Finding absolutely and relatively periodic orbits in the equal mass 3-body problem with vanishing angular momentum

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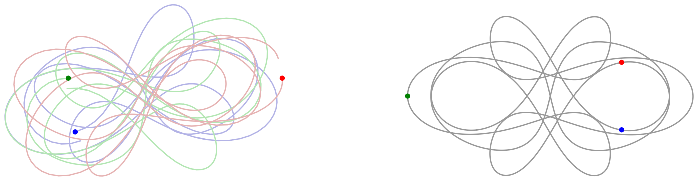
Introduction

Basic ideas:

- Relative vs. absolute periodic orbits,
- 3-body problem in reduced, regularised coordinates,
- Discrete symmetry,
- Geometric phase,
- Theorem on geometric phase.

Relative and absolute periodic orbits in the 3-body problem



- Three point masses in the plane, $m_j \in \mathbb{R}^+$, $j = 1, 2, 3$.
- Each position denoted by $X_j \in \mathbb{C}$.
- Each momentum denoted by $P_j \in \mathbb{C}$.
- Centre of mass $O = \frac{1}{m} \sum m_j X_j$ (with $m = \sum m_j$),
- Angular momentum $p_\phi = \operatorname{Im} \sum \bar{X}_j P_j$.

The 3-body problem

Described by the Hamiltonian:

$$H = \sum \frac{|P_j|^2}{2m_j} - \sum \frac{m_k m_l}{|X_l - X_k|} \tag{1}$$

producing Hamilton's equations

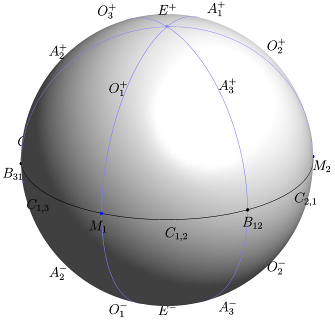
$$z' = J\nabla H(z) = F(z), \tag{2}$$

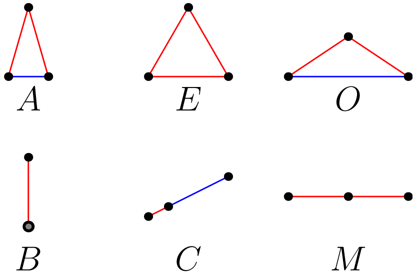
where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$
$$z = (X_1, X_2, X_3, P_1, P_2, P_3)^T \in \Omega,$$

and $\Omega = \mathbb{C}^6$ is the phase space.

Reduce to the shape sphere





"Shape space" $(w_1, w_2, w_3) \in \mathbb{R}^3$. "Shape sphere" $w_1^2 + w_2^2 + w_3^2 = 1$. Features when $m_1 = m_2 = m_3$:

- Equilateral points (Lagrange configurations): $E^{\pm}$, $(0, 0, \pm 1)$.
- Isosceles curves: $A_j^{\pm}$ (acute), $O_j^{\pm}$ (obtuse).
- Collinear curves: $C_{j,k}$ $w_3 = 0$.
- Isosceles collinear points (Euler configurations): M_j.
- Binary collision points: B_{kl}.

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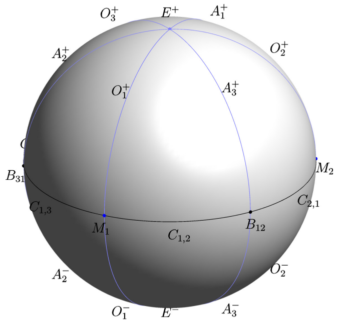

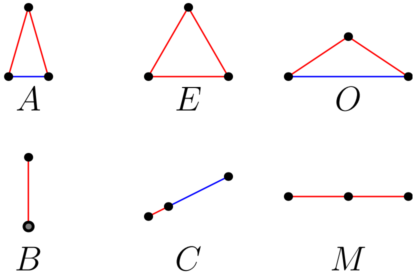

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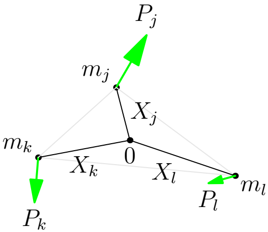
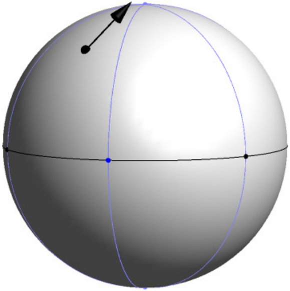
Discrete symmetries

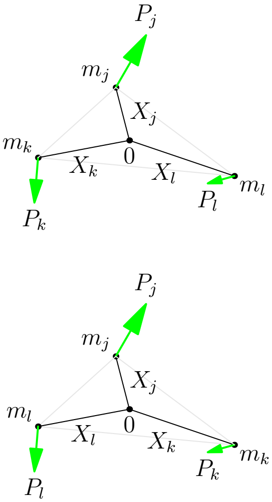
Original configuration.

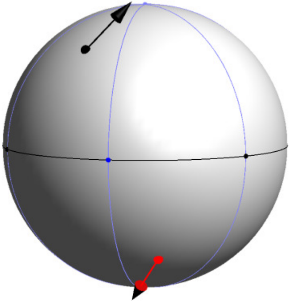




Discrete symmetries

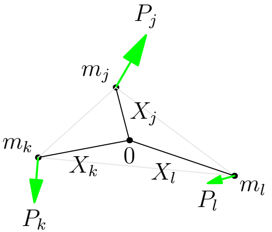
σ_j swaps indices k, l ($m_k = m_l$).

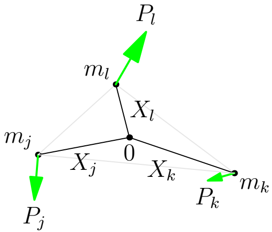


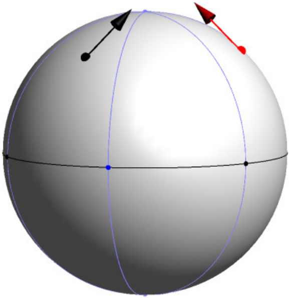


Discrete symmetries

$c = \sigma_l \circ \sigma_k$ cycles indices: $(1, 2, 3) \to (2, 3, 1)$ $(m_1 = m_2 = m_3)$.

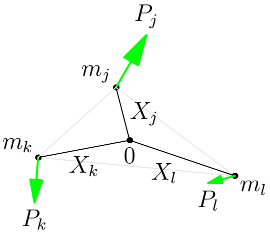


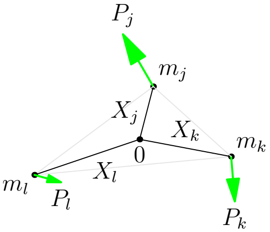


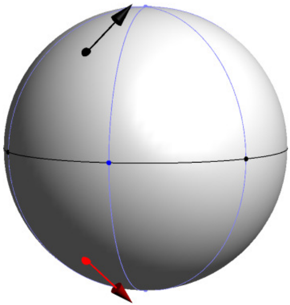


Discrete symmetries

ρ reflects whole configuration in space (any masses).

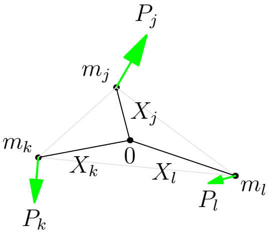


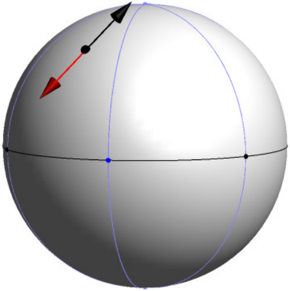


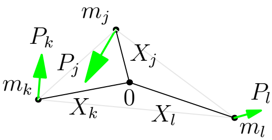


Discrete symmetries

τ reflects in time: $P_j \to -P_j$, each j (any masses).







Reversing symmetries

- Define $S:\Omega \longrightarrow \Omega$: symmetry of vector field $F(z)$ iff $S \circ F(z) = F \circ S(z)$.
- Define $\mathfrak{S}_S = \{I, \sigma_1, \sigma_2, \sigma_3, c, c^2, \rho, \rho\sigma_1, \rho\sigma_2, \rho\sigma_3, \rho c, \rho c^2\} \cong S_3 \times Z_2$ (order 12), a group under composition.
- Observe that $\tau \circ F(z) = -F \circ \tau(z)$ means τ is an *antisymmetry* of F.
- We call τ a *reversing symmetry*. Composition $R = \tau \circ S$ is also a reversing symmetry.

We now have a *reversing symmetry group* $\mathfrak{S}_R \cong S_3 \times Z_2^2$ (order 24). Note that $Z_2^2 = V_4 = \{I, \rho, \tau, \tau\rho\}$ is the centre of $\mathfrak{S}_R$.

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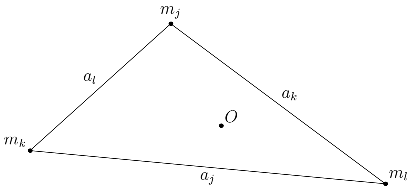
Regularisation



Simultaneous regularisation of all binary collisions (due to Lemaître [1]):

- New coordinates $\alpha_j \in \mathbb{R}$ such that $a_j = \alpha_k^2 + \alpha_l^2$, $a_j \geq 0$ side length opposite m_j.
  - $\alpha_j = 0$ gives collinearity with m_j in eclipse.
  - $\alpha_k = \alpha_l = 0$ gives collision between m_k and m_l.
  - *Signed* area $S = \alpha_1 \alpha_2 \alpha_3 \alpha$, where $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$.
- Canonical momenta $\pi_j \in \mathbb{R}$.

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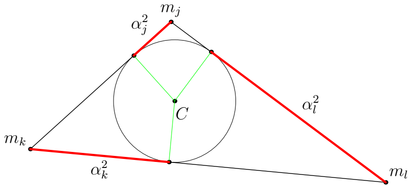

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Regularisation

- Define fictional time τ by $\frac{dt}{d\tau} = a_1 a_2 a_3$, then
- define new Hamiltonian $K = (H - h)a_1 a_2 a_3 \equiv 0$, h is physical energy.
- Shape changes by $\dot{\alpha}_j$, $\dot{\pi}_j$. New phase space is $\Omega = \mathbb{R}^6$.
- *Shape dynamics alone govern rotation dynamics when* $p_\phi = 0$*.*

Discrete symmetries in regularised coordinates

Preserve physical meanings of symmetries. With $z = (\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_2, \pi_3)$, choose:

$$\begin{aligned}
\sigma_1(z) &= -(\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_3, \pi_2), \text{ etc.,} \\
c(z) &= (\alpha_2, \alpha_3, \alpha_1, \pi_2, \pi_3, \pi_1), \\
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Subgroup $\{I, s_1, s_2, s_3\} \cong V_4$. Elements interact with S_3 by semidirect product $S_3 \rtimes V_4 = S_4$ (order 24). Elements written uniquely as composition $S \circ s_j$, $S \in S_3$.

New (reversing) symmetry group $\widetilde{\mathfrak{G}}_R \cong S_4 \times Z_2^2$ (order 96), with same centre as before.

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Reversing fixed sets

Fixed set of symmetry S: $\{z \in \Omega \colon S(z) = z, S \in \widetilde{\mathfrak{G}}_R\}$.

- Fixed sets of reversing symmetries are not invariant.
- Solution with points in fixed sets of reversing involutions *run in reverse possibly with some other symmetry applied after that instant in time*.

Theorem

A solution connecting two points in the fixed sets of reversing involutions R_1, R_2 is periodic, if $(R_2 R_1)$ has finite order.

Proof.

Suppose $z(0) \in \operatorname{Fix} R_1$ and $z(\tau_0) \in \operatorname{Fix} R_2$. Observe that $z(2\tau_0) \in \operatorname{Fix} R_1 R_2 R_1 = \operatorname{Fix} R_1 S$, where S is non-reversing of order k, as $\widetilde{\mathfrak{G}}_R$ is finite. If $R_1 = R_2$ then $S = I$ and orbit is periodic with period $2\tau_0$. Else periodic with period $2k\tau_0$. □

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Example reversing orbit

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  $\mathrm{Fix}\, R_1 = (0, \alpha_2, \alpha_2, \pi_1, \pi_2, -\pi_2)$ (which looks like...)
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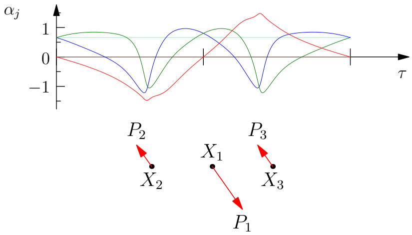

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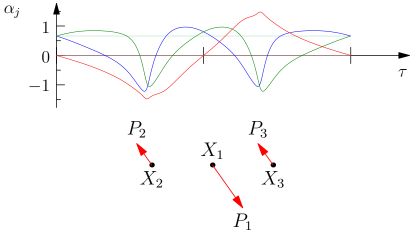

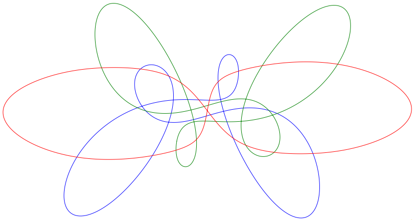

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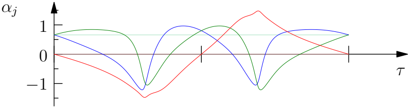
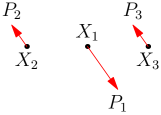
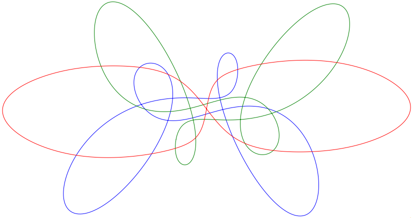
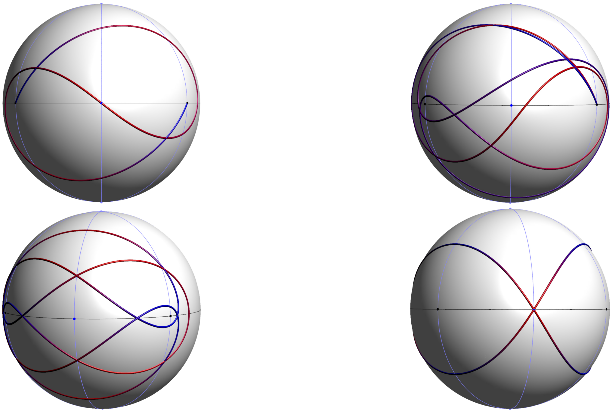






Reversing fixed sets

Five classes of reversing fixed sets in regularised system: 1. Collinear ($\tau\rho s_j$), 2. Isosceles ($\tau\rho\sigma_j$ or $\tau\rho\sigma_j s_j$), 3. Isosceles collinear ($\tau\sigma_j$ or $\tau\sigma_j s_j$), 4. Brake-collision (τs_j), 5. Brake (τ, example in [3])



Montgomery's formula for geometric phase

Montgomery [2] shows calculation of geometric phase. "Area enclosed by a loop on the shape sphere."

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dG &= -\frac{1}{2} w_3 d\theta, \text{ where } \theta = \arg(w_1 + i w_2) \\
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We calculate geometric phase over an orbit of period T by

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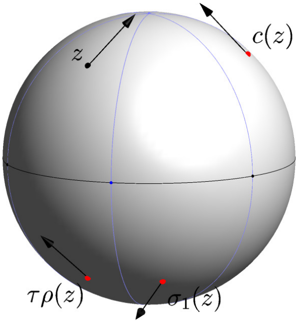
Geometric interpretation: symmetries/antisymmetries of U

Consider $S \in S_4$.

- $U \circ S(z) = U \circ (\tau \circ \rho \circ S)(z) = U(z)$.
- Symmetries S and reversing symmetries $\tau \circ \rho \circ S$ leave dG invariant.
- $U \circ (\tau \circ S)(z) = U \circ (\rho \circ S)(z) = -U$.
- Symmetries $\rho \circ S$ and reversing symmetries $\tau \circ S$ send $dG \to -dG$.
- Antisymmetries of U have even order.



Geometric interpretation: symmetries/antisymmetries of U

Consider $S \in S_4$.

- $U \circ S(z) = U \circ (\tau \circ \rho \circ S)(z) = U(z)$.
- Symmetries S and reversing symmetries $\tau \circ \rho \circ S$ leave dG invariant.
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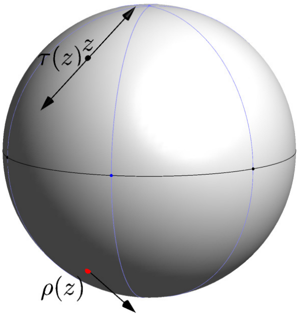

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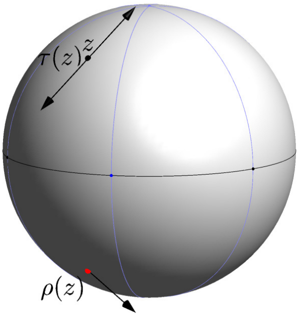


Cancellation of geometric phase

Define isotropy subgroup of T-periodic solution $z(\tau)$ by $\Sigma_z = \{S \in \widetilde{\mathfrak{G}}_R \colon S(z) = z\}$.

Theorem

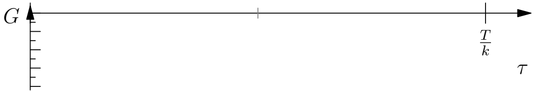
If a T-periodic solution $z(\tau)$ of the regularised equations of motion has isotropy subgroup Σ_z containing any antisymmetry of U, then the geometric phase $G(T) = \int_0^T U(z(\tau))d\tau = 0$.

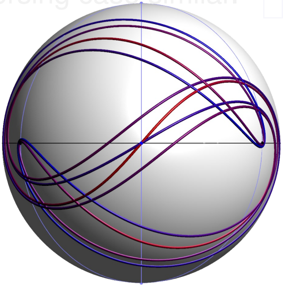
Outline of proof

Consider orbit with isotropy subgroup generated by reversing involutions R_1, R_2 such that $(R_2R_1)^k = I$. W.l.o.g. at least R_1 antisymmetry of U and $R_1(z(\tau)) = z(\frac{T}{2k} - \tau)$.
Consider $0 \leq \tau \leq \frac{T}{k}$.

$$G(\tfrac{T}{k}) = \int_0^{\frac{T}{2k}} U(z(\tau))d\tau + \int_{\frac{T}{2k}}^{\frac{T}{k}} U(z(\tau))d\tau = \ldots = 0.$$

Now whether or not (R_2R_1) (of order k) is an antisymmetry of U, result follows for reversing case. Non-reversing case similar. □





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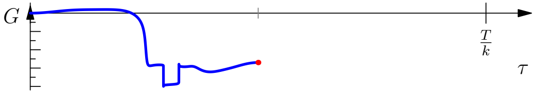

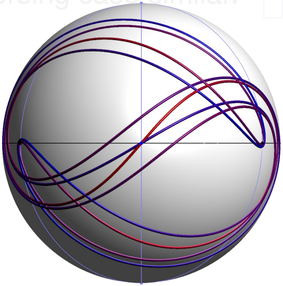

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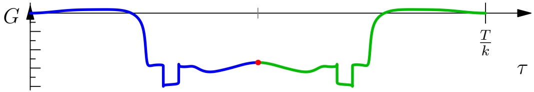

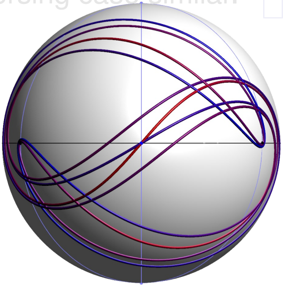

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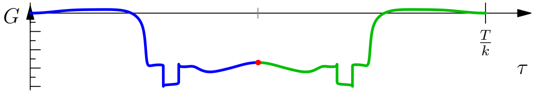

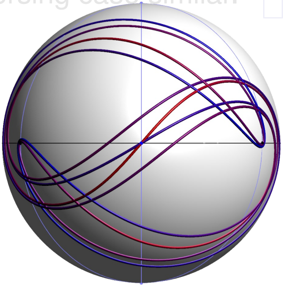

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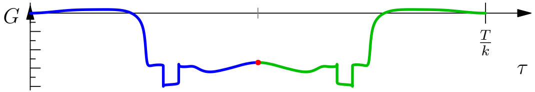

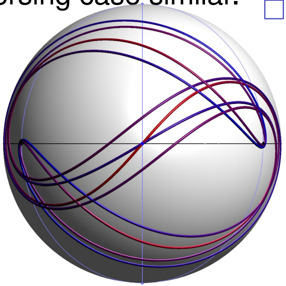

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Orbits whose isotropy subgroups contain antisymmetries of U are the only ones whose geometric phase is forced to vanish.

Supported by extensive numerical evidence. 363 orbits obeying Theorem 2 or its converse.

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Conclusion

- Regularised system has reversing symmetry group $\widetilde{\mathfrak{G}}_R \cong S_4 \times Z_2^2$.
- Antisymmetries of U present in isotropy subgroups of periodic orbits dictate that geometric phase vanishes, by Theorem 2.
- Can use Theorem 2 to choose symmetries to impose to obtain absolute periodic orbits.
- Choosing other symmetries allows relative periodic orbits with vanishing angular momentum.

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