Introduction to Schubert calculus and flag varieties.

§0. References.

- Michel Brion "Lectures on the geometry of flag varieties".
- Felice Ronga "Schubert calculus according to Schubert".

§1. Motivation.

I. Solution to Enumerative problems.

In 1879, Schubert proposed a method to solve enumerative problems in geometry.

- Choose basic obj: planes, lines, points $\mathbb{CP}^3$.
- Conditions represented by symbols: $x, y, z, \cdots$

$\wedge \quad \vee$

$$\begin{cases} x \cdot y := \text{both are satisfied} \\ x + y := \text{one of } x \text{ or } y \text{ are satisfied or both} \end{cases}$$

Hidden computations in cohomology.

e.g. Points in $\mathbb{CP}^3$.

generic position

Conditions

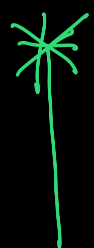
- p_H : the point lye in a plane
- p_ℓ : the point lye in a line
- p : the point is given (specific point)

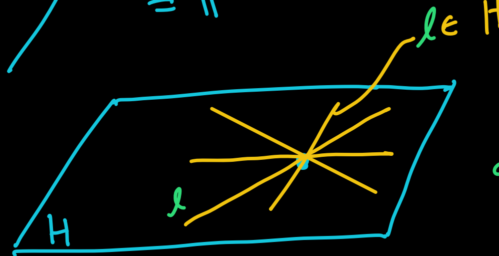
p

$$p_H^2 = p_\ell, \quad p_H^3 = p_\ell p_H = p.$$

e.g. Lines in $\mathbb{C}\mathbb{P}^3$

For this example:



- line =  $= \mathbb{P}^1$
- pencil = 

dim = 1

Condition

3 $\quad g_\ell$: the line cuts a given line

2 $\quad g_H$: the line lies in a given plane

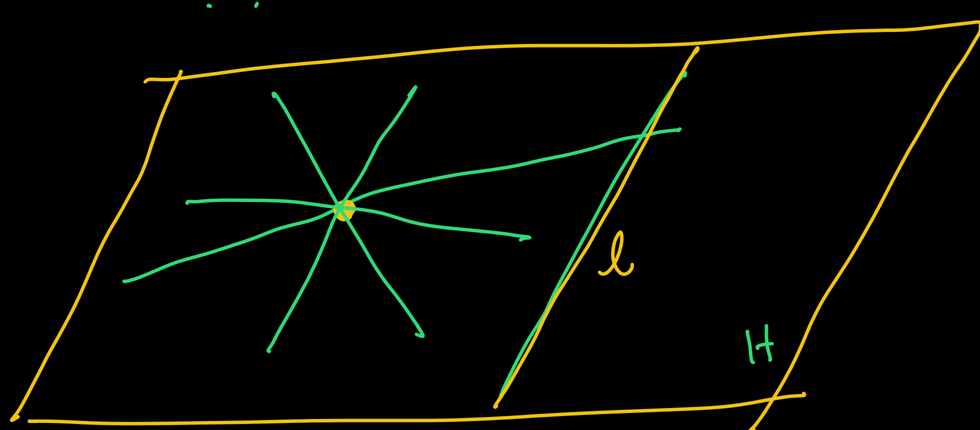
2 $\quad g_p$: the line pass through a point

1 $\quad \underline{g_c}$: the line belongs to a pencil

0 $\quad \underline{G}$: the line is given

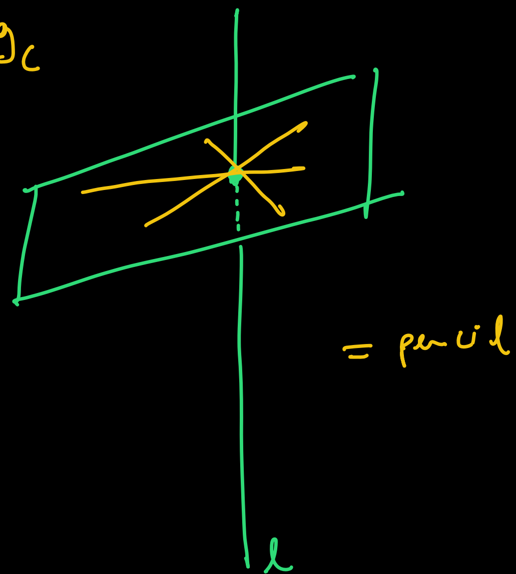
Relations

1) $g_\ell \cdot g_p = g_c$



= pencil

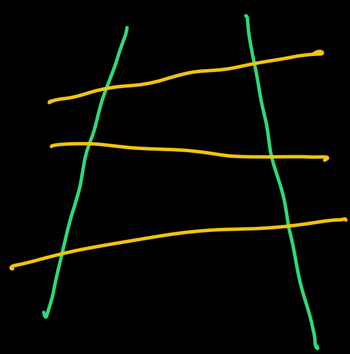
2) $g_\ell \cdot g_H = g_C$



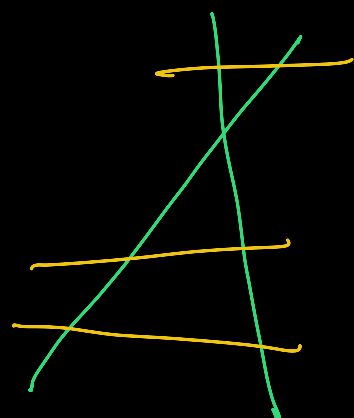
3) $g_\ell \cdot g_C = G$

4) $g_p \cdot g_H = 0$

5) $g_\ell^2 =$



- you can suppose both lines intersect
- justification for this requires work!



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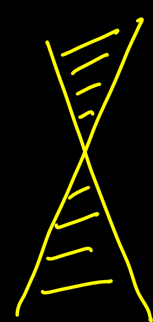
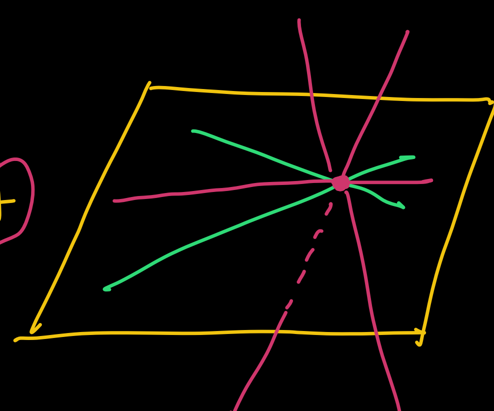


ruled surface

$=$

$+$



$$g_\ell^2 = g_H + g_p$$

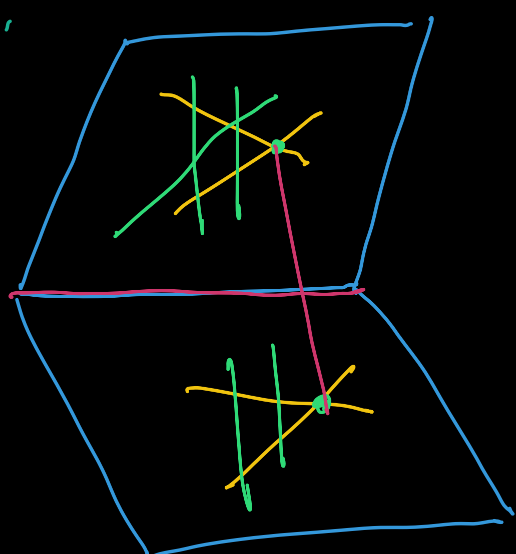
Question. How many lines intersect 4 generic lines? 2

$$g_\ell^4 = g_\ell g_\ell g_\ell^2 = g_\ell g_\ell (g_H + g_P)$$

$$= \underline{g_\ell^2 \cdot g_\ell^2} \quad = g_\ell \underbrace{(g_\ell g_H)}_{g_c} + g_\ell \underbrace{(g_\ell g_P)}_{g_c}$$

$$= 2 \underbrace{g_\ell g_c} = 2G$$

Alternatively,



2

Hilbert 15th problem. Foundation for this!

II. Modular representation theory.

G s.s. linear alg. group $\mathbb{C}$ $\quad (SL_n(\mathbb{C}))$

$G \supset B \supset T$

Borel $\quad$ torus

G/B

1-dim T-module $\longrightarrow$ 1-dim B-module $\longrightarrow$ G-module

$H \subset G \qquad s \cdot [G:H]$

$V_{g_i} \xrightarrow{\varphi_{ij}} V_{g_j}$

$V \in \mathrm{Rep}\, H \longrightarrow \bigoplus_{g_i \in G/H} V_{g_i}$

$g \cdot v = \varphi_{ij}(h \cdot v)$

$G/H = \{g_1 H, \dots, g_n H\}$

$g \cdot v$

$v \in V_{g_i} \qquad g\, g_i = g_j h$

e.g. $G = SL_2(\mathbb{C})$, $X(T) \cong \mathbb{Z}$

$$m \in \mathbb{Z} \longmapsto \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \longmapsto x^m \right)$$

Rep G

char 0

$$\text{1-dim } T \text{ module} \longrightarrow \text{1-dim } B \text{ mod} \longrightarrow$$

$$W(m) \qquad \left(\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \longmapsto x^m \right)$$

$$B = UT$$

$$\mathrm{Ind}_B^G(W_m) = \Gamma(G/B, \underline{G \times^B W_m})$$

$$= \Gamma(\mathbb{P}^1, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[x,y]_m & m \geq 0 \\ 0 & o/w \end{cases}$$

simple $\underline{m+1}$ $Sl_2(\mathbb{C})$

$\boxed{\text{Char} = p}$ $\mathrm{Ind}_B^G(W(\lambda)) \supset$ simple.

§2. The Grassmannian $Gr(d,n)$.

V v.s over $\mathbb{C}$ $\qquad \Lambda V := T(V) / \langle x \otimes x \mid x \in V \rangle$

$x, y \in V$, $x \wedge y = -y \wedge x$.

$k \in \mathbb{N}$ $\quad \underline{\Lambda^k(V)} \subset \Lambda(V)$ spanned

$x_1 \wedge \cdots \wedge x_k$ totally decomposable vector in $\Lambda^k V$

If $\{e_1, \dots, e_n\}$ basis of $V = \mathbb{C}^n$

$$\{ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}$$

is basis of $\Lambda^k(V)$ $\leftarrow$ $\dim = \binom{n}{k}$

The Grassmannian $Gr(d,n)$ is the set

$$\{ E \subset \mathbb{C}^n \mid \dim_{\mathbb{C}} E = d \}$$

$\{v_i\}^d, \{w_i\}^d$ are two basis $E \in Gr(d,n)$

$Q \in \underline{GL(E)}$ $\quad Q v_i = w_i \quad Q$

$$v_1 \wedge v_2 \wedge \cdots \wedge v_d = \overbrace{\det(Q)}^{\neq 0} w_1 \wedge w_2 \wedge \cdots \wedge w_d .$$

$\Rightarrow \underline{\mathbb{P}(\Lambda^d \mathbb{C}^n)}$, $\quad [v_1 \wedge v_2 \wedge \cdots \wedge v_d] = [w_1 \wedge \cdots]$

$$\iota : Gr(d,n) \longrightarrow \mathbb{P}\left(\Lambda^d \mathbb{C}^n\right) \cong \mathbb{P}^{\binom{n}{d}-1}$$

$$E \longmapsto [v_1 \wedge \cdots \wedge v_d]$$

Plücker embedding. Injection $\Lambda \mathbb{C}^n$

$$v_1 \wedge \cdots \wedge v_d \in \Lambda^d \mathbb{C}^n$$

$$\underline{v_1 \wedge \cdots \wedge v_d} = \sum p_{i_1 i_2 \cdots i_d} \cdot e_{i_1} \wedge \cdots \wedge e_{i_d}$$

$$[\, p_{123\cdots d} : p_{134\cdots} : \quad]$$

$Im(\iota)$ = totally decomposable vectors $\Longrightarrow$ $x \wedge x = 0$

convolution

$n=4,\ d=2$

$\Lambda^k V,\ \Lambda^k V^*$

$$v_1 \wedge \cdots \wedge v_n \wedge x = 0$$

$$\varphi \circ f = 1$$

Useful criteria

e.g. $\mathbb{P}(\Lambda^2 \mathbb{C}^4)$ $\quad n=4, \; d=2$, $\quad x = e_1\wedge e_2 + e_3 \wedge e_4 = \underline{f} \wedge \underline{g}$ $\quad f,g \in V$

$$x \wedge x = e_1\wedge e_2 \wedge e_3 \wedge e_4 + \underbrace{(-1)^4}_{1} e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$$

$\Rightarrow x$ is not t.d.

Theorem $Gr(d,n)$ is a proj. variety.

e.g. $d=1$, $Gr(d,n) = \mathbb{P}^{n-1}$

$d=2, n=4$, $\quad Gr(2,4) \subset \mathbb{P}^5 \qquad \binom{4}{2} = 6$

$$\Lambda^2 \mathbb{C}^4 \xrightarrow[\text{basis}]{} \left\{ \begin{matrix} e_1\wedge e_2, & e_2 \wedge e_3 \\ e_1 \wedge e_3, & e_2 \wedge e_4 \\ e_1 \wedge e_4, & e_3 \wedge e_4 \end{matrix} \right\}$$

$$x = P_{12}\, e_1 \wedge e_2 + P_{13}\, e_1 \wedge e_3 + \cdots$$

$x \wedge x = 0$

$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0$$

homogeneous polynomial.

$GL_n(\mathbb{C}) =: G \curvearrowright X = Gr(d,n)$ transitive action

$E_{123\ldots d} = \langle e_1, e_2, \ldots, e_d \rangle$

$$P = \mathrm{Stab}_G E_{12\ldots d} = \left\{ \begin{pmatrix} \boxed{d\times d} & \boxed{*} \\ \boxed{\begin{matrix} 0 \\ n-d\times d \end{matrix}} & \boxed{*} \end{pmatrix} \right\} \supset \text{upper } \Delta \text{ matrices} = B$$

$G/P \simeq X$ smooth

$\dim X = d(n-d) = \dim G - \dim P$

$T = \{\mathrm{diag}(a_{ii})\} \subset B$
$a_{ii} \neq 0$

$G \curvearrowright X$

$G \curvearrowright V = \bigwedge^d \mathbb{C}^n$ more useful

$$T \curvearrowright V \Rightarrow V = \bigoplus_{\lambda \in X(T)} V_\lambda$$

simultaneous diagonalisation

$V_\lambda = \{ v \in V \mid t \cdot v = \lambda(t) v \ \forall t \in T \}$

$X(T) \cong \mathbb{Z}^n$

$$\begin{array}{ccc} \mathrm{diag}(a_{ii}) & \longleftarrow\!\!\!\longrightarrow & (m_1, m_2, \ldots, m_n) \\ \downarrow & & \\ a_{11}^{m_1} a_{22}^{m_2} \cdots a_{nn}^{m_n}. & & \end{array}$$

e.g. $n=4$, $d=2$ $\qquad \binom{4}{2}=6 \qquad \bigwedge^2 \mathbb{C}^4 \qquad e_i \wedge e_j$

$$t = \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix}, \quad t \cdot e_1 \wedge e_2 = te_1 \wedge te_2$$

$$= ae_1 \wedge be_2$$

$$= ab \; e_1 \wedge e_2$$

$$= a^1 b^1 c^0 d^0 \, e_1 \wedge e_2$$

$$\lambda = (1,1,0,0) \in \underset{X(T)}{\underset{\parallel}{\mathbb{Z}^4}}$$

$$t = \begin{pmatrix} ab & & & & & \\ & ac & & & & \\ & & ad & & & \\ & & & bc & & \\ & & & & bd & \\ & & & & & cd \end{pmatrix}$$

$$\bigwedge^2 \mathbb{C}^4 = \bigoplus_{\lambda \in X(T)} V_\lambda$$

$(1,1,0,0)$

$(1,0,0,1)$

$\dots$

Thm. $T \subset G \curvearrowright V$, $V = \bigoplus_{\lambda \in X(T)} V_\lambda$. Then

$[v] \in \mathbb{P}(V)$ is T-fixed $\iff$ $v \in V_\lambda$ for some $\lambda \in X(T)$

Cor. $X = Gr(d,n)$ $\quad I = \{1 \le i_1 < i_2 < \cdots < i_d \le n\}$

$X^T = \{E_I\}$, $\quad E_I := \langle e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \rangle$

Also, $X = \bigsqcup_I BE_I$, let us see some B-orbits.

e.g. $I = \{1,2,3\}$

$\begin{pmatrix} * \\ 1 \\ 0 \\ 0 \end{pmatrix} \uparrow$

$E_I = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{B-}{\rightsquigarrow} \begin{pmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow BE_I = E_I$$

dim $= 0$

$I = \{1,3,4\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{B}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow BE_I$$

dim $= 2$

$I = \{i_k\}$, $|I| = \sum i_k - k$, then $BE_I \simeq \mathbb{A}_{\mathbb{C}}^{|I|}$

$$\begin{array}{ccc} 1 & 3 & 4 \\ 1 & 2 & 3 \\ \hline & 1 & 1 \end{array} = 2$$

$B E_I =: C_I$ the Schubert cell

$X_I := \overline{C_I}$ the Schubert variety

Define $\{i_k\} = I \geq J = \{j_k\}$ if $i_k \geq j_k \ \forall k$.

We have

$$X_I = \bigsqcup_{I \geq J} C_J = \bigsqcup_{I \geq J} B E_J .$$

<u>Proposition</u>. $X = Gr(d,n)$ $n_j^I = \#\{k \mid 1 \leq k \leq d : i_k \leq j\}$.

- $C_I = \left\{ E \in Gr(d,n) \;\middle|\; \begin{array}{l} \dim E \cap \langle e_1, \dots, e_j \rangle = n_j^I \\ \text{for } j = 1, \dots, n \end{array} \right\}$

- $X_I = \left\{ E \in Gr(d,n) \;\middle|\; \begin{array}{l} \dim E \cap \langle e_1, \dots, e_j \rangle \geq n_j^I \\ \text{for } j = 1, \dots, n \end{array} \right\}$

<u>e.g.</u> $I = \{1,3\}$ $d=2, n=4$, $|I| = 1$

$C_{13} \subset X_{13}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$C_I \cong \mathbb{A}^1_{\mathbb{C}}$

$E_{12} = \langle e_1, e_2 \rangle$

Can obtain $X_{13} = \overline{C_I}$

$X_{13} = E_{12} \sqcup C_{13}$

$E_{12} \in X_{13}$ because:

$\dim E_{12} \cap \langle e_1 \rangle = 1 \geq 1 = n_1$

$\dim E_{12} \cap \langle e_1, e_2 \rangle = 2 \geq 1 = n_2$

$\dim E_{12} \cap \langle e_1, e_2, e_3 \rangle = 2 \geq 2 = n_3$

$\dim E_{12} \cap \langle e_1, e_2, e_3, e_4 \rangle = 2 \geq 2 = n_4$

Useful trick: Zariski closure of C_I $\Leftrightarrow$ Analytic closure of C_I.

$x_n \to x$

$\Rightarrow$

$m \in \mathbb{N}$ $$\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1/m & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[m\to\infty]{} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{12}$$

$\Rightarrow X_{13} = C_{13} \sqcup C_{12}$

poset

Gr(2,4)

$$
\begin{tikzcd}[column sep=small]
& C_{34} = X_{34} = X \arrow[d, dash] & \\
& X_{24} \arrow[dl, dash] \arrow[dr, dash] & \\
X_{14} \arrow[dr, dash] & & X_{23} \arrow[dl, dash] \\
& X_{13} \arrow[d, dash] & \\
& X_{12} = E_{12} &
\end{tikzcd}
$$

$G/B \hookrightarrow$

dim	
4	$\lvert 34 \rvert = 4$
3	
2	
1	
0	

<u>E_{12} is singular in X_{24}</u>.

$X_{24} \supset C_{24}$

$\dim X = 4$

$\dim X_{24} = 3$.

$$\begin{pmatrix} e_4 \\ e_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

$[e_1 \wedge e_4 + e_2 \wedge e_4] \in X_{24}$ etc...

E_{34}

$e_3 \wedge e_4 \notin X_{24}$

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

p_{ij} $\quad e_i \wedge e_j$

$$p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} = 0.$$

We have <u>$p_{34} = 0$</u> then

<u>$X = \mathrm{Proj}$</u>$\left(\mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] \middle/ \langle p_{12} p_{34} - p_{13} p_{24} + p_{14} p_{23} \rangle \right)$

$+ \underline{p_{34} = 0} \to X_{24}$

$E_{12} = e_1 \wedge e_2 = [1:0:0:0:0:0]$

Take $U_{12} = \{ p_{12} \neq 0 \} \ni E_{12}$, $E_{12} = (0,0,0,0,0) \in \boxed{\mathbb{A}^5} \cong U_{12}$

$T_{E_{12}} U_{12} = T_{E_{12}} \mathbb{A} \cap \{ \underline{f} \bmod m^2 = 0 \}$ $\quad m/m^2$

$p_{12} = 1$

$p_{34} = 0$ $\quad f = p_{34} - p_{13} p_{24} + p_{14} p_{23}$

$\Rightarrow T_{E_{12}} U_{12} = \{ p_{34} = 0 \}$ $\quad \dim X_{24} = 3$

Miracle, $\quad X_{24} = X \cap T_{E_{12}} X$

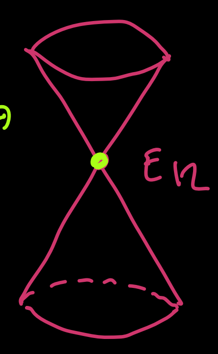
$p_{13} p_{24} = p_{14} p_{23}$

quadratic cone

$\mathbb{A}^5$

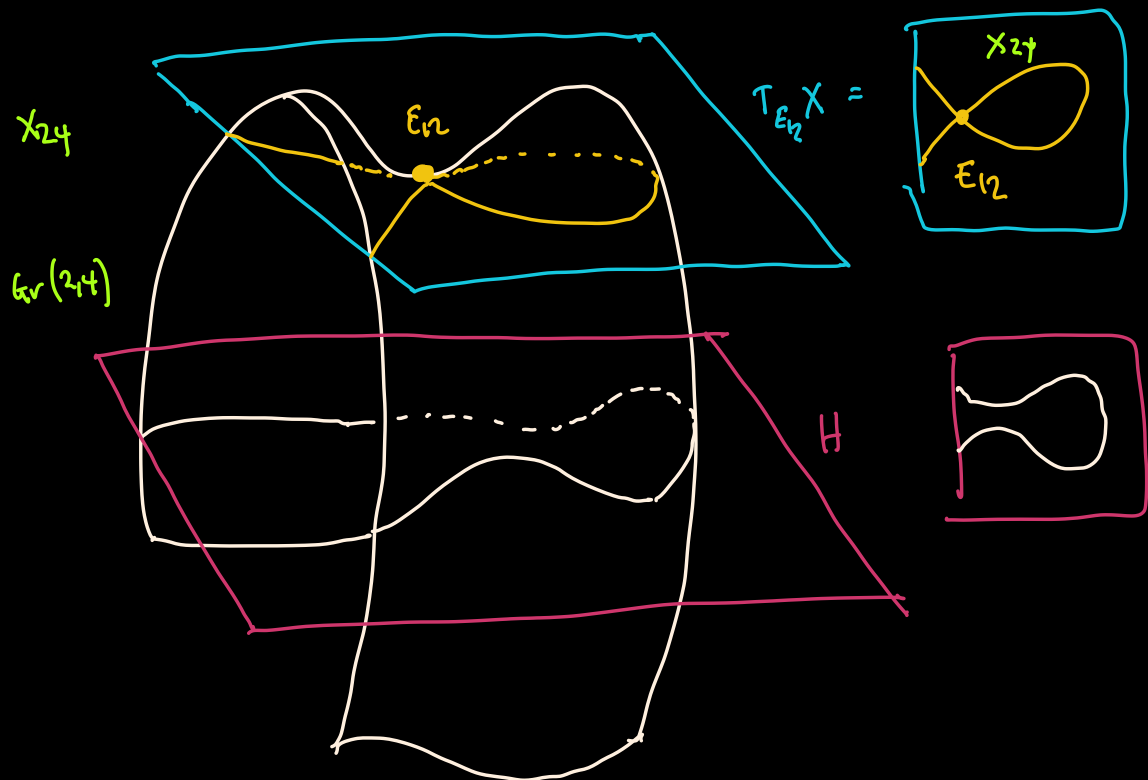
tangent

$T_{E_{12}} Gr(2,4)$



$\underline{X_{24}}$

We can imagine the picture by reducing one dimension



$0 = V_0 \subset V_1 \subset \cdots \subset V_d = \mathbb{C}^n \leftarrow \quad 0 \subset V \subset \mathbb{C}^n$

$G \supset P \supset B$

$\dim V_{i+1}/V_i -$

$G/P(d_1, \ldots, d_m)$