

$$B_{s_1} \otimes \cdots \otimes B_{s_n}(m)$$

and SBimod consists of direct summands of direct sums of such.

- <u>Thm (Abe, 2019):</u> Assuming V satisfies an extra technical condition (e.g. in 3.3 of EW16), there is an equivalence of categories

$$\mathcal{D} \xleftrightarrow{\cong} \text{SBimod}$$

arising from $\mathcal{D}_{BS} \xleftrightarrow{\cong} \mathcal{BS}$.

Return to G w/ $(\Delta \subseteq X, \Delta^\vee \subset X^\vee)$ and (W_{aff}, S_{aff}).

§4 An Action on G,T-modules

image in W_f

- Choosing the rep: Consider $\Lambda = \{\lambda : \mathcal{A} \to X \mid \lambda(xA) = \bar{x}\lambda(A)$ for all $x \in W_{aff}\}$

For any $A \in \mathcal{A}$, have:

$$\Lambda \xrightarrow{\cong} X, \quad \lambda \mapsto \lambda_A = \lambda(A),$$

inverse $\alpha \mapsto \alpha^A$.

If $s \in S_{aff}$, $A \in \mathcal{A}$, write As for alcove $\neq A$ sharing s-wall.

Then $W_{aff} \curvearrowright \Lambda$, $w(\lambda)(A) = \lambda(Aw)$.

- Notation: For $A \in \mathcal{A}$, $s \in S_{aff}$, take $\alpha \in \Delta^+$ such that $s_{\alpha,n} A = As$ (some $n \in \mathbb{Z}$). Put $\alpha_s = \alpha^A \in \Lambda_k$

$\alpha_s^\vee = (\alpha^\vee)^A \in \Lambda_k^*$.

- Lemma [Abe]: (1) $(\alpha_s, \alpha_s^\vee)$ does not depend on A, up to sign.

(2) $(\Lambda_k, \{\alpha_s\}, \{\alpha_s^\vee\}) = V$ as a rep. for (W_{aff}, S_{aff}) satisfies the conditions from §3. $\rightsquigarrow$ cat. SBimod. over $R = S(\Lambda_k)$.

- Goals: (1) Find a realisation of the essential image of $\mathcal{V}: \mathrm{Proj}(\mathrm{Rep}_0(G,T)) \longrightarrow K(\mathcal{A}) = K$ (fully faithful)

(2) Find SBimod $\curvearrowright$ realisation s.t. $B_s \curvearrowright$ as Θ_s.

$\Rightarrow$ SBimod $\curvearrowright \mathrm{Proj}(\mathrm{Rep}_0(G,T))$ through WC functors $\xRightarrow[\text{categories argument}]{\text{derived}}$ SBimod $\curvearrowright$ $\mathrm{Rep}_0(G,T)$.

- Def: $\widetilde{K}'$ is the category defined as follows:
  - objects: M is a graded (S,R)-bimodule, f.g. and torsion free as a left S-module, together with a decomp.

$$M^\emptyset = \bigoplus_{A \in \mathcal{A}} M_A^\emptyset, \quad mf = f_A m, \; m \in M_A^\emptyset, \; f \in R.$$

  - morphisms: degree-zero (S,R)-bimodule hom. such that $f: M \to N$ $\quad f(M_A^\emptyset) \subseteq \bigoplus_{A' \geq A} N_{A'}^\emptyset$.

- <u>Action:</u> Let $B \in \mathrm{SBimod}$. Then have $B^\emptyset = \bigoplus_{x \in W_{aff}} B_x^\emptyset$ s.t. $Q \otimes_{R^\emptyset} B_x^\emptyset = B_x^Q$. Define

$$M * B = M \otimes_R B, \quad (M * B)_A^\emptyset = \bigoplus_{x \in W_{aff}} M^\emptyset_{Ax^{-1}} \otimes_{R^\emptyset} B_x^\emptyset$$

for $M \in \tilde{K}'$, $B \in \mathrm{SBimod}$. Thus $\mathrm{SBimod} \curvearrowright \tilde{K}'$ on the right.

- <u>Thm:</u> There is a (technically specified) full SBimod-submodule category $\tilde{K}_P \subseteq \tilde{K}'$ such that:

  (1) Every object of $\tilde{K}_P$ is isomorphic to a direct sum of shifts of indecomposables $Q(A)$, $A \in \mathcal{A}$.
  (2) The split Grothendieck group $[\tilde{K}_P]$ is isomorphic to a well known submodule $P^\circ \subseteq P$ = periodic Hecke.

- <u>Detour [Lus 80]:</u> (1) Periodic Hecke $P = \bigoplus_{A \in \mathcal{A}} \mathbb{Z}[v^{\pm 1}]A \curvearrowleft H$ by

$$A \underline{s} = \begin{cases} As, & \text{if } As > A \\ As + (v^{-1} - v)A, & \text{if } As < A. \end{cases}$$

  (2) If λ is an integral weight, let $e_\lambda = \sum_{\lambda \in \bar{A}} v^{-\ell(A)} A$, ℓ a length function $\mathcal{A} \to \mathbb{Z}$. Then P° is the H-submodule generated by all e_λ.

- <u>Functor into combinatorics:</u> Modify $\tilde{K}_P \rightsquigarrow K_P$: same objects but

$$\mathrm{Hom}_{K_P}(M,N) = \frac{\mathrm{Hom}_{\tilde{K}_P}(M,N)}{\{\varphi \mid \varphi(M_A^\emptyset) \subseteq \bigoplus_{A' > A} N_{A'}^\emptyset\}}.$$

Then K_P has all the same properties as $\tilde{K}_P$ but we can define $\mathcal{F} : K_P \to K(\mathcal{A})\text{-}K$ by

$$\mathcal{F}(M)(A) = M_A^\emptyset, \quad \mathcal{F}(M)(A_\alpha) = M^\alpha \cap (M_A^\emptyset \oplus M^\emptyset_{\alpha\uparrow A}).$$

- <u>Thm:</u> (1) $\mathcal{F}$ is fully faithful.
  (2) There are isomorphisms $\mathcal{F}(M * B_s) \cong \Theta_s \mathcal{F}(M)$ natural in M.

- <u>Rmk:</u> (1) is complicated and requires localisation techniques similar to last week.
  The first step in (2) is not too bad...

$$\begin{aligned}
\mathcal{F}(M * B_s)(A) &= (M * B_s)^{\emptyset}_{A} \\
&= M^{\emptyset}_{A} \otimes (B_s)^{\emptyset}_{1} \oplus M^{\emptyset}_{As} \otimes (B_s)^{\emptyset}_{s} \\
&\cong M^{\emptyset}_{A} \oplus M^{\emptyset}_{As} = \mathcal{F}(M)(A) \oplus \mathcal{F}(M)(As) \\
&= (\Theta_s \mathcal{F}(M))(A).
\end{aligned}$$

- <u>Final piece of the puzzle:</u> The method of AJS (94) shows that, if we let $\mathcal{L} = \mathcal{F}(K_p)$ and form $\mathcal{L}^f$ w/ same objects as $\mathcal{L}$ but

$$\mathrm{Hom}_{\mathcal{L}^f}(M,N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{L}}(M, N(i)),$$

then $k \otimes_S \mathcal{L}^f \cong \mathrm{Proj}(\mathrm{Rep}_0(G,T))$. So SBimod $\curvearrowright \mathcal{L}$ lifts to SBimod $\curvearrowright \mathrm{Proj}(\mathrm{Rep}_0(G,T))$ as desired!