Derived categories and functors

Plan

1. Recollections and motivation
2. Localisation
3. Derived functors

§1 Recollections and motivation

- A triangulated category $\mathcal{T}$ is an additive category equipped with
$$\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T},$$
along with a class of distinguished (exact) triangles,
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \qquad (\Delta)$$
satisfying four axioms.

- A triangulated functor $F : (\mathcal{T}, \Sigma) \longrightarrow (\mathcal{T}', \Sigma')$ satisfies
$$F\Sigma \cong \Sigma' F$$
and $F(\mathcal{T}\text{-exact } \Delta) = \mathcal{T}'\text{-exact } \Delta$.

- Prototypical example:

$\mathcal{A}$ additive category

e.g. $\mathcal{A} = Ab$
$\mathcal{A} = R\text{-mod}$
$\mathcal{A} = Sh(X)$

$\rightsquigarrow Ch(\mathcal{A})$ additive category of chain complexes in $\mathcal{A}$.

$\rightsquigarrow K(\mathcal{A})$ homotopy category of $\mathcal{A}$ with

$Ob(K(\mathcal{A})) = Ob(Ch(\mathcal{A}))$.

$\mathrm{Hom}_{K(\mathcal{A})}(X,Y) = \mathrm{Hom}_{Ch(\mathcal{A})}(X,Y) / \text{homotopy equivalence}$.

Then $K(\mathcal{A})$ is triangulated with exact triangles those isomorphic to ones of the form

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow \Sigma X.$$

Also $F: \mathcal{A} \to \mathcal{A}'$ additive

$\rightsquigarrow F: K(\mathcal{A}) \longrightarrow K(\mathcal{A}')$ triangulated.

- Goal: If $\mathcal{A}$ is abelian, "localise" $K(\mathcal{A})$ to get triangulated category $D(\mathcal{A})$ which is the "right place" to study left/right exact functors on $\mathcal{A}$:

  - $F: \mathcal{A} \to \mathcal{A}'$ left exact $\rightsquigarrow RF: D(\mathcal{A}) \to D(\mathcal{A}')$ with $H^i RF(A) = R^i F(A)$ for $A \in \mathcal{A}$
  - Complicated homological facts (e.g. expressed by spectral sequences) about F simplify via RF.

- $D(Sh(X))$ will also be the home of perverse sheaves on X.

§2 Localisation [Weibel, 10.3]

- $\mathcal{C}$ a category, $S \subseteq \mathrm{Mor}(\mathcal{C})$.

  A **localisation** of $\mathcal{C}$ at S is a category $S^{-1}\mathcal{C}$ with a functor

  $$q: \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$$

  such that: (1) $q(s) \in \mathrm{Isom}(S^{-1}\mathcal{C})$ for all $s \in S$ ($q(s)$ is an iso)

  (2) Any $F: \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(s) \in \mathrm{Isom}(\mathcal{D})$ for all $s \in S$ admits a factorisation,

$$\begin{tikzcd} C \arrow[r, "q"] \arrow[dr, "F"'] & S^{-1}\mathcal{C} \arrow[d, "\exists!"] \\ & D \end{tikzcd}$$

$S^{-1}\mathcal{C}$ unique up to equivalence

- If $Ob(\mathcal{C})$ or S is a set, then $S^{-1}\mathcal{C}$ exists. More generally, it is a delicate set-theoretic issue, which we will avoid.

- Example: Let R = comm. ring and consider the category $\mathcal{C}_R$:

$$\mathrm{Hom}(*,*) = R$$

$$\begin{tikzcd} * \arrow[loop above, "R"] \end{tikzcd}$$

Recall that if $S \subseteq R$ is multiplicatively closed, then we can form $S^{-1}R$.

Then $S^{-1}\mathcal{C}_R \cong \mathcal{C}_{S^{-1}R}$. (exercise)

- Example: R and S as above. Then S determines a collection of morphisms

$$S = \{ f \in Mor(R\text{-Mod}) : S^{-1}(f) \text{ an iso} \}.$$

Then $S^{-1}(R\text{-mod}) \cong (S^{-1}R)\text{-mod}$.

(exercise)

- If S has good properties, we can describe $S^{-1}\mathcal{C}$ explicitly.

- S is called a **multiplicative system** in case:

(1) $1_X \in S$ for all $X \in \mathcal{C}$ and S is closed under composition.

(2) Given a situation with $t \in S$ and g in $\mathcal{C}$

$$\begin{tikzcd} & Z \arrow[d, "t"] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \quad \overset{\exists f \text{ in } \mathcal{C}}{\underset{\exists s \in S}{\rightsquigarrow}} \quad \begin{tikzcd} W \arrow[r, "f"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[d, "t"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

Ore condition: "$t^{-1}g = fs^{-1}$"

moreover, the symmetric statement is true.

(3) Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$,

$(\exists t \in S \text{ with } ft = gt) \Leftrightarrow (\exists s \in S \text{ with } sf = sg)$.

- **Theorem [Gabriel–Zisman]:** Let S be a (locally small) multiplicative system in $\mathcal{C}$. Then $S^{-1}\mathcal{C}$ exists and admits an explicit description as follows:

(1) $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.

(2) $$\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) = \left\{ \begin{matrix} \text{left fractions} \\ X \xleftarrow{s} X' \xrightarrow{f} Y \end{matrix} \right\} \Big/ \sim$$

in S

where $(X \leftarrow X' \rightarrow Y) \sim (X \leftarrow X'' \rightarrow Y)$

iff there is a fraction $X \leftarrow Z \rightarrow Y$ such that the following commutes

$$\begin{tikzcd}
& X' \arrow[dl] \arrow[dr] & \\
X & Z \arrow[l] \arrow[u] \arrow[r] \arrow[d] & Y \\
& X'' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Commutative diagram.

(3) Composition relies on the Ore condition. (exercise)

- Key situation:

  $\mathcal{T}$ = triangulated category

  $\mathcal{A}$ = abelian category.

Then $H: \mathcal{T} \to \mathcal{A}$ is cohomological if it sends exact triangles (Δ) to LES

$$\cdots \to H(\Sigma^i X) \to H(\Sigma^i Y) \to H(\Sigma^i Z)$$

$$\to H(\Sigma^{i+1} X) \to \cdots$$

Notation: $H^i = H\Sigma^i$.

Examples: (1) $H = H^0 : K(\mathcal{A}) \to \mathcal{A}$

(2) $H = \mathrm{Hom}_{\mathcal{T}}(X, -) : \mathcal{T} \to Ab$ if $X \in \mathcal{T}$.

- Proposition: Let $H: \mathcal{T} \to \mathcal{A}$ cohomological. If S = collection of morphisms in $\mathcal{T}$ such that

  $H^i(s)$ is an iso. for all i,

  then (1) S is a multiplicative system

  (2) $S^{-1}\mathcal{T}$ is a triangulated category, with $\mathcal{T} \to S^{-1}\mathcal{T}$ a triangulated functor.

- Now the derived category $D(\mathcal{A})$ = localisation of $K(\mathcal{A})$ arising from $H = H^0: K(\mathcal{A}) \to \mathcal{A}$.
(we invert quasi-isomorphisms)

- Alternative construction: $D(\mathcal{A})$ is a "Verdier quotient" of $K(\mathcal{A})$ by the full subcat. generated by acyclic complexes X (i.e. $H^i(X) = 0$ for all i)

[Chambert-Loir]

- Subcategories:
  - Bounded below categories $Ch^+(\mathcal{A})$, $K^+(\mathcal{A})$, $D^+(\mathcal{A})$ with $H^i(A) = 0$ for $i \ll 0$ for all A.
  - Bounded above categories $Ch^-(\mathcal{A})$, $K^-(\mathcal{A})$, $D^-(\mathcal{A})$ with $H^i(A) = 0$ for $i \gg 0$ for all A.
  - Intersections $Ch^b(\mathcal{A})$, $K^b(\mathcal{A})$, $D^b(\mathcal{A})$ of the preceding, the bounded categories.

§3 Derived functors

- $\mathcal{A}, \mathcal{B}$ abelian categories, $F : \mathcal{A} \to \mathcal{B}$ additive.

  $\rightsquigarrow$ $F : K(\mathcal{A}) \to K(\mathcal{B})$. "uninteresting"

  If furthermore F is exact, then

  $\rightsquigarrow$ $F : D(\mathcal{A}) \to D(\mathcal{B})$.

- More interesting: what can we say if F is exact on only one side? To fix notation, assume F is left exact.

- Definition: A right derived functor of F is a functor

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

along with natural transformation η as follows

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "F"] \arrow[d, "q"'] \arrow[dr, phantom, "\swarrow\eta"] & K^+(\mathcal{B}) \arrow[d, "q"] \\ D^+(\mathcal{A}) \arrow[r, "RF"'] & D^+(\mathcal{B}) \end{tikzcd}$$

such that RF with η is universal in this setup.

- <u>Theorem</u>: If $\mathcal{A}$ has enough injectives, then RF exists.

Sketch proof: Let $\mathcal{I} \subseteq \mathcal{A}$ be the full subcategory generated by injective objects. Every $X \in K^+(\mathcal{A})$ admits a quasi-isomorphism

$$X \longrightarrow I \in K^+(\mathcal{I})$$

and also $S^{-1}K^+(\mathcal{I}) \subseteq D^+(\mathcal{A})$ is a fully faithful embedding. This proves that

$$S^{-1}K^+(\mathcal{I}) \cong D^+(\mathcal{A})$$

Since quasi-isomorphisms in $K^+(\mathcal{I})$ are already isomorphisms, $K^+(\mathcal{I}) \cong S^{-1}K^+(\mathcal{I})$, so

$$\Psi: K^+(\mathcal{I}) \cong D^+(\mathcal{A}).$$

Now we take $RF = D^+(\mathcal{A}) \xrightarrow{\Psi^{-1}} K^+(\mathcal{I}) \xrightarrow[F]{} K^+(\mathcal{B}) \xrightarrow[q]{} D^+(\mathcal{B})$

So $RF = qF\Psi^{-1}$. Exercise: construct η.

- Theorem: If $\mathcal{A}$ has enough injectives, then
$$H^i RF(X) \cong \mathbb{R}^i F(X).$$ hypercohomology

In particular, if $X \in \mathcal{A}$, then
$$H^i RF(X) \cong R^i F(X).$$

- Examples: A = commutative ring, $\mathcal{A} = A$-mod.

(1) If M is an A-module, then can form
$$F = \mathrm{Hom}_A(M, -)$$
left exact and
$$H^i RF(N) = R^i F(N) = \mathrm{Ext}^i_A(M,N).$$

(2) The functor
$$G = M \otimes_A - : \mathcal{A} \longrightarrow \mathcal{A}$$
right exact $\rightsquigarrow LG : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$
is written $LG(N) = M \otimes^L_A N$ and
$$H^{-i} LG(N) = \mathrm{Tor}_i^A(M,N).$$
(works because $\mathcal{A}$ has enough projectives.)

- Remark: We have $R\mathrm{Hom}(-,-)$ and $- \otimes^L -$ where one input is a module and the other is a complex. A theory of derived bifunctors permits both variables to be complexes.