Arbeitsgemeinschaft Analysis: closing workshop
University of Zurich, 28 May 2015

Uniform convergence of solutions to Robin & Neumann boundary problems on domains with shrinking holes

Daniel Hauer

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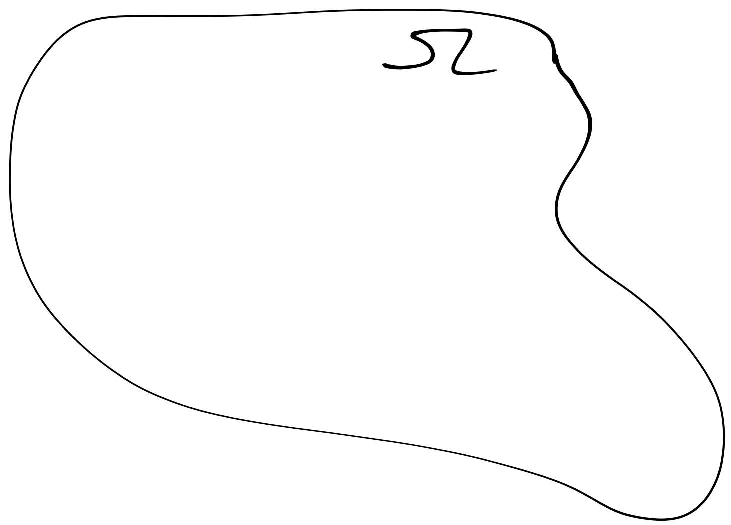
Initial situation:



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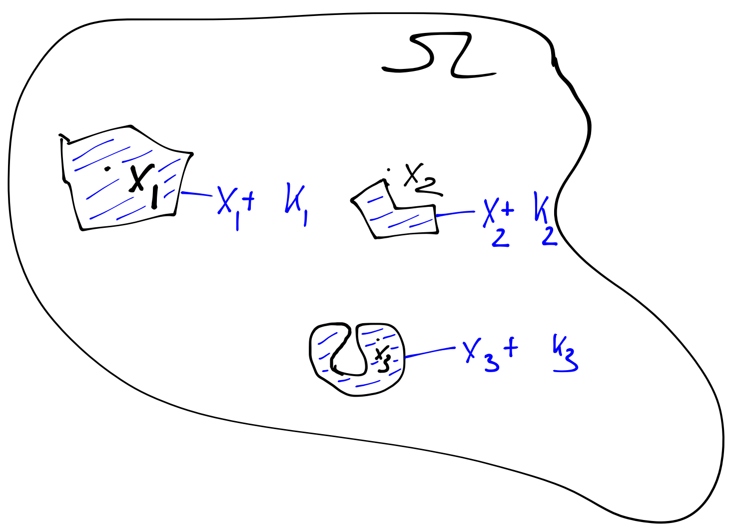

Let $\Omega \subseteq \mathbb{R}^d$ be a bdd. Lipschitz domain, $d \geq 2$, $\{x_1, \dots, x_m\} \subseteq \Omega$ & $K_1, \dots, K_m \subseteq \mathbb{R}^d$ nonempty compact sets with Lipschitz bdry ∂K_j s.th.

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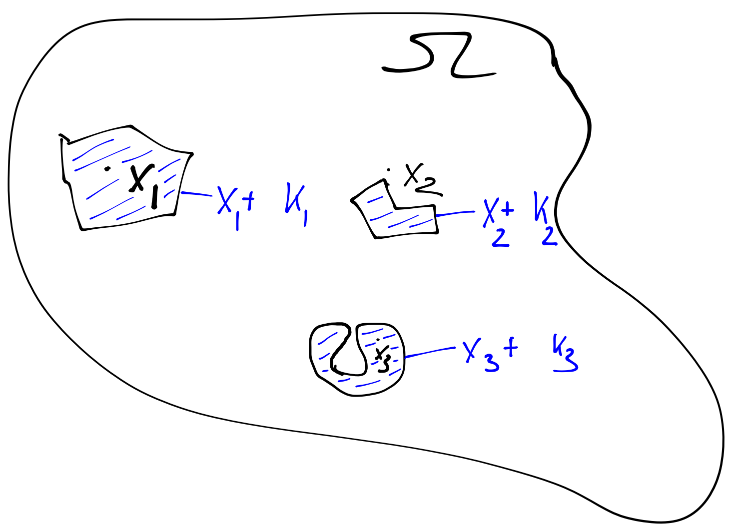

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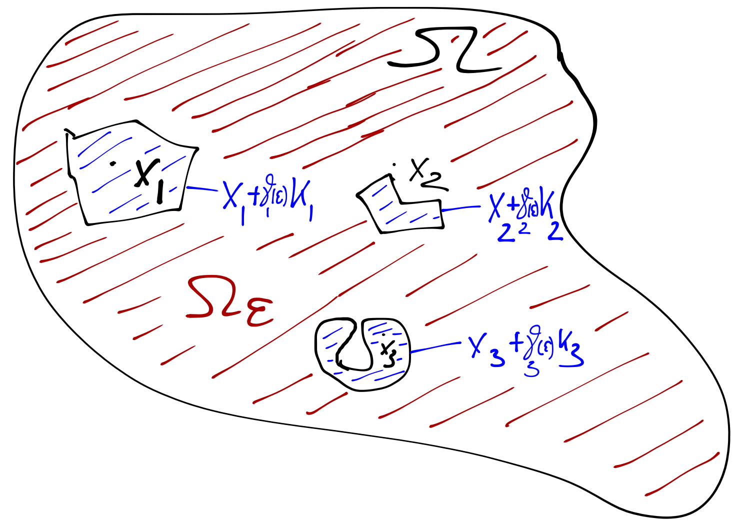

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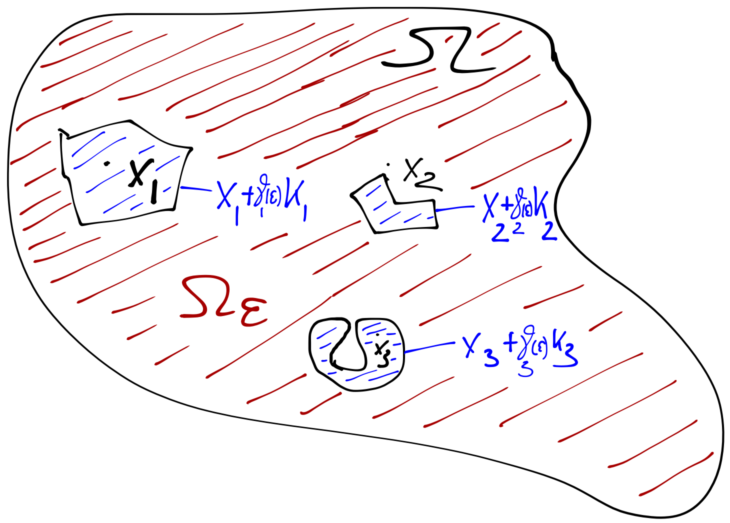

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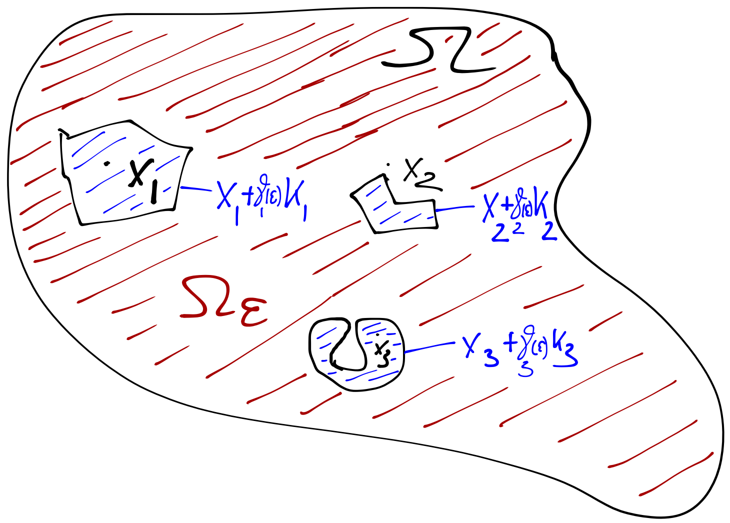

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For every $\varepsilon \in (0,1]$, let $H_\varepsilon := \bigcup_{j=1}^{m} (x_j + \delta_j(\varepsilon) K_j) \subseteq \Omega$ & set $\Omega_\varepsilon := \Omega \setminus H_\varepsilon$.

In $d=2$, let ∂K_j be a closed curve & of class $C^{1,\theta}$ for some $\theta \in (0,1)$.

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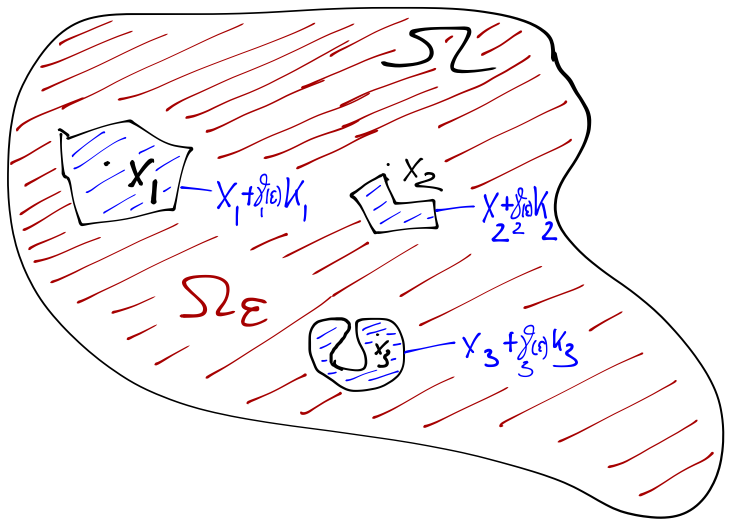

For $f_\varepsilon \in L^p(\Omega_\varepsilon)$ let u_ε be the weak solution of

$$(1)\quad \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ \quad u_\varepsilon = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon \end{cases}$$

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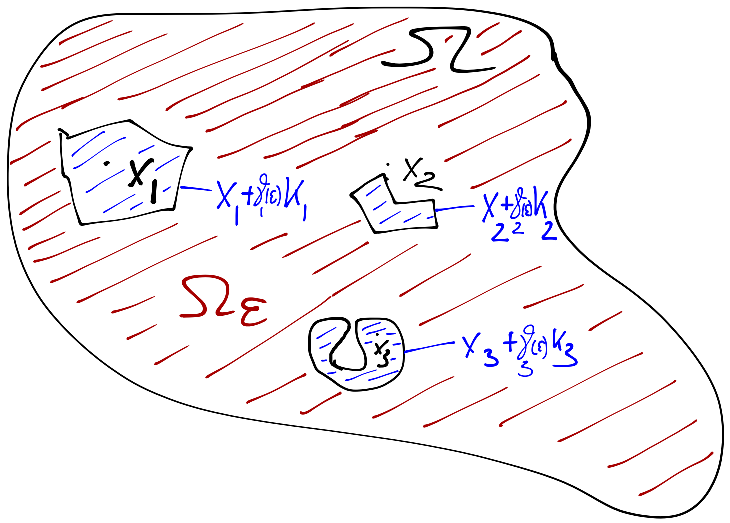

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and for $f \in L^p(\Omega)$ let u be the unique weak solution of

$$(2)\quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} .$$

Initial situation:

The aim is to prove that if $f_\varepsilon \to f$ in $L^p(\Omega)$ for some $p > d/2$, then the weak solutions u_ε of (1) converge uniformly to the unique weak solution u of (2)

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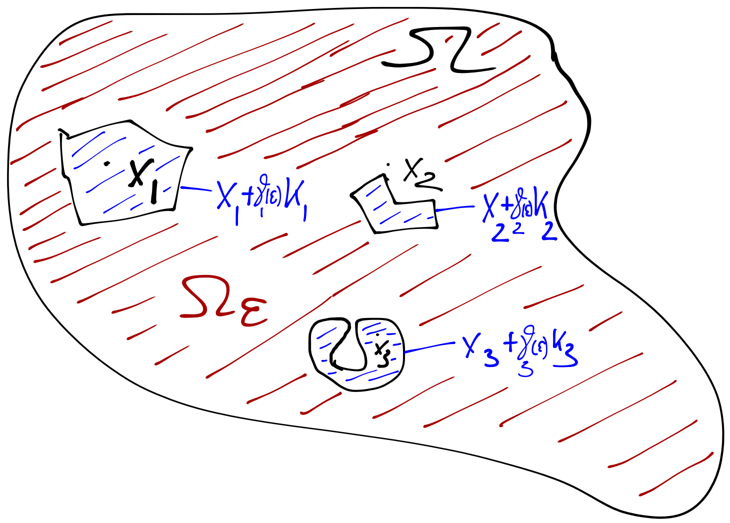

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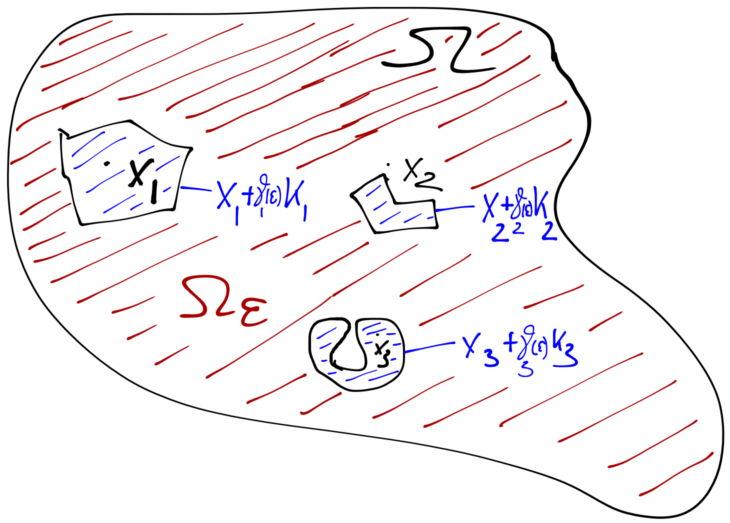

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Note: A similar result for Dirichlet problems cannot be true!

Remarks

- L^2- or L^p-convergence of solutions to elliptic equations on domains with shrinking holes has been studied by many authors as, for instance,
  - Dancer '90
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  - Daners '99, '08
  - Bucur–Varchon '00, '02
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- In contrast to this, we have established L^∞-convergence of solutions & we do not just treat Neumann bdry conditions, but also Robin bdry conditions where β_ε may be positive or negative.

1. Theorem (Dancer, Daners, H. '14)

Suppose the bdry-function $\beta_\varepsilon : \partial H_\varepsilon \to \mathbb{R}$ satisfy Assumption 1. If $f_\varepsilon \to f$ in $L^p(\Omega)$ for some $p > d/2$, then the weak solutions u_ε of

$$(1) \quad \begin{cases} -\Delta u_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \partial H_\varepsilon, \end{cases}$$

converges uniformly to the unique solution u of

$$(2) \quad \begin{cases} -\Delta u = f & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where "uniform convergence" means

$$\lim_{\varepsilon \to 0+} \sup_{x \in \overline{\Omega_\varepsilon}} |u_\varepsilon(x) - u(x)| = 0.$$

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- Solutions of elliptic bdry-value problems have a global smoothing property in Lebesgue spaces provided the underlying space of solutions $H^1(\Omega_\varepsilon)$ admits a "Sobolev-type inequality":

$$\|v\|_{L^{d^*}(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon)\,\|v\|_{H^1(\Omega_\varepsilon)} \qquad \forall\, v \in H^1(\Omega_\varepsilon),$$

where $d^* = \frac{2d}{d-2}$ if $d \geq 3$, & $d^* < \infty$ if $d = 2$ &

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2. Theorem (Dancer, Daners, H., 14)

Suppose that Assumption 1 is satisfied. Let $p > d/2$ and $f_\varepsilon \in L^p(\Omega_\varepsilon)$ for all ε.

Then $\exists\, \varepsilon_0 > 0$ & $M > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$ such that the solution u_ε of

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satisfies

$$\|u_\varepsilon\|_\infty \leq M \cdot \|f_\varepsilon\|_p \qquad \text{for all } \varepsilon \in (0, \varepsilon_0]$$

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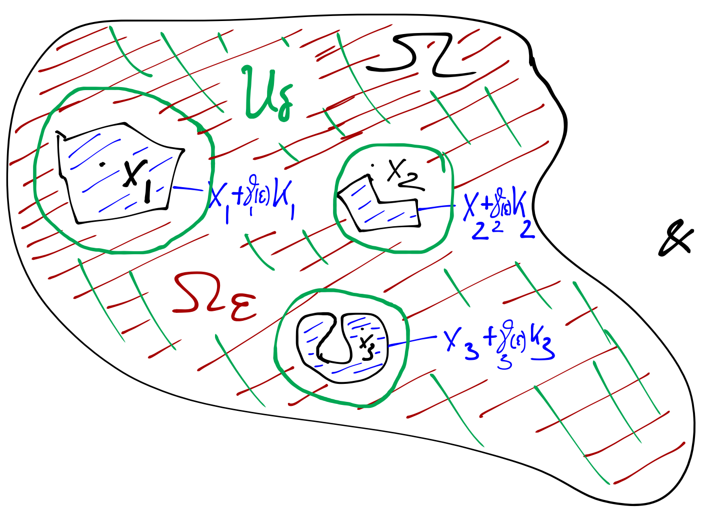
This means that the initial problem becomes a "local" problem.

$$\sup_{x \in \Omega_\varepsilon \setminus U_\delta} |u_\varepsilon(x) - u(x)| \to 0 \quad \text{as } \varepsilon \to 0+ .$$



& $\forall\, \delta > 0$ let

$$U_\delta := \Omega \setminus \bigcup_{j=1}^{m} B(x_j, \delta) .$$

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- We may assume that u_ε & u are harmonic close to the holes.

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By translation, we may assume that $x_1 = 0$.

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$\Longrightarrow$ We need to deal with only hole K_1 at a time.

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The new problem which need to consider instead of problem (1) is:

$$(\tilde{1}) \quad \begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \tilde{\Omega}_\varepsilon := B(0,1) \setminus \varepsilon K_1 \\ \dfrac{\partial u_\varepsilon}{\partial \gamma} + \beta_\varepsilon u_\varepsilon = 0 & \text{on } \varepsilon \partial K_1 \end{cases}$$

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$\rightsquigarrow$ v_ε is a weak solution of

$$(3) \quad \begin{cases} -\Delta v_\varepsilon = 0 & \text{in } \varepsilon^{-1}\tilde{\Omega}_\varepsilon = B(0, 1/\varepsilon) \setminus K_1 \\ \dfrac{\partial v_\varepsilon}{\partial \nu} + \varepsilon \tilde{\beta}_\varepsilon v_\varepsilon = 0 & \text{on } \partial K_1 \end{cases}$$

satisfying $\|v_\varepsilon\|_\infty = \|u_\varepsilon\|_\infty \leq M \quad \forall\, \varepsilon \in (0,1]$.

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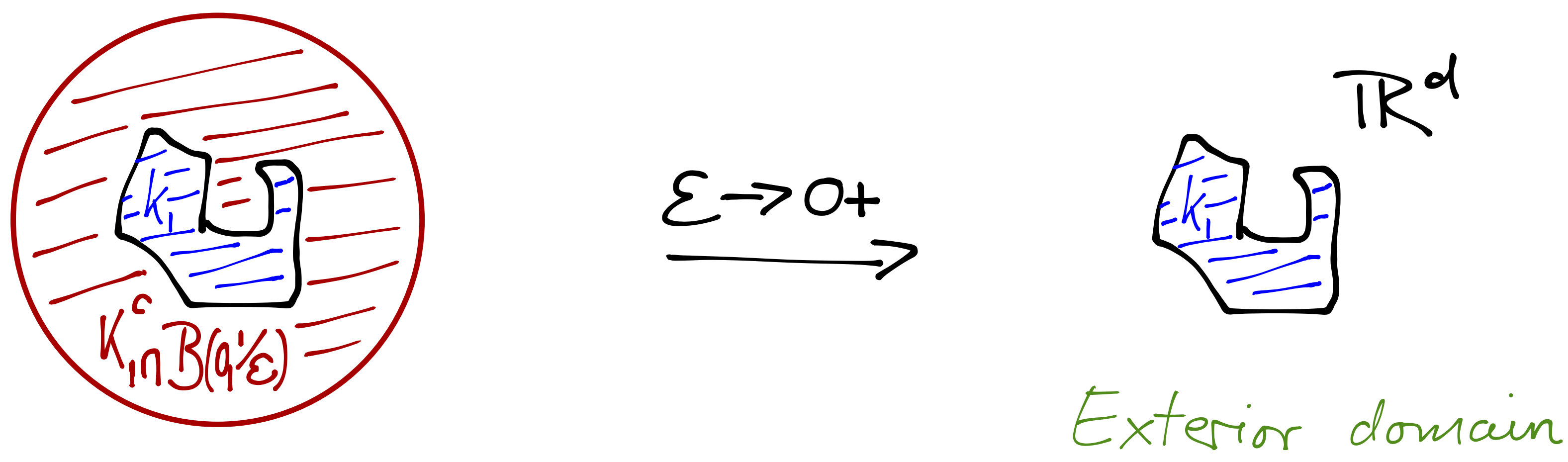
Letting $\varepsilon \to 0+$ in the rescaled problem



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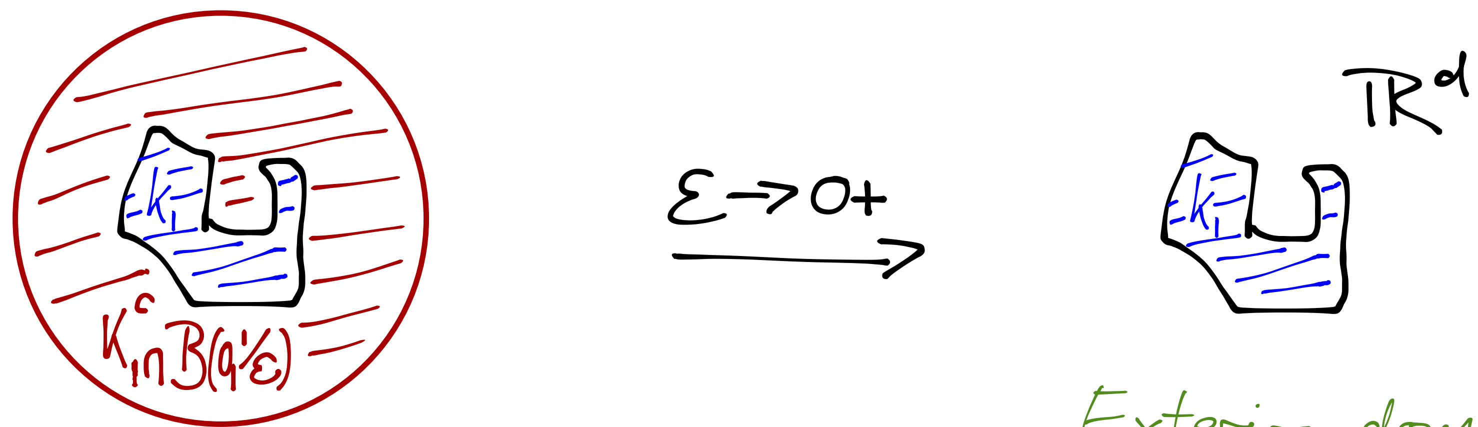

Exterior domain

Since $\|v_\varepsilon\|_\infty \le M$ $\forall \varepsilon > 0$ & by Assumption 1 on β_ε:

$\hookrightarrow$ (v_ε) is bdd in H^1_{loc}, $\hookrightarrow$ $v_{\varepsilon'} \to v$ in H^1_{loc}

v is a weak solution

of $$\begin{cases} -\Delta v = 0 & \text{in } K_1^c \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial K_1 \end{cases}$$

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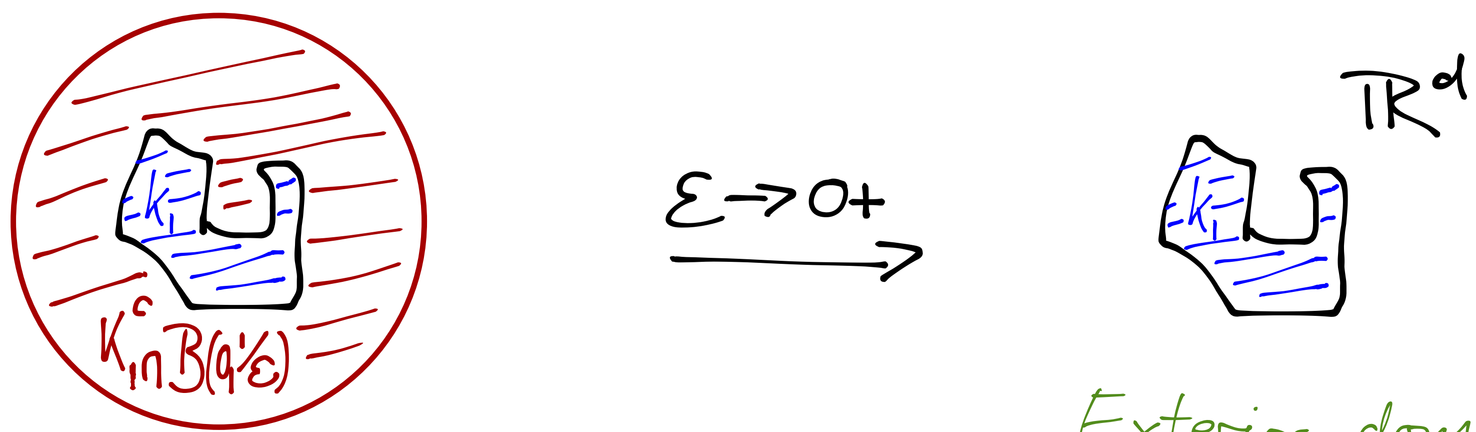

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Liouville thm ([Dancer, Daners, H. '13])

$\Longrightarrow$ $v \equiv v_0$ on K_1^c for some $v_0 \in \mathbb{R}$.

To prove our main Theorem we need to identify v_0.

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At this point, we need to treat the case $d \geqslant 3$ and $d = 2$ separately.

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Let $x = (r, \theta)$ where $r = |x|$ & $\theta \in \mathbb{S}_d$ being the unit sphere in $\mathbb{R}^d$.

If v is a harmonic function on an annulus $A_{r_1, r_2} := \{x \in \mathbb{R}^d \,|\, r_1 < |x| < r_2\}$ for some $0 < r_1 < r_2$,

then $\displaystyle \bar{v}(r) := \int_{\mathbb{S}_d} v(r, \theta)\, d\theta \quad \forall\, r \in (r_1, r_2)$

is a solution of $\displaystyle \frac{d}{dr}\left(r^{d-1} \frac{dv}{dr}\right) = 0$ on (r_1, r_2).

Proof of Theorem 1 for $d \geq 3$:

▷ We apply a transformation in polar coordinates to $v_\varepsilon, u_\varepsilon$ & u:

$\Longrightarrow$ $\bar{v}_\varepsilon$ satisfies $\frac{d}{dr}\left(r^{d-1}\frac{d\bar{v}}{dr}\right) = 0$ on $(R_1, 1/\varepsilon)$

where $R_1 := \sup\{|y| \mid y \in K_1\} < 1$ by assumption

$\bar{u}_\varepsilon$ & $\bar{u}$ satisfy this equation on $(\varepsilon R_1, 1)$.

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$\bar{u}_\varepsilon$ & $\bar{u}$ satisfy this equation on $(\varepsilon R_1, 1)$.

Since $\{1, E_d\}$ is a basis of the space of radially symmetric harmonic functions on any annulus & $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$

$$\Longrightarrow \quad \bar{v}_\varepsilon(r) = a_\varepsilon + b_\varepsilon E_d(r)$$

$$\bar{u}_\varepsilon(r) = a_\varepsilon + b_\varepsilon E_d(r/\varepsilon) = a_\varepsilon + b_\varepsilon \varepsilon^{d-2} E_d(r)$$

$$\bar{u}(r) = u(0) \quad (\text{Mean value property})$$

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& hence $\overline{v}_{\varepsilon_2'}(r) \to v_0$ $\forall r > 1$. (1)

Furthermore, $\overline{u}_\varepsilon(r) \to \overline{u}(r) = u(0)$ $(\varepsilon \to 0+)$ (2)

From (1) & since $\{1, E_d\}$ is linear independent

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Furthermore, $\bar{u}_\varepsilon(r) \to \bar{u}(r) = u(0) \quad (\varepsilon \to 0+)$ (2)

From (1) & since $\{1, E_d\}$ is linear independent

$\Longrightarrow b_{\varepsilon_2'} \to 0$ & $a_{\varepsilon_2'} \to v_0$

& so also $\varepsilon^{d-2} b_{\varepsilon_2'} \to 0$.

Using $\bar{u}_\varepsilon(r) = a_\varepsilon + b_\varepsilon \varepsilon^{d-2} E_d(r) \overset{(2)}{\Longrightarrow} \underline{\underline{u(0) = v_0}}$

$\square$

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We can complete the proof if we assume that

$$\mathbb{R}^2 \setminus K_1' \cong \mathbb{R}^2 \setminus B(0,1)$$

& by using the boundry cond. on $\partial \varepsilon K_1$. $\square$

Thank You!