The Mercator and stereographic projections, and many in between*

Daniel Daners

Revised Version, July 29, 2011

Abstract

We consider a family of conformal (angle preserving) projections of the sphere onto the plane. The family is referred to as the Lambert conic conformal projections. Special cases include the Mercator map and the stereographic projection. The techniques only involve elementary calculus and trigonometry.

1 Introduction

The starting point for this exposition is the Mercator map designed by the Flemish/German cartographer Gerardus Mercator in 1569. The map is probably the most commonly used map of the world; see Figure 1. It was originally designed for navigation, and is still used for that purpose. The map is also useful for plotting meteorological or oceanographic data. We explain why this is the case and discuss a whole family of related maps which includes the Mercator map and also the stereographic projection as limit cases.

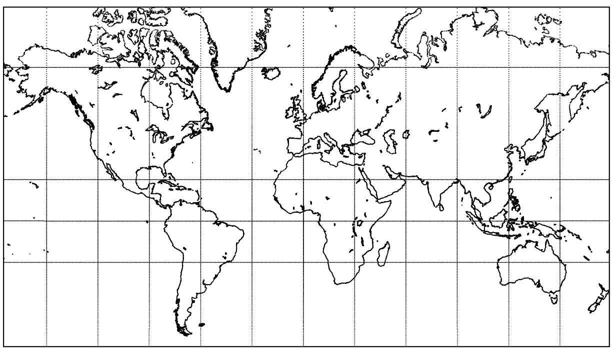

Figure 1: Mercator map.

On a rectangular map east–west is usually the horizontal and north–south the vertical direction. Other lines of constant compass bearing (often

*To appear in American Mathematical Monthly called loxodromes) do not necessarily correspond to straight lines. The Mercator map is designed such that *all* lines of equal compass bearing α from due north on the sphere become straight lines of angle α from the vertical on the map. Hence, it is very easy to plot or read off directions of travel, ocean currents, wind, barometric pressure gradients, and other data. The Mercator map is therefore a special angle preserving or *conformal map*. We give a construction in Section 2.

A second, seemingly unrelated projection is the *stereographic projection*, not usually to map the earth, but to map the sky. It has been used on astrolabes to measure and display astronomical observations more than 2000 years ago. Many astrolabe clocks such as the famous one from 1410 on the clock tower of the old Town Hall in Prague display the movement of the planets, the sun, and the zodiac. Figure 2 shows the first astrolabe watch. It was built by the author's father Richard Daners [8] in 1981 for Gübelin AG Lucerne, Switzerland.

The stereographic projection is a conformal map as well. In complex analysis it is used to represent the extended complex plane (see for instance [2, Chapter I]). The stereographic projection has the property that all circles on the sphere are mapped onto circles or straight lines on the plane, and therefore it is easy to map astronomical observations. We include a construction in Section 3.

Photo courtesy Gübelin AG, Switzerland.

Figure 2: Astrolabe watch.

Up to the late 18th century the Mercator and stereographic projections were treated as completely unrelated. It was Johann Heinrich Lambert (1728–77) who, in his seminal work [5] from 1772, changed the way map projections were approached. He started with desired properties of the map like conformality and the shape of the projection surface, and then constructed whole families of projections. One of the projection surfaces he considered was a cone. The corresponding maps are now known as *Lambert conic conformal projections*. We give a construction of these projections in Section 4. Lambert then observed that the Mercator and stereographic projections are limit cases of these conic projections [5, §49, 50]. The idea is that the cylinder and plane are limit cases of cones as shown in Figure 3. This is well known amongst experts in cartography (see [1, 7]). The purpose

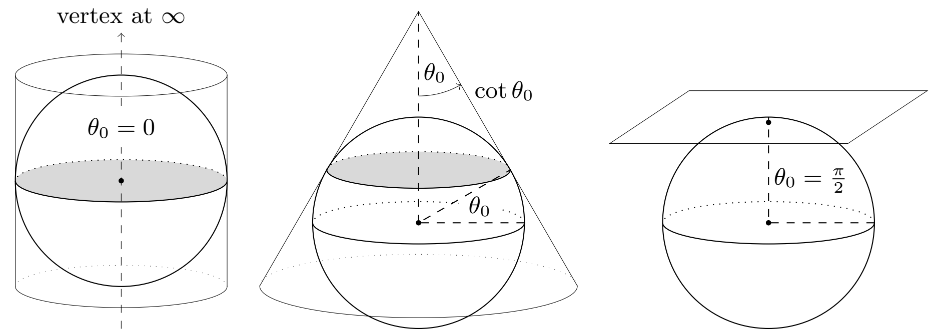

Figure 3: Cone and limit cases corresponding to Mercator and stereographic projection, respectively.

of this article is to make this nice part of cartography accessible to anyone knowing only elementary calculus. We consider these limits in Section 5. In Section 6 we construct conic projections where the lengths of two parallels are preserved. Finally, we provide more on the history in Section 7.

There are other interesting relationships between Mercator and stereographic projections. It is shown in [9] that the complex exponential function acts as a bijection between the two. This has counterparts in hyperbolic geometry; see [10]. There are many other map projections we do not discuss here. In particular we do not discuss area preserving maps, the gnomonic map, which maps all great circles onto straight lines, and many others. We refer to [3] or the more specialized book [1] on cartography for a wealth of information.

2 The Mercator projection

As outlined in the introduction, Mercator's idea was to map the sphere onto the plane such that the following properties hold:

(i) the north–south direction is the vertical direction;

(ii) the east–west direction is the horizontal direction with the length of the equator preserved;

(iii) all paths of equal compass bearing on the sphere are straight lines.

For simplicity we assume the sphere has radius one. The first two conditions imply that the image of the sphere lies in a strip of width 2π. Moreover, the meridians are mapped onto vertical lines and the parallels onto horizontal lines. Hence we only need to determine the spacing of the parallels. We parametrize the unit sphere by spherical coordinates

$$x = \cos\varphi\cos\theta, \quad y = \sin\varphi\cos\theta, \quad z = \sin\theta,$$

where $\varphi \in [-\pi, \pi]$ is longitude and $\theta \in [-\pi/2, \pi/2]$ is latitude. For mathematical purposes it is more convenient to measure latitude and longitude in radians rather than degrees. On the plane we introduce a rectangular coordinate system with $u = u(\varphi, \theta)$ the horizontal direction and $v = v(\varphi, \theta)$ the vertical direction.

We now consider a line of constant compass bearing on the sphere. Assume that the bearing from due north is α. By (ii) we have $u = \varphi$. Consider a small rectangle at (φ, θ) with $\Delta\varphi$ and $\Delta\theta$ determined by α as shown in Figure 4. Because the parallel at latitude θ has radius $\cos\theta$, the edge along the

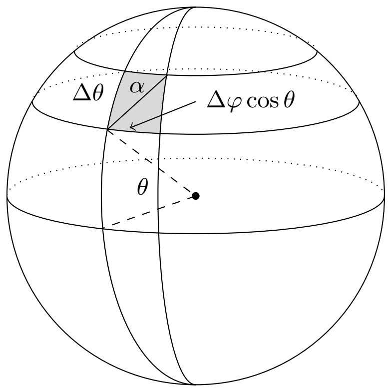

Figure 4: Small rectangle on the sphere.

parallel has approximate length $\Delta\varphi\cos\theta$. The edge parallel to the meridian has length $\Delta\theta$ (see Figure 4). Hence

$$\cot\alpha \approx \frac{\Delta\theta}{\Delta\varphi\cos\theta}.$$

The image of that path on the map is a straight line with angle α from the v-axis as shown in Figure 5. To satisfy (iii) we require that

$$\cot\alpha \approx \frac{\Delta v}{\Delta u} = \frac{\Delta v}{\Delta\varphi}.$$

Equating the two expressions for $\cot\alpha$ we get

$$\frac{\Delta\theta}{\cos\theta} = \Delta\theta\sec\theta \approx \Delta v.$$

If we let $\Delta\theta$ tend to zero we get

$$\frac{dv}{d\theta} = \sec\theta. \tag{2.1}$$

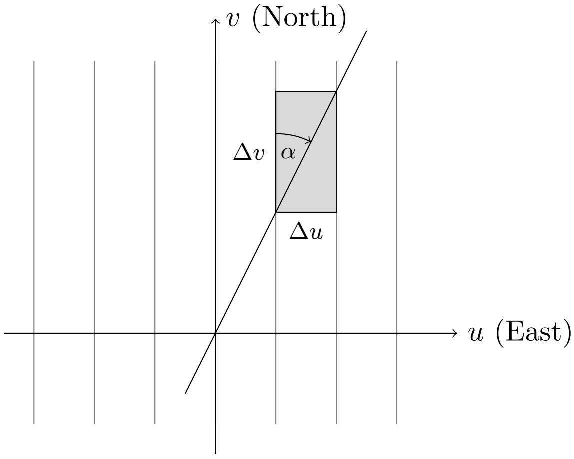

Figure 5: Grid for the Mercator projection.

Integrating we get

$$v(\theta) = \log(\tan\theta + \sec\theta) + C$$

for some constant C. Since we require that $v(0) = 0$ we get $C = 0$. This means that the map

$$\begin{cases} u(\varphi, \theta) = \varphi, \\ v(\varphi, \theta) = \log(\tan\theta + \sec\theta) = \log\Big(\tan\Big(\dfrac{\theta}{2} + \dfrac{\pi}{4}\Big)\Big) \end{cases} \tag{2.2}$$

has the properties (i)–(iii) required above. An alternative construction is to make sure that the north–south distortion of length is the same as the east–west distortion. See for instance [9] or the comments in [7] for that approach.

3 The stereographic projection

The stereographic projection maps the sphere from one of the poles onto a plane parallel to the equator. The most common choices are the plane containing the equator or the plane tangent to the sphere at the pole opposite to the pole from which we project. For our purpose it is best to project from the south pole S onto the plane tangent at the north pole N. If P is a point on the sphere, its projection is the intersection Q of the line through the south pole and the point P with that plane. A cross section is shown in Figure 6. To get the coordinates of Q we only need to compute its distance $r = \overline{NQ}$ from the north pole as a function of the latitude θ. The triangles $\triangle SPT$ and $\triangle SQN$ are similar and therefore, since $\overline{OT} = \sin\theta$, $r = \overline{NQ}$,

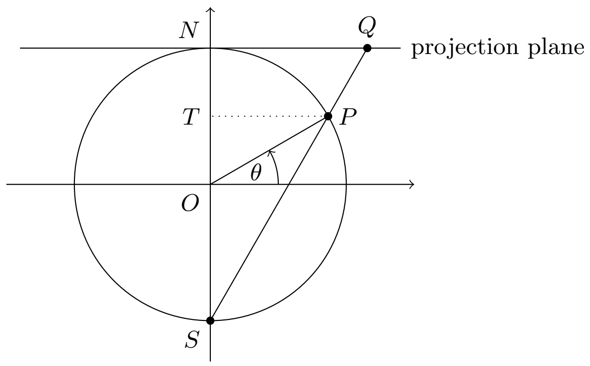

Figure 6: Stereographic projection of a point with $\theta \in (0, \pi/2)$.

$\overline{PT} = \cos\theta$, and the radius of the sphere is 1,

$$\frac{r}{2} = \frac{\cos\theta}{\sin\theta + 1} = \frac{1}{\tan\theta + \sec\theta}.$$

We want to write down the projection in cartesian coordinates, where u is the horizontal axis and v the vertical axis. The negative v-axis should represent the null meridian. We measure the longitude φ from it in the counterclockwise direction, so that the origin corresponds to the north pole. Hence, the stereographic projection is given by

$$\begin{cases} u(\varphi, \theta) = \dfrac{2}{\tan\theta + \sec\theta}\sin\varphi, \\ v(\varphi, \theta) = -\dfrac{2}{\tan\theta + \sec\theta}\cos\varphi. \end{cases} \tag{3.1}$$

It is not very common to use stereographic projections for maps of the world, but nevertheless there is one in Figure 7. As expected the distortion gets huge on the southern hemisphere.

4 A family of conical projections

We now map the sphere onto a cone such as the one in Figure 3. In order to construct a map the cone is cut open and flattened. We assume that the meridians correspond to uniformly spaced straight lines from the vertex of the cone and that the parallel of latitude θ corresponds to an arc of the circle of radius $\rho(\theta)$ centered at $(0, \rho_0)$, as shown in Figure 8. We make the design so that the parallel of latitude θ_0 passes through the origin, that is, $\rho(\theta_0) = \rho_0$. The opening angle $2\pi t$ of the cone is determined by the parameter $t \in (0, 1]$. Hence the equations are of the form

$$\begin{cases} u(\varphi, \theta) = \rho(\theta)\sin(t\varphi), \\ v(\varphi, \theta) = \rho_0 - \rho(\theta)\cos(t\varphi). \end{cases} \tag{4.1}$$

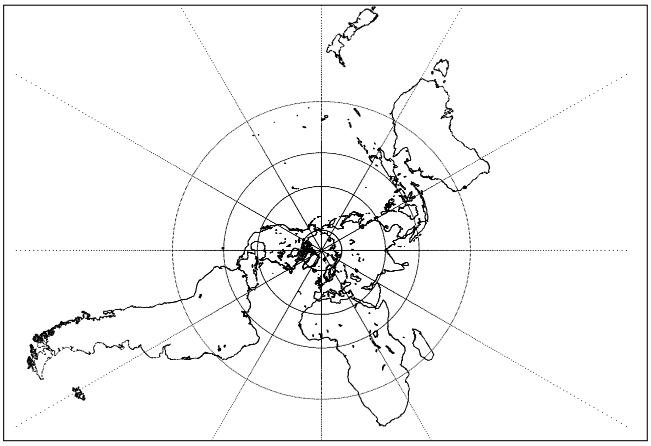

Figure 7: Stereographic projection of the earth.

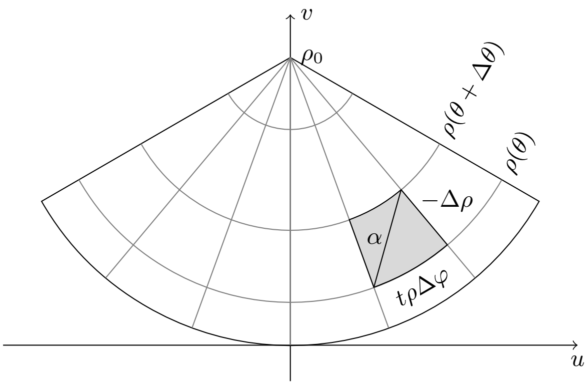

Figure 8: A conical projection.

The aim is to determine the spacing of the parallels so that the map becomes conformal. As in the construction of the Mercator map we look at a path of equal compass bearing of angle α from due north. On that path consider a small rectangle on the sphere at (φ, θ) with side lengths $\Delta\varphi \cos\theta$ and $\Delta\theta$ as in Figure 4. Hence, as in Section 2

$$\cot\alpha \approx \frac{\Delta\theta}{\Delta\varphi\cos\theta}.$$

The corresponding rectangle on the map has edges of lengths $t\rho\Delta\varphi$ and $-\Delta\rho$, where

$$\Delta\rho = \rho(\theta + \Delta\theta) - \rho(\theta)$$

(shaded in Figure 8). We therefore require

$$\cot\alpha \approx -\frac{\Delta\rho}{t\rho\Delta\varphi}.$$

The minus sign comes from the fact that $\rho(\theta)$ decreases as θ increases. Equating the two we get

$$\frac{\Delta\theta}{\Delta\varphi\cos\theta} \approx -\frac{\Delta\rho}{t\rho\Delta\varphi}.$$

This leads to the differential equation

$$\frac{d\rho}{d\theta} = -\frac{t\rho}{\cos\theta} = -t\rho\sec\theta$$

with initial condition $\rho(\theta_0) = \rho_0$. This is a linear differential equation for ρ and the solution is given by

$$\begin{aligned}\rho(\theta) &= \rho_0 \exp\Big(-t\int_{\theta_0}^{\theta} \sec\gamma\, d\gamma\Big) \\ &= \rho_0 \exp\Big(-t\log\Big(\frac{\tan\theta + \sec\theta}{\tan\theta_0 + \sec\theta_0}\Big)\Big) = \rho_0\Big(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\Big)^t.\end{aligned}$$

That solution can be obtained by separation of variables. There are two parameters we can play with, namely the parallel θ_0 and the opening angle determined by t. We choose t such that the length of the parallel of latitude θ_0 is preserved. That parallel has length $2\pi\cos\theta_0$ on the sphere. Hence we require that $2\pi\cos\theta_0 = 2\pi\rho_0 t$, that is,

$$t = \frac{\cos\theta_0}{\rho_0} \in (0, 1]. \tag{4.2}$$

We say that the parallel of latitude θ_0 is a *standard parallel*. The conical projection with standard parallel at latitude θ_0 is therefore given by

$$\rho(\theta) = \rho_0\Big(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\Big)^{\frac{\cos\theta_0}{\rho_0}} \tag{4.3}$$

with the only restriction that $\rho_0 \geq \cos\theta_0$ as otherwise (4.2) cannot be satisfied. One natural choice for ρ_0 is such that the cone is tangent to the sphere at latitude θ_0 as in the middle diagram in Figure 3. Another natural choice for ρ_0 is so that the length of a second parallel of latitude θ_1 is preserved. We discuss this in Section 6.

If the cone is tangential to the sphere, then $\rho_0 = \cot\theta_0$ (see Figure 3) and therefore (4.3) becomes

$$\rho(\theta) = \cot\theta_0\Big(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\Big)^{\sin\theta_0}. \tag{4.4}$$

If we use (4.2) and (4.1) we get

$$\begin{cases} u_{\theta_0}(\varphi, \theta) = \cot\theta_0\Big(\dfrac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\Big)^{\sin\theta_0}\sin(\varphi\sin\theta_0), \\ v_{\theta_0}(\varphi, \theta) = \cot\theta_0 - \cot\theta_0\Big(\dfrac{\tan\theta_0 + \sec\theta_0}{\tan\theta + \sec\theta}\Big)^{\sin\theta_0}\cos(\varphi\sin\theta_0). \end{cases} \tag{4.5}$$

In the next section we show that by taking the limits $\theta_0 \to 0$ and $\theta_0 \to \pi/2$ we can recover the Mercator and stereographic projections as suggested by Figure 3.

5 Stereographic and Mercator projection as limit cases

We first show that the limit case $\theta_0 \to \pi/2^-$ in (4.5) reduces to the stereographic projection. We start by observing that

$$1 - \sin\theta_0 \leq \cos\theta_0 \leq 1$$

for all $\theta_0 \in (0, \pi/2)$. By the squeeze law

$$\lim_{\theta_0 \to \pi/2^-} (\cos\theta_0)^{1-\sin\theta_0} = 1$$

if we use that $s^s \to 1$ as $s \to 0^+$ with $s = 1 - \sin\theta_0$. Rearranging (4.4),

$$\rho(\theta) = \frac{(\cos\theta_0)^{1-\sin\theta_0}}{\sin\theta_0}\Big(\frac{1+\sin\theta_0}{\tan\theta + \sec\theta}\Big)^{\sin\theta_0} \to \frac{2}{\tan\theta + \sec\theta}$$

as $\theta_0 \to \pi/2^-$ because then $\sin\theta_0 \to 1$. Further note that $\cot\theta_0 \to 0$ as $\theta_0 \to \pi/2^-$. Hence (4.5) reduces to the stereographic projection (3.1). We can do a similar calculation for $\theta_0 \to -\pi/2^+$ to get the stereographic projection from the north pole.

We next show that (4.5) reduces to the Mercator projection (2.2) if $\theta_0 \to 0$. Rewriting (4.5) we get

$$u_{\theta_0}(\varphi, \theta) = \varphi(\cos\theta_0)^{1-\sin\theta_0}\Big(\frac{1+\sin\theta_0}{\tan\theta + \sec\theta}\Big)^{\sin\theta_0} \frac{\sin(\varphi\sin\theta_0)}{\varphi\sin\theta_0} \to \varphi$$

as $\theta_0 \to 0$ if we use that $\sin(s)/s \to 1$ as $s \to 0$ with $s = \varphi\sin\theta_0$. As the limit does not depend on the latitude θ, the meridians become vertical lines as $\theta_0 \to 0$. Note next that $\rho_0 = \cot\theta_0 \to \infty$ as $\theta_0 \to 0$. This in particular means that the circular arcs representing the parallels in the conical projection will approach horizontal straight lines.

To simplify the calculations we introduce the function

$$g(\theta) := \log(\tan\theta + \sec\theta),$$

which appears in the Mercator projection (2.2). Then we can rewrite (4.5) in the form

$$v_{\theta_0}(\varphi, \theta) = \frac{1 - e^{(g(\theta_0) - g(\theta))\sin\theta_0}\cos(\varphi\sin\theta_0)}{\tan\theta_0}. \tag{5.1}$$

The limit as $\theta_0 \to 0$ can be computed by L'Hôpital's rule since numerator and denominator converge to zero. We observe that $g'(\theta) = \sec\theta$ and so

$$\frac{d}{d\theta_0}\big(g(\theta_0)\sin\theta_0\big) = g(\theta_0)\cos\theta_0 + \sin\theta_0\sec\theta_0 = g(\theta_0)\cos\theta_0 + \tan\theta_0.$$

Note that we have used $g'(\theta) = \sec\theta$ already by solving (2.1) to construct the Mercator projection.

We first deal with the case $\varphi = 0$. Using that $g(\theta_0) \to 0$ and l'Hôpital's rule we get

$$\begin{aligned}\lim_{\theta_0\to 0} v_{\theta_0}(0,\theta) &= \lim_{\theta_0\to 0} \frac{(g(\theta_0) - g(\theta))\cos\theta_0 + \tan\theta_0}{\sec^2\theta_0} e^{(g(\theta_0)-g(\theta))\sin\theta_0} \\ &= g(\theta) = \log(\tan\theta + \sec\theta).\end{aligned}$$

For the general case note that (5.1) can be written as

$$v_{\theta_0}(\varphi,\theta) = v_{\theta_0}(0,\theta)\cos(\varphi\sin\theta_0) + \frac{1-\cos(\varphi\sin\theta_0)}{\tan\theta_0}.$$

Applying L'Hôpital's rule we get

$$\lim_{\theta_0\to 0}\frac{1-\cos(\varphi\sin\theta_0)}{\tan\theta_0} = \lim_{\theta_0\to 0}\frac{\sin(\varphi\sin\theta_0)\varphi\cos\theta_0}{\sec^2\theta_0} = 0.$$

Combining everything we conclude that $v_{\theta_0}(\varphi,\theta) \to g(\theta) = \log(\tan\theta + \sec\theta)$ as $\theta_0 \to 0$. Hence u and v are exactly as in the Mercator projection (2.2). Figure 9 illustrates the continuous deformation of maps from the Mercator to the stereographic map as θ_0 increases from 0 to $\pi/2$.

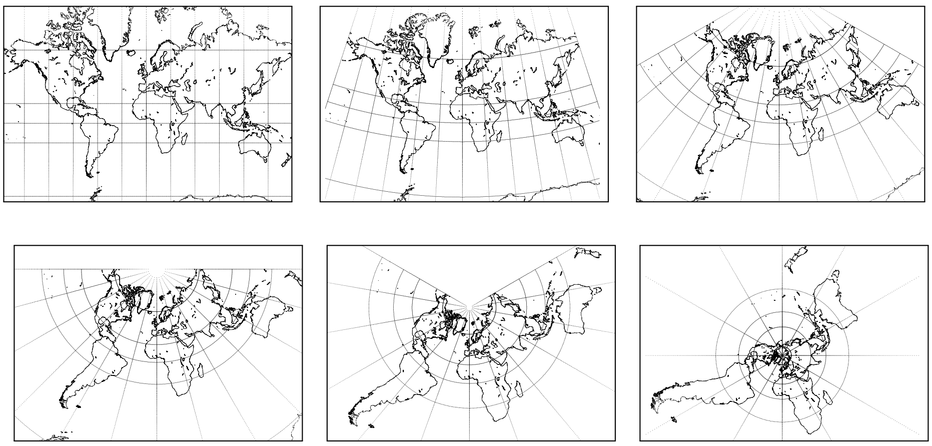

Figure 9: From Mercator to stereographic map via conic conformal maps.

6 Projections with two standard parallels

To derive (4.3) we made sure that the length of the parallel of latitude θ_0 was preserved. There is still one free parameter, namely ρ_0. We show that we can choose ρ_0 so that there is a second standard parallel, that is, a parallel of latitude θ_1 whose length is preserved. The advantage of having two standard parallels is that we can construct a conformal map with minimal distortion of area and length over a moderately large area as is frequently done for maps of the USA, Europe, or Australia. This is already emphasized by Lambert [5, §52]. Figure 10 shows a map of Europe with standard parallels at 40° and 60° north.

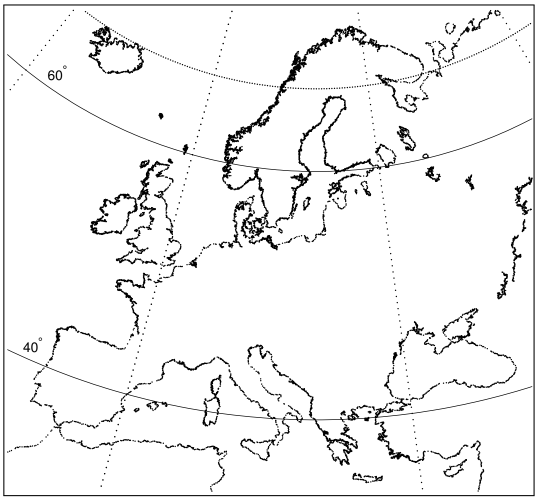

Figure 10: A conic conformal map of Europe with standard parallels at 40° and 60° north.

To make sure θ_1 is a standard parallel we need to choose ρ_0 such that

$$t = \frac{\cos\theta_0}{\rho_0} = \frac{\cos\theta_1}{\rho(\theta_1)},$$

so that the opening angle of the cone in Figure 8 defined by the two parallels is the same. Hence

$$\rho_0 = \rho(\theta_0) = \frac{\cos\theta_0}{\cos\theta_1}\rho(\theta_1), \tag{6.1}$$

and so from (4.3)

$$\rho(\theta_1) = \rho_0\left(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_1 + \sec\theta_1}\right)^{\frac{\cos\theta_0}{\rho_0}} = \rho(\theta_1)\frac{\cos\theta_0}{\cos\theta_1}\left(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_1 + \sec\theta_1}\right)^{\frac{\cos\theta_0}{\rho_0}}.$$

Therefore

$$1 = \frac{\cos\theta_0}{\cos\theta_1}\left(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_1 + \sec\theta_1}\right)^{\frac{\cos\theta_0}{\rho_0}}$$

and taking logarithms on both sides

$$0 = \log\Big(\frac{\cos\theta_0}{\cos\theta_1}\Big) + \frac{\cos\theta_0}{\rho_0}\log\Big(\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_1 + \sec\theta_1}\Big).$$

Solving the equation for ρ_0 we get

$$\rho_0 = \rho(\theta_0) = -\cos\theta_0 \frac{\log\big(\frac{\tan\theta_0+\sec\theta_0}{\tan\theta_1+\sec\theta_1}\big)}{\log\big(\frac{\cos\theta_0}{\cos\theta_1}\big)}. \tag{6.2}$$

By using the relationship between $\cos\theta_0$ and $\cos\theta_1$ from (6.1) we get

$$\rho(\theta_1) = -\cos\theta_1 \frac{\log\big(\frac{\tan\theta_0+\sec\theta_0}{\tan\theta_1+\sec\theta_1}\big)}{\log\big(\frac{\cos\theta_0}{\cos\theta_1}\big)} = -\cos\theta_1 \frac{\log\big(\frac{\tan\theta_1+\sec\theta_1}{\tan\theta_0+\sec\theta_0}\big)}{\log\big(\frac{\cos\theta_1}{\cos\theta_0}\big)},$$

so the formulas for $\rho(\theta_0)$ and $\rho(\theta_1)$ are symmetric in θ_0 and θ_1.

We can view θ_0 and θ_1 as parameters and consider limit cases. In particular, if $\theta_0 \neq 0$, then L'Hôpital's rule shows that $\rho(\theta_1) \to \cot\theta_0$ as $\theta_1 \to \theta_0$, which is consistent with (4.4). We can also let θ_0 and θ_1 go to 0 one after the other or simultaneously. The limit is again the Mercator projection, but more effort is required to compute it. Similarly, if θ_0 and θ_1 approach $\pi/2$ (or $-\pi/2$), then the limit is the stereographic projection from the south pole (or the north pole).

7 Historical Comments

Gerardus Mercator (1512–1594) was born in Belgium from German parents. He later moved to Germany to escape the religious conflict between catholics and protestants in Belgium. His original map was a rather large wall map (202 by 124 cm or 80 by 49 inches). Quite a good facsimile can be seen at [4]. Mercator designed his map long before calculus even existed. Later the mathematician Edward Wright derived, in a purely graphical manner, a table of the spacing of the parallels in his book *Certaine Errors in Navigation* in 1599. This was done by graphically integrating $\sec\theta$ to make sure that the north–south distortion on the rectangular grid is the same as the east–west distortion (see [7, pp. 63–67]). The table is very accurate; see the $10°$ intervals listed in [7, p. 68]. Another English mathematician, Thomas Harriot, looked at the problem in a cleaner fashion but did not publish his results. His findings anticipated a discovery of Henry Bond, who noticed a striking similarity between Wright's table and a table of logarithms of tangents, namely

$$\log\Big(\tan\Big(\frac{\theta}{2} + \frac{\pi}{4}\Big)\Big),$$

which happens to be the correct formula (2.2). This opened a much more precise way for computing the spacing of the parallels.

Finally, Johann Heinrich Lambert (1728–77), born in Alsace, and a member of the Prussian Academy of Sciences during the time of Frederick the Great, took a different point of view in his 1772 exposition [5] (re-edited in 1894 with illustrations, historical comments, and an appendix in [6]). Rather than treating each projection (Mercator, stereographic, and others) separately, he unified the approach to include them as special cases of a larger family of maps. His point of view was to prescribe properties like conformality and the projection surface, and then to use calculus to derive the formulas. This includes the family of projections we discuss in the present article and our approach is not that far from his. Lambert further generalized the approach. Rather than looking at conical projections on which the meridians are straight lines passing through one point (the vertex of the cone), Lambert also looked at conformal maps where the meridians are circles passing through two points (the poles) and the parallels are circles perpendicular to the meridians. He provided tables suitable to display the continents, in particular for Europe, North and South America, and Asia. Furthermore, Lambert treated area preserving maps in the same way. More details of the interesting history of these map projections, and in particular the Mercator projection, can be found in [7].

References

[1] L. M. Bugayevskiy and J. P. Snyder, *Map Projections: A Reference Manual*, Taylor & Francis, London, 1995.

[2] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978, book.

[3] T. G. Feeman, *Portraits of the Earth*, Mathematical World: A mathematician Looks at Maps, vol. 18, American Mathematical Society, Providence, RI, 2002.

[4] F. W. Krücken, Ad maiorem Gerardi Mercatoris gloriam, available at http://www.wilhelmkruecken.de/.

[5] J. H. Lambert, Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten, in *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, vol. 3, Buchhandlung der Realschule, Berlin, 1772, available at GDZ http://resolver.sub.uni-goettingen.de/purl?PPN590414488, 104–199.

[6] ———, *Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten. Von J.H. Lambert (1772)*, Ostwald's Klassiker der exakten Wissenschaften, vol. 54, ed. A. Wangerin, W. Engelmann, Leipzig, 1894, available at http://name.umdl.umich.edu/ABR2581.0001.001.

[7] M. Monmonier, *Rhumb Lines and Map Wars: A Social History of the Mercator Projection*, University of Chicago Press, Chicago, 2004.

[8] Musée International d'Horlogerie, *Richard Daners: Sein Werk/Son Oevre*, Éditions Institut l'homme et le temps, La Chaux-de-Fonds, Switzerland, 2007.

[9] W. Pijls, Some properties related to Mercator projection, *Amer. Math. Monthly* **108** (2001) 537–543.

[10] H. Rummler, Mercatorkarte und hyperbolische Geometrie, *Elem. Math.* **57** (2002) 168–173.

Daniel Daners
School of Mathematics and Statistics
The University of Sydney, NSW 2006, Australia
daniel.daners@sydney.edu.au