Symplectic integration of the reduced, zero angular momentum 3-body problem in regularised coordinates

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Hamiltonian of the planar 3-body Problem in Cartesian Coordinates

Let (j,k,l) be cyclic permutations of $(1,2,3)$. Let X_j be the complex cartesian coordinates of m_j and P_j be its canonically conjugated momentum. The Hamiltonian of the planar 3-body problem in these coordinates is

$$H = \sum \frac{|P_j|^2}{2m_j} + \frac{1}{2}\sum \frac{m_k m_l}{|X_l - X_k|}.$$

Following Waldvogel [3], introduce symmetric coordinates a_j, ϕ and canonically conjugated momenta p_j, p_ϕ, where a_j is the length of the side opposite m_j. ϕ is the angle of orientation of the triangle, of interest in discovering the geometric phase of relative periodic orbits in the regularised coordinates. When $p_\phi = 0$ its equation of motion is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{2}{3}\sum \frac{S}{m_j a_k a_l}\left(\frac{p_k}{a_l} - \frac{p_l}{a_k}\right),$$

where $S = \sqrt{\sigma(\sigma - a_1)(\sigma - a_2)(\sigma - a_3)}$ is the signed area of the triangle and $\sigma = \frac{1}{2}(a_1 + a_2 + a_3)$.

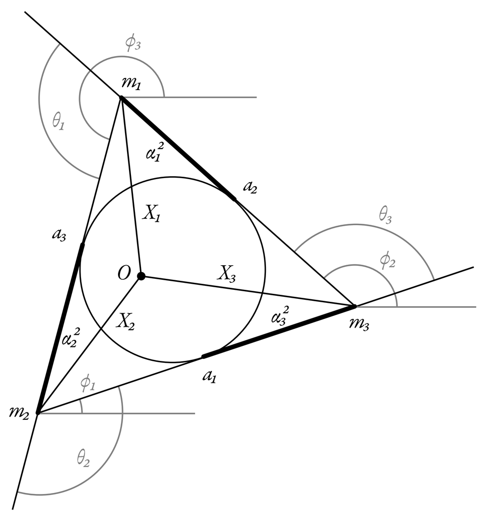


Figure 1: Geometry of the physical coordinates X_j, symmetric coordinates a_j and regularised coordinates α_j, including the angles θ_j and ϕ_j involved in the transformations.

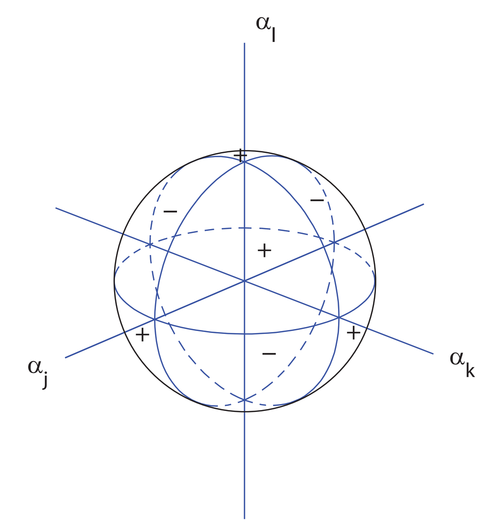


Figure 2: α-space, showing the orientation of triangles in the octants.

Regularised Coordinates

Introduce α_j such that $a_j = \alpha_k^2 + \alpha_l^2$ and canonically conjugated momenta π_j. In these coordinates, each non-degenerate oriented triangle is represented four times (Figure 2). Degenerate triangles are given by:

- $\alpha_j = 0$, α_k, $\alpha_l \neq 0$ a collinear configuration with m_j between m_k and m_l;
- $\alpha_j = \alpha_k = 0$, $\alpha_l \neq 0$ a binary collision between m_j and m_k; and
- $\alpha_j = \alpha_k = \alpha_l = 0$ the triple collision.

See Waldvogel for details of the transformations and their inverses. Note: care must be taken in the conversion back to Cartesian coordinates. The exterior angles θ_j must be adjusted so that $\sum \theta_j = 0$. Pick $\theta_j = \theta_k + \theta_l - 2\pi$ for the initial configuration and label this state $s = j$. At a collinearity, α_j changes sign, so label this transition $t = j$. The table below shows to which state the system moves with each transition:

t \ s	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

E.g., start in state 1 and α_1 changes sign, then remain in state 1. Continue with $\theta_1 = \theta_2 + \theta_3 - 2\pi$. If next α_2 changes sign $(t = 2)$, we go from $s = 1$ to $s = 3$ and reconstruct the cartesian coordinates by using $\theta_3 = \theta_1 + \theta_2 - 2\pi$..

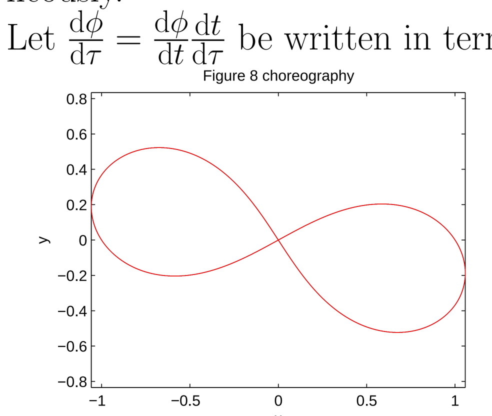
Hamiltonian of 3-body Problem in Regularised Coordinates

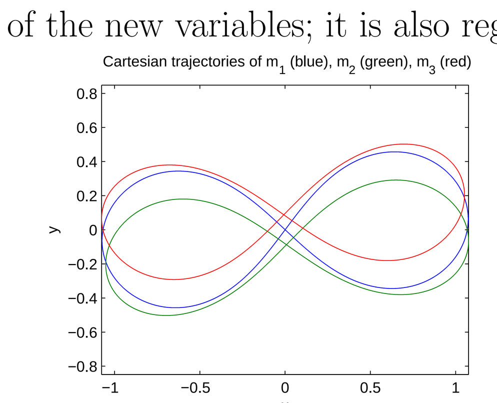
Rescale time such that $\frac{\mathrm{d}t}{\mathrm{d}\tau} = a_1 a_2 a_3$. Then let the new Hamiltonian $K = a_1 a_2 a_3 (H - h)$, where h is the physical energy of the system, calculated from the initial conditions, and only solutions with $K = 0$ have physical meaning. Introduce 3-vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$, $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)^T$ and write $K = K_0(\boldsymbol{\alpha}, \boldsymbol{\pi}, p_\phi) - h a_1 a_2 a_3$, where in the case of $p_\phi = 0$,

$$K_0(\boldsymbol{\alpha}, \boldsymbol{\pi}, 0) = \frac{1}{8}\sum\left(\frac{a_j}{m_j}\left(\alpha^2 \pi_j^2 + (\alpha_k \pi_l - \alpha_l \pi_k)^2\right) - m_k m_l a_k a_l\right),$$

where $\alpha^2 = \sum \alpha_j^2$. The equations of motion, $\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\tau} = \frac{\partial K}{\partial \boldsymbol{\pi}}$, $\frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}\tau} = -\frac{\partial K}{\partial \boldsymbol{\alpha}}$ are regularised in every binary collision simultaneously.

Let $\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau}$ be written in terms of the new variables; it is also regularised at every binary collision.





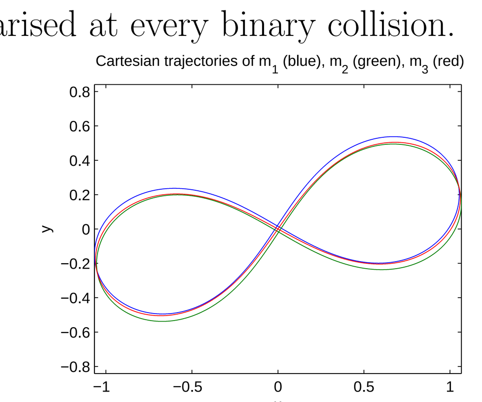


Figure 3: Figure 8 choreography integrated using regularised coordinates. Time step $\delta\tau = 10^{-5}$.

Figure 4: Continued figure 8 choreography integrated using regularised coordinates. $m_1 = 0.995$, $\delta\tau = 10^{-5}$.

Figure 5: A periodic orbit that lies near the figure 8. Equal masses $m_1 = m_2 = m_3 = 1$, $\delta\tau = 10^{-5}$.

Explicit Symplectic Splitting Integrator

If $H = \sum H_i$, each H_i exactly integrable, then the flow of H can be approximated to first order in time step t by following the flow of each H_i for time t, [2]. Reversing the order in which each flow is applied gives the adjoint of this map. The first order flow and its adoint can be composed with half steps to produce a generalised midpoint integrator. This method is reversible and second order, so Yoshida's trick [4] can be used to build higher even order integrators.

Solution Forms for Monomial Hamiltonians

Channell & Neri [1] provide a theorem that a monomial Hamiltonian is integrable. The p-th term of $H = \sum H_i$ is

$$H_p = A_p q^{m_p} p^{n_p},$$

where $m_p, n_p \in \mathbb{Z}^+$. It has integrals $I_{pj} = q_j^{m_{pj}} p_j^{n_{pj}}$. When $m_p \neq n_p$

$$q(t) = q_0\,(1 + (n_p - m_p)\,A_p\,q_0^{m_p - 1}\,p_0^{n_p - 1}\,t)^{\frac{n_p}{n_p - m_p}}$$
$$p(t) = p_0\,(1 + (n_p - m_p)\,A_p\,q_0^{m_p - 1}\,p_0^{n_p - 1}\,t)^{\frac{m_p}{m_p - n_p}}.$$

When $m_p = n_p$,

$$q(t) = q_0 \exp(m_p\,A_p\,(q_0\,p_0)^{m_p - 1}\,t)$$
$$p(t) = p_0 \exp(-m_p\,A_p\,(q_0\,p_0)^{m_p - 1}\,t).$$

In a system with M degrees of freedom, consider each pair (q_i, p_i) by itself and hide every other pair inside $A_p = B_p \prod_{j \neq i}^{M} I_{pj}$, where B_p is the actual constant coefficient of the p-th term of the full polynomial.

With $z = (q, p)^T$, the full solution for the p-th monomial is like

$$z_p(t) = (\ldots, q_i(t), \ldots, q_k(t), \ldots, p_i(t), \ldots, p_k(t), \ldots)^T,$$

where

$$q_i(t) = q_{i,0}\exp(m_{pi}\,B_p \prod_{j \neq i} I_{pj}\,(q_{i,0}\,p_{i,0})^{m_{pi} - 1} t)$$
$$p_i(t) = p_{i,0}\exp(-m_{pi}\,B_p \prod_{j \neq i} I_{pj}\,(q_{i,0}\,p_{i,0})^{m_{pi} - 1} t)$$
$$q_k(t) = q_{k,0}(1 + (n_{pk} - m_{pk})\,B_p \prod_{j \neq k} I_{pj}\,q_{k,0}^{m_{pk} - 1}\,p_{k,0}^{n_{pk} - 1} t)^{\frac{n_{pk}}{n_{pk} - m_{pk}}}$$
$$p_k(t) = p_{k,0}(1 + (n_{pk} - m_{pk})\,B_p \prod_{j \neq k} I_{pj}\,q_{k,0}^{m_{pk} - 1}\,p_{k,0}^{n_{pk} - 1} t)^{\frac{m_{pk}}{m_{pk} - n_{pk}}}.$$

The terms I_{pj} must be calculated anew at each stage of each step.

Represent the solution above by $z_p(t) = \psi_p^t z_0$, where $z(0) = z_0$ is the initial condition.

Explicit Symplectic Integrator

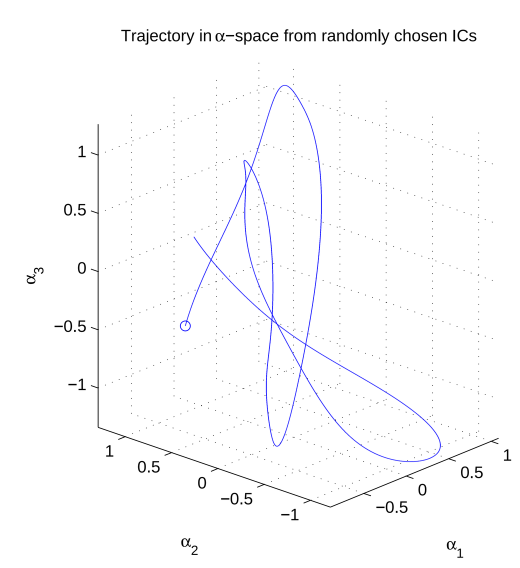
Let $z(t) = \psi_N^t\,\psi_{N-1}^t\,\ldots\,\psi_2^t\,\psi_1^t\,z_0 + O(t^2)$, denoted by ψ^t, be a first order approximation of H. The adjoint of this method is $\left(\psi^t\right)^*$, where the solutions ψ_p^t are applied in the reverse order.

A reversible second order approximation is given by $z(t) = \left(\psi^{\frac{t}{2}}\right)^* \psi^{\frac{t}{2}}\,z_0 + O(t^3) = \psi_1^{\frac{t}{2}}\ldots\psi_{N-1}^{\frac{t}{2}}\,\psi_N^t\,\psi_{N-1}^{\frac{t}{2}}\ldots\psi_1^{\frac{t}{2}}\,z_0 + O(t^3)$, denoted by ϕ_2^t.

Implementation

MATLAB's symbolic algebra toolbox was used to represent the Hamiltonian of the planar 3-body problem in the regularised coordinates, and the integrator was built as above. Simplifications may be made when several monomials are functions of coordinates or momenta only. K is a polynomial with 34 terms; collecting terms so reduces the number of stages from 34 to 22.

A trajectory with several close encounters, one a near collision with separation $\sim 10^{-4}$, is shown in $\boldsymbol{\alpha}$-space in Figure 6, and its behaviour vs scaled time τ is shown in Figure 7. The shape of the corresponding orbit in physical space is shown in Figure 8, and its progression in physical time t in Figure 9.



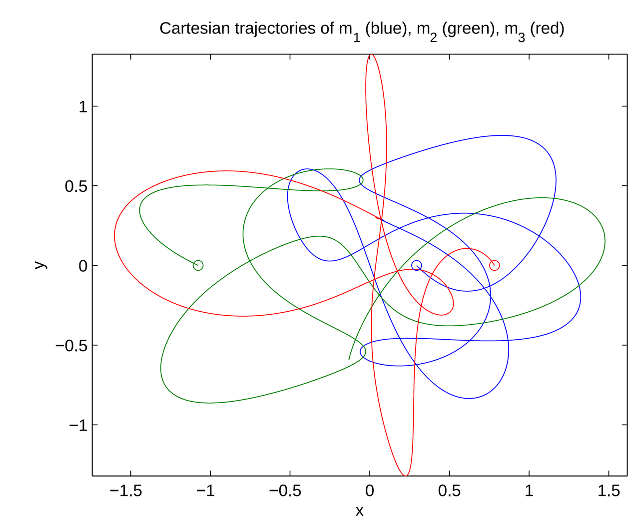
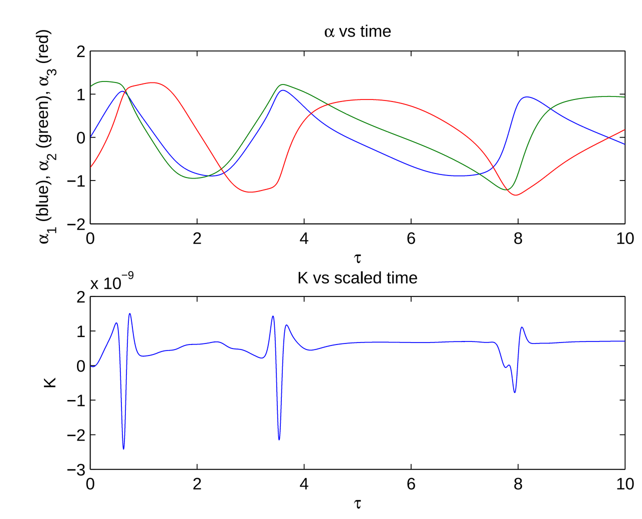


Figure 6: Trajectory in $\boldsymbol{\alpha}$-space. Figures 7, 8 and 9 are based on the same integration of 10^6 time steps at $\delta\tau = 10^{-5}$.

Figure 8: Physical trajectories of the three bodies integrated from initial conditions in Figure 6, with $\phi(0) = 0$.



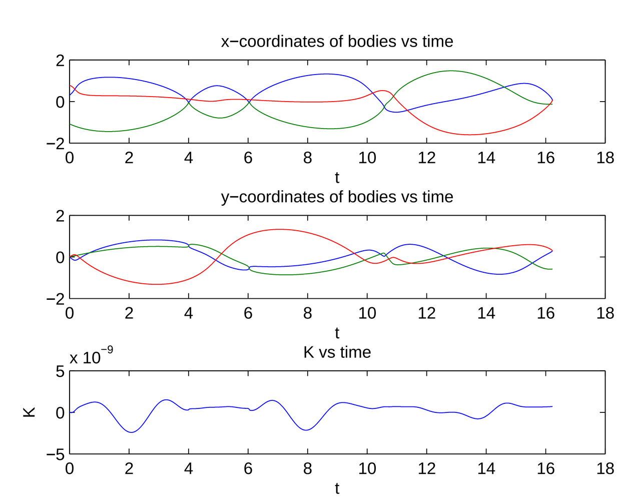


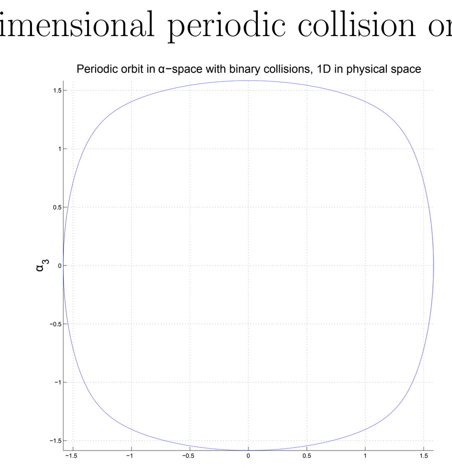
Figure 7: α and error in K versus scaled time τ for the initial conditions in Figure 6. There are close encounters between m_1 and m_2 at $\tau \approx 1$, 2 and a near collision (distance between bodies $\approx 10^{-4}$) between m_1 and m_3 at $\tau \approx 9.7$.

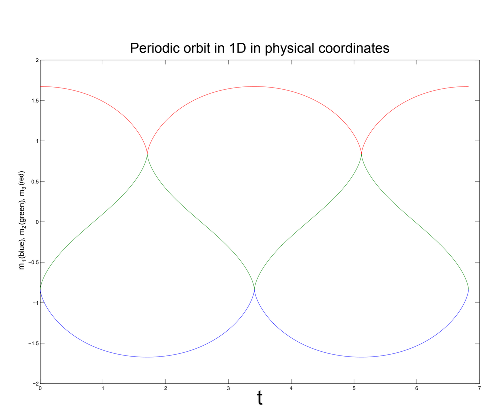
Figure 9: Physical x and y coordinates and error in K vs physical time t for the three bodies from Figure 6. Note the time scaling coming into effect near close encounters compared to K vs τ in Figure 7.

Application and results

Newton's method is used on the Poincaré section to discover periodic orbits in the regularised coordinates. Starting from the classic figure-8 choreography (Figure 3), one of the masses is modified, and a nearby periodic orbit is found. This was repeated to discover the orbit shown in Figure 4, with $m_1 = 0.995$.

A 1-dimensional periodic collision orbit is shown in Figures 10 and 11, and the energy error is shown in Figure 12.





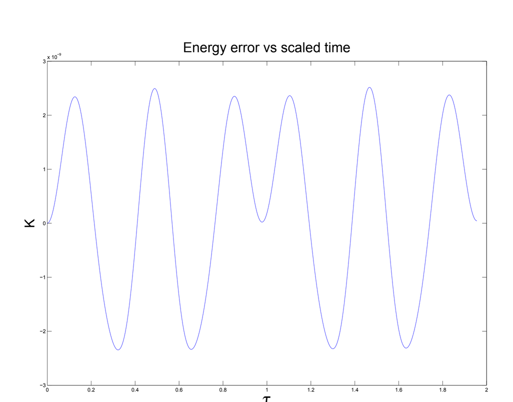


Figure 10: Trajectory in $\boldsymbol{\alpha}$-space of periodic collision orbit in one dimension. Integrated with time step $\delta\tau = 10^{-5}$.

Figure 11: Physical trajectories of the three bodies in one dimension vs time.

Figure 12: Energy error for one dimensional periodic collision orbit vs scaled time.

References

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[4] Haruo Yoshida. Construction of higher order symplectic integrators. *Physics Letters A*, 150:262–268, 1990.