Humphreys's book

§7.4 Hecke algebras and inverses

$T_s T_w = T_{sw}$ $\ell(sw) > \ell(w)$ $s\in S$ $w\in W$

$T_s^2 = (q-1)T_s + qT_e.$ $\forall s\in S$

$T_s^{-1} = q^{-1}T_s - (1-q^{-1})T_e$

$= q^{-1}(T_s - (q-1)T_e)$

$T_w = T_{s_1}T_{s_2}\cdots T_{s_r}$ if $(s_1,\dots,s_r)$ is rex

Lemma $sw<w$, $x<w$

(a) $sx<x \Rightarrow sx<sw$

(b) $sx>x \Rightarrow sx\le w$ and $x\le sw$

Either case $sx\le w$.

$\varepsilon_w = (-1)^{\ell(w)}$, $q_w = q^{\ell(w)}$

Prop. For $w\in W$,

$$(T_{w^{-1}})^{-1} = \varepsilon_w q_w^{-1}\sum_{x\le w}\varepsilon_x R_{x,w}(q)T_x,$$

where $R_{x,w}\in\mathbb{Z}[q]$ w/ $\deg = \ell(w)-\ell(x)$ in q

and $R_{w,w}(q) = 1$.

e.g. $w=e.$ $T_{e^{-1}}^{-1} = T_e$ $\overset{\ell(e)=0.}{\varepsilon_w = 1 = q_w = \varepsilon_e = q_e}.$

$T_s^{-1} = T_{s^{-1}}^{-1} = q^{-1}T_s - (1-q^{-1})T_e.$

$= \varepsilon_s q_s^{-1}(\varepsilon_s R_{s,s}(q)T_s + \varepsilon_e R_{e,s}(q)T_e)$

$= q^{-1}T_s \underbrace{+(-1)q^{-1}R_{e,s}(q)}_{(-1)(1-q^{-1})}T_e.$ $= -q^{-1}(-T_s + (q-1)T_e).$

$R_{e,s} = (q-1).$

$R_{x,w}\neq 0$ iff $x\le w$.

§7.5 Computing the R-polynomials

$R_{w,w}=1$, $R_{x,w}$ we want. Assume we know $R_{y,z}$ $\forall z$ $\ell(z)<\ell(w)$. Fix $s\in S$. $sw<w$

$(sw<w)$ Two configs:

(A) $x<w$, $sx<x$ (forcing $sx<sw$).

$R_{x,w} = R_{sx,sw}$. Known!

(B) $x<w$, $x<sx$ (forcing $sx\le w$ and $x\le sw$)

$R_{x,w} = (q-1)R_{x,sw} + qR_{sx,sw}$

Deodhar's formula:

$$R_{x,w} = \sum_{\sigma\in D(x)} \underbrace{(q-1)^{n(\sigma)}q^{m(\sigma)}}_{p(\sigma)}$$

Better feeling:

$\ell(w)-\ell(x)=1$. $w=s_1\cdots s_r$, $x=s_1\cdots\hat{s}_i\cdots s_r.$

e.g. $w=sts$, $x=st$, ⒶⒶ

$sx=t<x.$

$R_{x,w} = R_{st,sts} = R_{t,ts} = (q-1) = R_{1,s}$

$T_s^{-1} = q^{-1}T_s - (1-q^{-1})T_e$

$T_{st}^{-1} = T_{(ts)^{-1}}^{-1} = T_t^{-1}T_s^{-1} = (q^{-1}T_t - (1-q^{-1}))(q^{-1}T_s - (1-q^{-1}))$

$= q^{-2}\underbrace{T_tT_s}_{T_{ts}} - q^{-1}(1-q^{-1})T_t - q^{-1}(1-q^{-1})T_s + (1-q^{-1})^2$

$= (-1)^2q^{-2}\Big((-1)^2\cdot\underset{R_{w,w}=1.}{1}\cdot T_{ts} - q(1-q^{-1})T_t - q(1-q^{-1})T_s + (q-1)^2\Big)$

$(-1)(q-1)T_t + (-1)(q-1)T_s$

$R_{t,sts} = (q-1)$

~~Other ways. (?) $t=sst<st=w$ $st>t$.~~

~~$R_{t,st} = (q-1)R_{t,t} + qR_{st,t}^{\to 0} = (q-1)$.~~

Other way: $R_{t,ts} = R_{1,s} = (q-1)$.

$R_{ts,sts} = R_{ts,tst} = R_{s,st} = R_{e,t} = (q-1),$

$R_{t,st} = (q-1)R_{t,t} + 0 = (q-1).$ $R_{t,t} = R_{e,e} = 1.$

$w = stmst\overset{\text{rex}}{ms\,r},$ $x = sust\overset{\text{rex expr.}}{ms\,r}\,st$ $\ell(x)=\ell(w)-1$

$R_{x,w} =$

$R_{ustus,tustus}$

$= (q-1)R_{x,x} + q^{\to 0}.$ $w=s_1\cdots s_r,$

$\ell(w)-\ell(x)=2$ $s_1\cdots\hat{s}_i\cdots\hat{s}_j\cdots s_r =: x$,

e.g. $x = stus$

$w = sututs.$

$\rightsquigarrow \overset{x}{\overset{\shortparallel}{tus}}$ $\overset{w}{\overset{\shortparallel}{ututs}}$ $ux = utus$, $uw = tuts$.

$R_{x,w} = (q-1)R_{x,uw} + qR_{ux,uw}$

~~$(x\not\le uw)\ \&\ (ux\not\le w)$~~

~~$= (q-1)\underbrace{R_{tus,tuts}}_{q-1} + q^{\to 0} = (q-1)^2$~~ .. no need!

$R_{x,w} = R_{tu,utut}.$

$= (q-1)R_{tu,tut} + qR_{utu,tut}.$

$= (q-1)^2 + 0.$

More on Deodhar's formula:

$w = s_1 \cdots s_r$. $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_r) \in W^{r+1}$.

$\sigma_0 = 1$ $\sigma_j = \sigma_{j-1}$ or $\sigma_j = \sigma_{j-1} s_j$ $(1 \leq j \leq r)$

$\boxed{\text{Bruhat stroll}}$

$\sigma = s_1 \dots \hat{s}_{i_1} \cdots \hat{s}_{i_p} \cdots s_r$

$\{i_1, \dots, i_p\} = \{j \mid \sigma_j = \sigma_{j-1}\}$

$n(\sigma) := p = \# D0 + \# U0 = \# 0's$.

$m(\sigma) := \mathrm{Card}\{j \mid \sigma_{j-1} > \sigma_j\} = \# D1's$.

"distinguished" subexpressions

$D(x) = \{\sigma \text{ s.t. } x = \sigma_r \ \& \ \sigma_j \leq \sigma_{j-1} s_j \ \forall_j\}$

e.g. $w = \overset{\circ}{s}t\overset{\circ}{s}tsts$, $b = (1,0,1,0,1,1,1)$

UUDUUUU.

$ssst s = sts$

$\sigma_0 = 1$ $\sigma_1 = s$, $\sigma_2 = s$, $\sigma_3 = ss = e$, $\sigma_4 = e$.

$\sigma_5 = s$, $\sigma_6 = st$, $\sigma_7 = sts$.

distinguished condition $\forall_j$ $\sigma_j \leq \sigma_{j-1} s_j$. $\forall_j \geq 1$

Is b distinguished?

$\sigma_1 = s \leq \sigma_0 s = s$ ✓.

0 $\sigma_2 = s \leq \sigma_1 t = st$ ✓.

~ $\sigma_3 = e \leq \sigma_2 s = e$ ✓.

0 $\sigma_4 = e \leq \sigma_3 t = t$ ✓.

$\sigma_5 = s \leq \sigma_4 s = s$ ✓

$\sigma_6 = st \leq \sigma_5 t = st$ ✓.

$\sigma_7 = sts \leq \sigma_6 s = sts$ ✓.

then => "distinguished"

~~$n(\sigma) = 0$~~ $= 2$

$n(\sigma) = \# 0's$

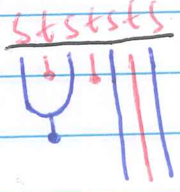
$= 2$.

$m(\sigma) = \# D1's$

$= 1$

$P(\sigma) = (v-1)^2 q$

stststs



defect $\# U0 - \# \underset{\sim}{D0}$.

no hubo D0!

→ Libedinsky light leaf!!

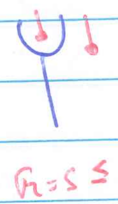
$w = stst$.

1 0 0 0. defecto = 1 = 2 - 1.

U U D U

$\sigma_0 = e$. $\sigma_2 = s$ $\sigma_4 = s$

$\sigma_1 = s$ $\sigma_3 = s$



$\sigma_0 = e$ $\sigma_1 = s \leq \sigma_0 e = s$ ✓

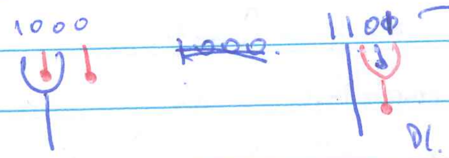
$\sigma_2 = s \leq \sigma_1 t = st$ ✓.

$\sigma_3 = s \leq \sigma_2 s = e = ss$ ✗. Not distinguished!

Note: No distinguished ⟺ Hay D0.

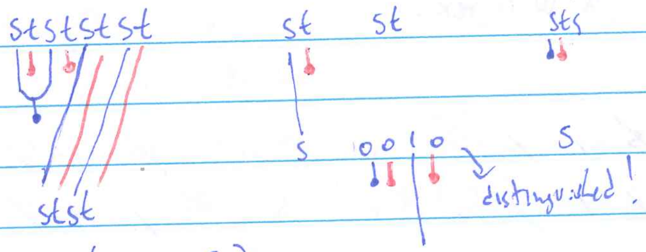
Thm: $x \leq w \Rightarrow D(x) \neq \emptyset$! 1101.

1000 ~~1000~~. 1100 → distinguished!



D1. for s.

ststststst st st sts

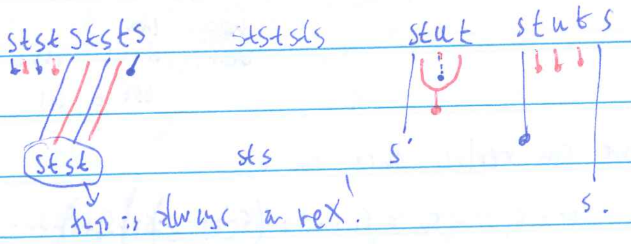


stst s 0010 s distinguished!

$\underline{w} = (s_1, \dots, s_r)$

$\underline{w}^e = (s_1^{e_1}, \dots, s_r^{e_r})$ $\underline{w}^e_{\bullet} = s_1^{e_1} \cdots s_r^{e_r}$.

ststststs ststsls stut stuts



stst → this is always a rex! sts s. s.

Lemma: [illegible]

$\underline{w}^e_{\bullet} < w$, pick $\exists$ a rex for $\underline{w}^e$.

$\underline{w}^e =$ [illegible]. ~~The~~ st. $\exists$ s_{i_j} $j \in \{1, \dots, r\}$

$s'_1 \cdots s'_r$.

and s.t. $s'_j = s_{i_j}$ and $j > j' \Rightarrow i_j > i_{j'}$.

"Proof": by induction. (Build a "super" distinguished)

i.e. no D1! no D0!?.

replace

e.g. $s_1 s_2 s_3 s_1 s_3$ by $s_1 s_2 s_3 s_1 s_3$

1 1 1 0 1 1 1 0 0 0.

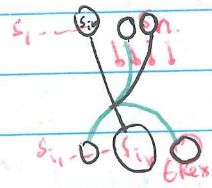
$\underline{w}^e_{\bullet} = s_1 s_2 s_3 s_3$ $\underline{w}^e = s_1 s_2$.

$= s_1 s_2$

this is the same as picking $\underline{w}^e = (s_1^{e_1}, \dots, s_n^{e_n})$

remove identities (only remain $e_i = 1$).

~~wre~~ $\underline{w}^e_{red} = (s_{i_1}, \dots, s_{i_r})$



$st^{00} = e$

$ut^{00} = e$

$st = ts$

sts = tst.

sts = ~~wtstu~~

~~st~~

wtsu.

$\not\!\!M$ st

ts

Not finished ☹.

e.g $\ell(w)-\ell(x)=1$. $1\,1\,1\,1\,1\,0\,1\cdots 1$.

$w = s_1 \cdots s_r$. , $x = s_1 \cdots \hat{s}_i \cdots s_r$.

and no cancelations. stst

that i is unique if w is rex.

Supp. $x = s_1 \cdots \hat{s}_i \cdots s_r$

and $x = s_1 \cdots \hat{s}_j \cdots s_r$.

$i \neq j$ (i.e.) $i<j$.

$\cancel{s_1} \cdots \cancel{s_{i-1}} \cdot \cancel{s_i} \cdot s_{i+1} \cdots s_{j-1} \cdot s_j \cdot \cancel{s_{j+1}} \cdots \cancel{s_r}$.

$= \cancel{s_1} \cdots \cancel{s_{i-1}} \cdot s_i \cdot s_{i+1} \cdots s_{j-1} \cdot \cancel{s_j}\, \cancel{s_{j+1}} \cdots \cancel{s_r}$.

$\Rightarrow s_{i+1} \cdots s_j = s_i \cdots s_{j-1} = w$

2 rex of w.

~~Counterexample?~~ ~~$w = stst$~~.

~~100 101.~~ ~~1~~ ~~0111~~ 1110

~~tst = sts~~

→ we can replace this in w.

$w = s_1 \cdots s_r = s_1 \cdots s_i \cdot (s_{i+1} \cdots s_j) s_{j+1} \cdots$

$= s_1 \cdots \cancel{s_i} (\cancel{s_i} \cdots s_{j-1}) s_{j+1} \cdots$

no rex ~~X~~.

∧ i unique ∴ $\# D(x) = 1$

∴ $R_{x,w} =$

there is only one 0. and is of course a U0.

since w is rex!.

$n(\sigma) = \#\, 0\text{'s} = 1 = \#\, U0\text{'s}$

$m(\sigma) = \#\, D1\text{'s} = 0$.

$$\begin{array}{c} \qquad\qquad\quad i \\ \underbrace{1 \cdots 1\,1\,1\,\,0}_{\substack{\text{since } w \\ \text{is rex!}}}\,\underbrace{1\,1\,1 \cdots 1\,1}_{\substack{\text{since no more} \\ \text{cancelations } \ell(w)-\ell(x)=1!}} \\ U \cdots U\,U\,U\,U\,U \cdots U\,U\,U \end{array}$$

∴ $R_{x,w} = (q-1)^1 q^0 = q-1$

$\ell(w)-\ell(x)=2$

$w = s_1 \cdots s_r$ $x = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$.

i, j unique! same reason as before! no cancelation

$1\,1\,1 \cdots 1\,0\,1 \cdots 1\,0\,1\,1\,1\,1$

$U\,U\,U\,U\,U\,U\,U\,U\,U\,U\,U\,U\,U\,U\,U$.

$n(\sigma)=2$, $m(\sigma)=0$.

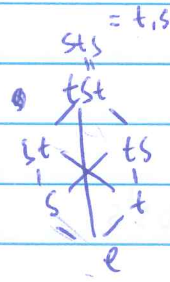
type A_2. $sts = tst$.

$W = S_3 = \langle s_1, s_2 \rangle = \langle s, t \rangle$

$R_{e,s} = R_{e,t} = (q-1)$ $\quad R_{e,st} = R_{e,ts} = R_{s,w_0} = R_{t,w_0} = (q-1)^2$

$= R_{s,st} = R_{t,ts}$

$= t,st = R_{s,ts} = R_{ts,w_0} = R_{st,w_0}$.



$R_{x,w} = (q-1)$ for 8 pairs.

$R_{x,w} = (q-1)^2$ for 4 pairs.

R_{e,w_0}.

sts.

101 and 000.

σ 1 1 1 σ

both distinguished!!

101 000

UUD UUU

$n(\sigma) = 1$ $\quad n(\sigma) = 3$

$m(\sigma) = 1$ $\quad m(\sigma) = 0$.

$R_{e,w_0} = (q-1)q + (q-1)^3$

Distinguished

no D0

means that:

$\#0\text{'s} = \#U0\text{'s}$

$\#D\text{'s} = \#D1\text{'s}$.

$\boxed{sw<w}$

A) $x<w$, $sx<x$

(forcing $sx<sw$)

$R_{x,w} = R_{sx,sw}$.

B) $x<w$, $sx>x$

(forcing $sx \leq w$, $x \leq sw$)

$R_{x,w} = (q-1) R_{x,sw} + q R_{sx,sw}$

$R_{e,sts}$. $s \leq ts$, $e<ts$.

$R_{e,w_0} = (q-1) R_{e,ts} + q R_{s,ts}$.

$= (q-1)(q-1)^2 + q(q-1)$

$= (q-1)^3 + q(q-1)$.

$s\,sts = stst = tstt = ts$ ✓.

(from Bjorner-Brenti book)

Kazhdan - Lusztig polynomials

Thm/Def: There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v\in W} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:

(i) $P_{u,v}(q)=0$, if $u\not\leq v$.

(ii) $P_{u,v}(q)=1$, if $u=v$.

(iii) $\deg(P_{u,v}(q)) \leq \frac{1}{2}(l(u,v)-1)$. if $u<v$.

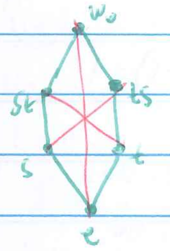
(iv)
$$q^{l(u,v)} P_{u,v}\left(\frac{1}{q}\right)=\sum_{a\in[u,v]} R_{u,a}(q)P_{a,v}(q) \text{ ; if } u\leq v.$$

Prop. $u\leq v \Rightarrow P_{u,v}(0)=1$.

Proof: Use Moebius function $\mu(u,a)$.

e.g. $P_{e,w_0}(q)$ in $S_3=A_2$.

$$q^3 P_{123,321}(q^{-1}) = \cancel{R_{e,e}(q)}P_{e,w_0}(q) + R_{e,st}(q)\cancel{P_{st,w_0}(q)}$$
$$+ R_{e,s}(q)\cancel{P_{s,w_0}(q)} + R_{e,ts}(q)\cancel{P_{ts,w_0}(q)}$$
$$+R_{e,t}(q)\cancel{P_{t,w_0}(q)} + R_{e,w_0}(q)\underbrace{\cancel{P_{w_0,w_0}(q)}}_{1}.$$
$$= P_{e,w_0}(q^{-1}).$$



$$q P_{st,sts}(q^{-1}) = \overset{1}{\cancel{R_{st,st}(q)}}P_{st,sts}(q)$$
$$+ R_{st,sts}(q)\cancel{P_{sts,sts}(q)}\ 1.$$

$$q P_{st,sts}(q^{-1}) = P_{st,sts}(q) + (q-1)$$

$$\deg(P_{st,sts}) \leq \frac{1}{2}(\underbrace{l(sts)-l(st)}_{1.} -1)=0.$$

$\therefore P_{st,sts} = ct. \Rightarrow 1.$ $\quad qc = c+(q-1) \Rightarrow c=1$

$c(q-1)=(q-1)$

~~$$q^2 P_{s,sts}\left(\tfrac{1}{q}\right) = R_{s,s}P_{s,sts} + R_{s,st}P_{st,w_0}$$~~
~~$$R_{s,w_0}P_{w_0,w_0} + R_{s,ts}P_{ts,w_0}$$~~
~~$$= P_{s,sts}(q) + \underbrace{2(q-1) + (q-1)^2}_{(q-1)(q-1+2)}$$~~

~~$$q^2 P_{s,sts}\left(\tfrac{1}{q}\right) = P_{s,sts}(q) + q^2-1.$$~~ is valid but long!

(sts = tst)

We check for $l(u,v)\leq 2$ that $P_{u,v}(q)=1$.

since, $l(u,v)-1 \leq 2-1=1$

$\therefore \frac{l(u,v)-1}{2} \leq \frac{1}{2} \therefore P=cte$
$\Rightarrow P=1$.

$\deg(P_{s,sts}) \leq \frac{1}{2}(2-1)=\frac{1}{2}$

$\Rightarrow P_{s,sts}=1$, $\deg(P_{e,w_0}) \leq \frac{1}{2}(3-1)=1$.

$\therefore P_{e,w_0}(q) = 1+cq$.

$$q^3 P_{e,w_0}\left(\frac{1}{q}\right) = P_{e,w_0}\left(\frac{1}{q}\right) + 2(q-1)^2 + 2(q-1)$$
$$+ q(q-1) + (q-1)^3$$

$$q^3 + cq^2 = 1+cq+(q-1)[(q-1)^2+q+2+2(q-1)]$$

$$= 1+cq+(q-1)[q^2-1+q+2]$$
$$(q-1)[q^2+1+q]$$

$$\cancel{q^3}+cq^2 = \cancel{1}+cq+\cancel{q^3}-\cancel{1} \Rightarrow c=0.$$

$P_{2134,4231}=?$ in S_4.

$2134=(12)$, $\quad 4231=(41)$

$P_{1324,3412}$

$1324=(23)$ $\quad 3412=(31)(42)=(13)$

$(31)=(13)$ $\qquad \binom{4}{2}=\frac{4!}{2!2!}=\frac{24}{4}=6$

$(13)=?$ $\quad \underset{s\ \cdot\ t\ \cdot\ s}{(23)(12)(23)}=(13)$

$$\overset{(13)(24)}{\overset{\shortparallel}{3412}} = (23)(12)(23)(34)(23)(34)$$
$$= t\cdot s\cdot t\cdot u\ tu = tstutu.$$
$$= s_2s_1s_2\,s_3s_2s_3 = tsu\cancel{tu}$$

$1324=23=t$. $\qquad 3412 = tsut$. $l=4$

tsut
| 3.
t

utstu.
| 4
s

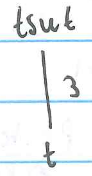
$4231=(14)=(34)(13)(34)$
$=(34)(23)(12)(23)(34)$
$=utstu$

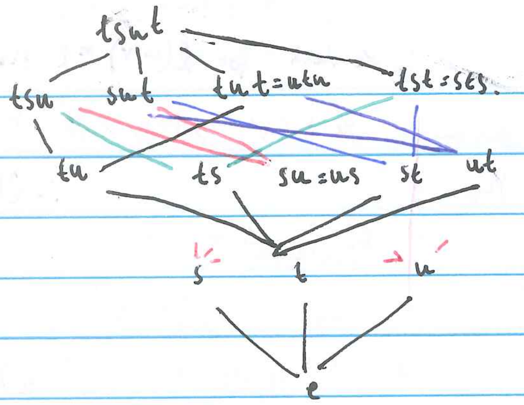
$s_3s_2s_1s_3s_2s_3 = utsutu$.

$tsu = (23)(12)(34) = 3142$

$su = us$

utsutu





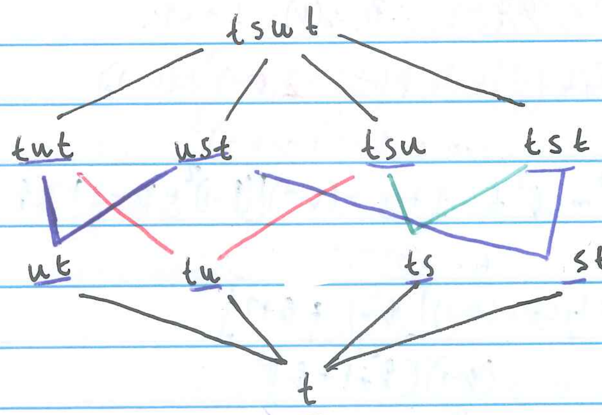
uts.

ust = 2413

3412

$tst = (23)(12)(23) = (13) = 3214$

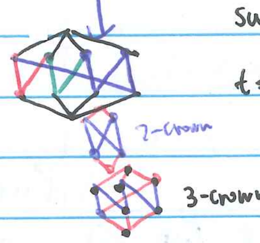
$tut = (24) = 1432$



$[t, tsut]$

≅

4-crown



$tu = (23)(34) = 1342.$

$ts = (23)(12) = 3124$

$st = (12)(23) = 2314$

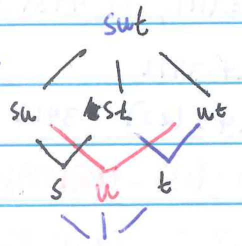
$su = us = (12)(34) = 2143$

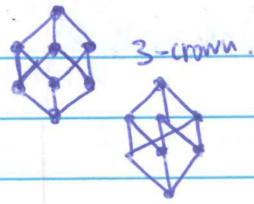
$t = (23) = 1324.$

Exercise for later: Compute $P_{t,tsut}(q)$.

$R_{t,tsut} = R_{e,sut}.$

$R_{s,ut} = 0.$





$R_{e,sut} = (q-1)R_{e,ut} + qR_{s,ut}.$

$= (q-1)^3 +$

$= R_{t,tsut}.$

$$q^3 P_{t,tsut}(q^{-1}) = \overset{1}{\cancel{R_{t,t}}} P_{t,tsut} + \overset{(q-1)}{R_{t,ut}} \overset{1}{\cancel{P_{ut,tsut}}} + \overset{(q-1)}{R_{t,tu}} \overset{1}{\cancel{P_{tu,tsut}}}.$$

$\boxed{P_{t,tsut} = P_{e,tsut}}$

$+ (q-1) + (q-1)$

$$= P_{t,tsut} + 4(q-1) + 4(q-1)^2 + R_{t,tsut} \overset{1}{\cancel{P_{tsut,tsut}}}$$

$cq^2 + q^3$

$q^2(q+c)$

$$= cq+1 + (q-1)^3 + 4(q-1)^2 + 4(q-1)$$

$c = 3+4+1 = \cancel{8}2$

$-q^2 + q.$

~~$= cq+1 + (q-1)((q-1)^2 + 4(q-1) + 4q + q)$~~

~~$q^2 - 2q + 1 + 4q - \cancel{4+4} + q$~~

~~$=$~~

~~$cq+1 + (q-1)(q^2+3q+1)$~~

~~$cq+1 + q^3 + 3q^2 + q - q^2 - 3q - 1.$~~

~~$= q^3 + 2q^2 + (c-2)q = q^3 + cq^2 \Rightarrow c=2$ and $c=2.$~~

~~$\therefore P_{t,tsut} = 2q+1$~~

$cq^2 + q^3 = q^3 + q^2 + (c-1)q \Rightarrow c=1,\ c=1$ ✓.

$R_{e,sut}.$

$\therefore P_{t,tsut} = q+1.$

$ustsu = utstu.$

$= sutus = stuts.$

$P_{s,utstu} =$ ~~[illegible]~~ $= P_{u,stuts}$

$= P_{su,utstu} = P_{su,ustsu} = P_{su,stuts}.$

$P_{t,tsut} = cq + 1.$

$\deg P_{t,tsut} \le \frac{1}{2}(3-1) = 1.$

σ UUU.

sut

↓↓↓

σ_{11} UoUoUo.

No Do! distinguish'!

$n(\sigma) = \#$ o's $= 3$

$m(\sigma) = \#$ D1's $= 0.$

$\therefore R_{e,sut} = (q-1)^3$

UoUoUo 1.

dist.

× no. dist.

TRY TO DO THIS FOR PARABOLIC (or spherical case)

(also from Bjorner-Brenti book)

More on KL-poly and R-poly

(W,S) any Coxeter system.

Prop 5.1.8 Let $u,v \in W$, $u \leq v$. If $s \in D_R(v)$. Then,

$$P_{u,v}(q) = P_{us,v}(q)$$

§5.3 R-polynomials

① Dyer's description
② Deodhar's description
} Both combinatorial

① Dyer's description

This description depends on a choice of a reflection order of Φ^+.

Prop 5.3.1. Let $u,v \in W$. Then, there exists a unique polynomial $\widetilde{R}_{u,v}(q) \in \mathbb{N}[q]$ such that

$$R_{u,v}(q) = q^{\frac{l(u,v)}{2}} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

Proof $u \not\leq v$ trivial. Sup $u \leq v$.

Uniqueity ✓ even for parabolic. (Same proof).

Existence

First a remark, since $\widetilde{R} \in \mathbb{N}[q]$ makes sense to give a combinatorial interpretation of this (and hence for R even though R may have some negative coefficients

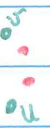
Induction on $l(v)$. $v=e$ ✓.

Let u,v, $u \leq v$ $l(v)>0$ $\exists s \in D_R(v)$

Case 1 $s \in D_R(u)$. $l(us,vs) = l(u,v)$

$$R_{u,v}(q) = R_{us,vs}(q)$$
$$\underset{\text{induction}}{=} q^{\frac{l(us,vs)}{2}} \widetilde{R}_{us,vs}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

Case 2 $s \notin D_R(u)$ $us > u$



$$R_{u,v}(q) = q R_{us,vs}(q) + (q-1) R_{u,vs}(q)$$
$$\underset{\text{induction}}{=} q \cdot q^{\frac{l(us,vs)}{2}} \widetilde{R}_{us,vs} + (q-1) q^{\frac{l(u,vs)}{2}} \widetilde{R}_{u,vs}$$

$l(us,vs) = l(u,v) ~~+1~~ -1-1 = l(u,v)-2$ ~~+1~~ -1

$$R_{u,v}(q) = q^{\frac{l(u,v)}{2}} \left[\widetilde{R}_{us,vs} + (q^{1/2} - q^{-1/2}) \widetilde{R}_{u,vs} \right]$$

$l(u,vs) = l(u,v) - 1$

$$\therefore (q-1) q^{\frac{l(u,v)-1}{2}} = (q-1) q^{-1/2} \cdot q^{\frac{l(u,v)}{2}}$$
$$= (q^{1/2} - q^{-1/2}) q^{\frac{l(u,v)}{2}}. \quad \square$$

only parabolic

(Case 3: $us \notin W^J$)

$$R^J_{u,v}(q) = \begin{cases} q R^J_{u,vs} & \text{if spherical} \\ -R^J_{u,vs} & \text{if anti-spherical} \end{cases}$$

$$q R^J_{u,vs} = q \cdot q^{\frac{l(u,vs)}{2}} \widetilde{R}_{u,vs}$$
$$= q \cdot q^{-1/2} \cdot q^{\frac{l(u,v)}{2}} \widetilde{R}^J_{u,vs}$$
$$= q^{1/2} \cdot q^{\frac{l(u,v)}{2}} \widetilde{R}^J_{u,vs} (q^{1/2} - q^{-1/2}) \quad X.$$

Cannot define $\widetilde{R}$ in the parabolic case!

$$\widetilde{R}_{u,v}(q) = \begin{cases} \widetilde{R}_{us,vs}(q) & \text{if } us<u. \\ \widetilde{R}_{us,vs} + q \widetilde{R}_{u,vs} & \text{o/w.} \end{cases}$$

Lemma: If $u,v \in W$ s.t. $u \longrightarrow v$, $s \in S \setminus \{u^{-1}v\}$. Then $us \longrightarrow vs$.

$\Delta = (a_0, \ldots, a_r)$ path in $\Omega_{(W,S)}$.

Let $<$ reflection ordering on Φ^+ (hence on T)

(i.e. $\forall \alpha, \beta \in \Phi^+$, $\lambda, \mu \in \mathbb{R}_{>0}$, s.t. $\lambda\alpha + \mu\beta \in \Phi^+$ we have either

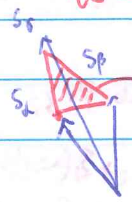
$$\alpha < \lambda\alpha + \mu\beta < \beta \quad \text{or}$$
$$\beta < \lambda\alpha + \mu\beta < \alpha.)$$

~~Lem~~ Prop Reflection orderings always exist on Φ^+.

Proof Fix total ordering on $S = \{s_1, \ldots, s_n\}$.

Let

$$\mathcal{U} \overset{\text{def}}{=} \left\{ \sum c_{s_i} \alpha_{s_i} \,\middle|\, \sum c_{s_i} = 1 \right\}$$ "convex" linear combinations



for every $\alpha \in \Phi^+ \subseteq \mathrm{Cone}(S\alpha_i)$.

$\therefore \exists! \lambda_\alpha \in \mathbb{R}_{>0}$ st $\lambda_\alpha \alpha \in \mathcal{U}$.

For $\alpha, \beta \in \Phi^+$, define $\alpha < \beta$ to mean $\lambda_\alpha \alpha <_{lex} \lambda_\beta \beta$.

$<_{lex}$ is def by $\sum c_{s_i} \alpha_{s_i} < \sum b_{s_i} \alpha_{s_i}$

if $(c_{s_1}, \dots, c_{s_n}) < (b_{s_1}, \dots, b_{s_n})$

lexicographically.

e.g. S_4

$\alpha_1, \alpha_2, \alpha_3$

$\langle s_1, s_2, s_3 \rangle = S_4$

$\overset{\text{rel}}{=}$ $\langle s, t, u \rangle$

non simple roots

do ± 1 $\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 \in \Phi^+ \\ \alpha_2 + \alpha_3 \in \Phi^+ \\ \alpha_1 + \alpha_2 \in \Phi^+. \end{cases}$

$T = \{ s, t, u, sts, tut,$

better

Supp $\alpha_1 > \alpha_2 > \alpha_3$

$)$

cone $(\alpha_1, \alpha_2, \alpha_3) = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$

$\alpha_1 + \alpha_2 \rightsquigarrow \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 = (\frac{1}{2}, \frac{1}{2}, 0)$

$\alpha_1 + \alpha_2 + \alpha_3 \rightsquigarrow (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$\alpha_2 + \alpha_3 \rightsquigarrow (0, \frac{1}{2}, \frac{1}{2})$

can permute that two and still be a refl. ordering

$\alpha_1 > \alpha_1 + \alpha_2 > \alpha_1 + \alpha_2 + \alpha_3 > \alpha_2 > \alpha_2 + \alpha_3 > \alpha_3$

$(1,0,0) > (\frac{1}{2}, \frac{1}{2}, 0) > (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) > (0,1,0) > (0, \frac{1}{2}, \frac{1}{2}) > (0,0,1)$

another example of reflection ordering ?

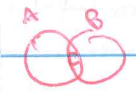
$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \alpha_2 < \alpha_2 + \alpha_3 < \alpha_3$

Let $\alpha, \beta \in \Phi^+$, $a, b \in \mathbb{R}_{>0}$ s.t. $\alpha < \beta$

and $a\alpha + b\beta \in \Phi^+$.

then $\lambda_{a\alpha+b\beta}\, a\alpha + b\beta = ? \in U$

$\overset{\shortparallel}{\sum c_i \alpha_{s_i}}$, $\sum c_i = 1$. $1 = \sum a_i = \sum b_i$



$\alpha = \sum_I a_i \alpha_{s_i}$ $a\alpha + b\beta = \sum a a_i \alpha_{s_i} + \sum b b_i \alpha_{s_i}$

$\beta = \sum_J b_i \alpha_{s_i}$ $= \sum_{i \in A} a a_i \alpha_{s_i} + \sum_{i \in B} b b_i \alpha_{s_i} + \sum_{i \in C} (a a_i + b b_i) \alpha_{s_i}$

$d_i = \begin{cases} a a_i \\ b b_i \\ a a_i + b b_i \end{cases}$

$a\alpha + b\beta = \sum d_i \alpha_{s_i}$

$\exists \lambda$ s.t. $\sum \frac{d_i}{\lambda} = 1$

$\therefore \lambda_{a\alpha+b\beta} = \sum \frac{d_i}{\lambda} \alpha_{s_i}$

$= \sum_A \frac{a a_i}{\lambda} \alpha_{s_i} + \sum_C \frac{a a_i + b b_i}{\lambda} \alpha_{s_i} + \sum_B \frac{b b_i}{\lambda} \alpha_{s_i}$

$= \lambda' \sum_{A \cup C} a a_i \alpha_{s_i} + \lambda' \sum_{B \cup C} b b_i \alpha_{s_i}$

$\sum \lambda' a a_i + \sum \lambda' b b_i = 1$.

$= \lambda' a \sum \cancel{a_i} + \lambda' b \sum \cancel{b_i} = 1$

$\lambda'(a+b) = 1$

$\lambda' a + \lambda' b = 1$

$= a' + b' = 1$, $b' = (1-a')$.

$a' + (1-a') = 1$ $a' = c$.

$\lambda' = \lambda_{a\alpha + b\beta}$

$\lambda'(a\alpha + b\beta) =$.

Better $\alpha = \sum a_i \alpha_{s_i} < \beta = \sum b_i \alpha_{s_i}$.

$\lambda_\alpha \sum a_i = 1$ $\lambda_\alpha \alpha = \lambda_\alpha \sum a_i \alpha_{s_i}$

$\lambda_\beta \sum b_i = 1$. $\lambda_\beta \beta = \lambda_\beta \sum b_i \alpha_{s_i}$ -

$a\alpha + b\beta = \sum c_i \alpha_{s_i}$

$= \lambda_\alpha a \sum a_i$

$+ \lambda_\beta b \sum b_i$.

$\overset{\lambda'}{\shortparallel}$ $\lambda_{a\alpha+b\beta}(a\alpha + b\beta) = \lambda'(a\alpha) + \lambda' b\beta$

$= 1$

$= \frac{1}{\cancel{\lambda_\alpha}} \lambda' a (\cancel{\lambda_\alpha} \alpha) + \frac{1}{\cancel{\lambda_\beta}} \lambda' b (\cancel{\lambda_\beta} \beta) =$

p.d. $\frac{1}{\lambda_\alpha} \lambda' a + \frac{1}{\lambda_\beta} \lambda' b = 1$. but

$\lambda' a \underbrace{\sum a_i}_{\frac{1}{\lambda_\alpha}} + \lambda' b \underbrace{\sum b_i}_{\frac{1}{\lambda_\beta}} = 1 = \frac{1}{\lambda_\alpha} \lambda' a + \frac{1}{\lambda_\beta} \lambda' b$

Deodhar's paper (parabolic polys)

(W,S) Coxeter system $\rightsquigarrow P_{x,y}$, $J \subset S$

W^J = set of minimal coset representatives in W/W_J.

Lemma

(i) $W^J = \{\sigma \in W \mid \ell(\sigma s) \geq \ell(\sigma) \ \forall s \in J\}$

(ii) Given $x \in W$ $\exists! \, \sigma \in W^J$ and $w_J \in W_J$ s.t.

$x = \sigma . w_J$. Furthermore ~~and~~ $\ell(x) = \ell(\sigma) + \ell(w_J)$.

(iii) $s \in S$, $\sigma \in W^J$ then one of three possibilities occur:

(a) $\ell(s\sigma) \leq \ell(\sigma)$. (This forces $s\sigma \in W^J$)

(b) $\ell(s\sigma) \geq \ell(\sigma)$ and $s\sigma \in W^J$

(c) $\ell(s\sigma) \leq \ell(\sigma)$ and $s\sigma \notin W^J$ this forces $s\sigma = \sigma s'$ for a unique $s' \in J$.

Let M^J be a free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module with $\{m^J_\sigma \mid \sigma \in W^J\}$ as a basis.

Let $u \in \{-1, q\}$. Let $s \in S$, define

$L(s) \in \mathrm{Hom}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]}(M^J)$ as

$$L(s)(m^J_\sigma) = \begin{cases} q m^J_{s\sigma} + (q-1) m^J_\sigma & \boxed{\text{if (a)}} \\ q m^J_{s\sigma} & \boxed{\text{if (b)}} \\ u \cdot m^J_\sigma & \boxed{\text{if (c)}} \end{cases}.$$

$\varphi_J : \mathcal{H} \to M^J$

$T_x \longmapsto u^{\ell(w_J)} \cdot m^J_\sigma$ $\quad (x = \sigma . w_J)$

Lemma: (i) φ_J onto.

(ii) $s \in S$

$$\begin{tikzcd} \mathcal{H} \arrow[r, two heads, "\varphi_J"] \arrow[d, "T_s \cdot"'] \arrow[rd, phantom, "\circlearrowleft"] & M^J \arrow[d, "L(s)"] \\ \mathcal{H} \arrow[r, two heads, "\varphi_J"'] & \mathcal{H}. \end{tikzcd}$$

Notation $\varphi_J(T_x) := T_x \cdot m^J_e \quad \forall x \in W$.

Let $\overline{q^{1/2}} = q^{-1/2}$

$\overline{m^J_\sigma} = \overline{T_\sigma} \cdot m^J_e$. $\left(\begin{array}{l} \text{In Soergel notation this is} \\ \quad \overset{T_\sigma}{} \; m^J_\sigma = 1 \otimes H_\sigma. \\ \quad \overline{1 \otimes H_\sigma} = 1 \otimes \overline{H_\sigma} \end{array} \right.$

$= \varphi_J(\overline{T_\sigma})$

$= \varphi_J(\overline{T_\sigma})$, $\overline{m^J_e} = m^J_e = \varphi_J(T_e) = \varphi_J(\overline{T_e})$

Def $R^J_{\tau,\sigma} \in \mathbb{Z}[q^{-1/2}, q^{1/2}]$ by

$$\overline{m^J_\sigma} = \sum_{\tau \in W^J} (-1)^{\ell(\sigma)+\ell(\tau)} \cdot q^{-\ell(\sigma)} R^J_{\tau\sigma} m^J_\tau$$

Lemma: (i) $R^{J's}_{\tau,\sigma}$ satisfy the recurrence: $\boxed{\text{for } s \in S, \ s\sigma < \sigma}$

$$R^J_{\tau,\sigma} = \begin{cases} R^J_{s\tau, s\sigma} & \text{if } ~~\dots~~ \ s\tau < \tau \\ (q-1) R^J_{\tau,s\sigma} + q \cdot R_{s\tau,s\sigma} & \text{if } s\tau > \tau \ \& \ s\tau \in W^J \\ (q-1-u) R^J_{\tau,s\sigma} & \text{if } s\tau > \tau \ \& \ s\tau \notin W^J \end{cases}$$

(ii) $R^J_{\tau,\sigma} \neq 0$ only if $\tau \leq \sigma$.

(iii) $R^J_{\tau,\sigma} \in \mathbb{Z}[q]$ and $\deg_q R^J_{\tau,\sigma} \leq \ell(\sigma) - \ell(\tau)$.

with equality in case $u = -1$.

(iv)

$$\sum_{\tau \leq \theta \leq \sigma} q^{-\ell(\theta)} \cdot R^J_{\tau,\theta} \cdot \overline{R^J_{\theta,\sigma}} = \delta_{\tau,\sigma} \cdot q^{-\ell(\sigma)}$$

Deodhar's formula (generalised version)

Fix $\sigma = s_1 \cdots s_k$ a rex. $(\sigma \in W^J)$

A J-subexpression is a sequence

$\theta = \{\theta_1, \dots, \theta_{k+1}\}$ of elements of W^J s.t.

(i) $\theta_{k+1} = e$

(ii) $\theta_p \in \{\theta_{p+1}, s_p \theta_{p+1}\}$ $\quad p \in \{1, \dots, k\}$.

θ is called (J-distinguished if

(iii) $\ell(s_p \theta_{p+1}) \geq \ell(\theta_{p+1})$ if $\theta_p = \theta_{p+1}$.

$\mathcal{D}^J = \{J\text{-dist subexpressions}\}$

$\pi : \mathcal{D}^J \to W^J$

$\theta = (\theta_1, \dots, \theta_{k+1}) \longmapsto \theta_1$. $\quad n_i : \mathcal{D}^J \to \mathbb{Z}$

$n_1(\theta) = \#\{p \mid \theta_p = \theta_{p+1} \ \& \ s_p \theta_{p+1} \in W^J\}$

$n_2(\theta) = \#\{p \mid \theta_p = \theta_{p+1} \ \& \ s_p \theta_{p+1} \notin W^J\}$

$m(\theta) = \#\{p \mid \theta_p = s_p \theta_{p+1} \ ~~\text{and}~~ \ \& \ \theta_p < \theta_{p+1}\}$

Theorem: $\tau, \sigma \in W^J$.

$$R^J_{\tau,\sigma} = \sum_{\substack{\theta \in \mathcal{D}^J \\ \pi(\theta)=\tau}} (q-1)^{n_1(\theta)} \cdot (q-1-u)^{n_2(\theta)} \cdot q^{m(\theta)}.$$

Prop $\tau, \sigma \in W^J$

$$R^J_{\tau,\sigma} = \sum_{w_J \in W_J} (-1)^{\ell(w_J)} \cdot u^{\ell(w_J)} \cdot R_{\tau w_J, \sigma}.$$

$\left(w_J \le \tilde{w}_J \text{ for some } \tilde{w}_J \in W_J \right)$

unique s.t. $\tau w_J \le \sigma$

$\Leftrightarrow \quad w_J \le \tilde{w}_J$.

§ Parabolic KL-polynomials

Build $P^J_{\tau,\sigma}$ using $R^J_{\tau,\sigma}$.

$J = \phi$ will give $P_{\tau,\sigma}$ (usual KL).

Prop $\exists!$ set $\{P^J_{\tau,\sigma} \mid \tau \le \sigma\}$ of polys in $\mathbb{Z}[q]$

s.t.

(i) $P^J_{\tau,\tau} = 1$ & $\deg_q P^J_{\tau,\sigma} \le \left(\frac{\ell(\sigma) - \ell(\tau) - 1}{2} \right)$ $\tau < \sigma$

(ii) $q^{\ell(\sigma)-\ell(\tau)} \overline{P^J_{\tau,\sigma}} = \sum_{\theta \in [\tau,\sigma]} R^J_{\tau,\theta} P^J_{\theta,\sigma}$.

Rmk: $u = -1$ "spherical" module $\mathcal{M}$ $(m_{x,y})$

$u = q$ "anti-spherical" module $\mathcal{N}$ $(n_{x,y})$

invariant element means w.r.t $(\bar{-})$.

Prop

(i) $\sigma \in W^J$, the element

$$C^J_\sigma = \sum_{\tau \in W^J} (-1)^{\ell(\sigma)+\ell(\tau)} q^{\left(\frac{\ell(\sigma)}{2}\right)} \cdot q^{-\ell(\tau)} \cdot \overline{P^J_{\tau,\sigma}} \cdot m^J_\tau$$

is s.t $\overline{C^J_\sigma} = C^J_\sigma$.

(ii) $P^J_{\tau,\sigma} \in \mathbb{Z}[q]$ are characterised by part (i) and

$\deg_q P^J_{\tau,\sigma} \le (\ell(\sigma) - \ell(\tau) - 1)/2$ if $\tau < \sigma$

(iii) $\{C^J_\sigma \mid \sigma \in W^J\}$ is a "basis" for invariants in M^J, i.e.,

if $C \in M^J$ $C = \overline{C}$ then $\exists!$

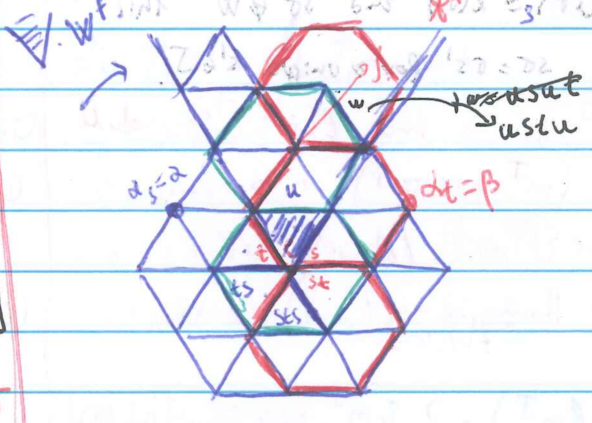
$\{f_\sigma\}_{\sigma \in W^J} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ s.t.

$f_\sigma = \overline{f_\sigma}$ and $C = \sum_{\sigma \in W^J} f_\sigma \cdot C^J_\sigma$.

Rmk $u = -1$, $P^J_{\tau,\sigma}(0) = 1$.

for $u = q$ $P^J_{\tau,\sigma}(0)$ can be 0.

e.g $\widetilde{A}_2 = \langle s,t,u \mid \quad \rangle$



$J = \{s,t\} \subset S = \{s,t,u\}$.

$W_J = A_2 \cong S_3 \cong D_6$ $6 = 2 \cdot 3$. D_{2n}.

$R_{e,sts} = (q-1)^3 + q(q-1)$.

$ustu = w$ $wt \notin W^J$ $t \in S$.

$wt > w$ bu

~~wt = ustu~~ $w = ustu$

$wt = ustut$

$sw = sustu$

$= usutu$

$= ustut = wt$.

$W_J = \langle s, t \mid \text{rels} \rangle$ $W = \tilde{A}_2 = \langle s, t, u \mid \sim \rangle$ $J = \{s, t\}$

$J \subset S$

$W^J = W_J \backslash W$ minimal length representative right cosets. $\boxed{W_J x.}$

$x \in W^J.$

$$W_J \times W^J \xrightarrow{\sim} W.$$

length additive pairing.

$\sigma s < \sigma.$ $\tau, \sigma \in W^J.$

$$R^J_{\tau,\sigma} = \begin{cases} R^J_{\tau s, \sigma s} & \text{if } \tau s < \tau. \\ (q-1) R^J_{\tau, \sigma s} + q R^J_{\tau s, \sigma s}, & \text{if } \tau s > \tau \ \boxed{\tau s \in W^J} \\ q R^J_{\tau, \sigma s}, & \tau s > \tau.\ \tau s \notin W^J. \end{cases}$$

Distinguish = no D0!

$n_1 = \#0$ y el ~~posthe~~ $\theta_{i+1}^{s_i} \in W^J.$

$n_2 = \#0$ y el ~~posthe~~ $\theta_{i+1}^{s_i} \notin W^J.$

$m = \#D1$ $\mathcal{D}^J = \{ J\text{-distinguished subexpressions} \}.$

Thm $\tau, \sigma \in W^J$,

$$R^J_{\tau,\sigma} = \sum_{\substack{\sigma \in \mathcal{D}^J \\ \pi(\sigma) = \tau}} (q-1)^{n_1(\theta)} (q-1-u)^{n_2(\theta)} q^{m(\theta)}$$

e.g.

$\ell(\tau, \sigma) = 1.$ $R_{\tau,\sigma} = (q-1)$?

$\sigma \in W^J$

$\sigma = \underbrace{s_1 \dots s_r}_{\text{rex}},$ $\tau = s_1 \dots \hat{s}_i \dots s_r.$

$U1 \dots U1 U0 U1 \dots U1$

can happen that $\theta_{i+1} s_i \notin W^J$?.

suppose happen then $\exists s$, s.t. $s \in J$

and $\theta_{i+1} s_i = s \theta_{i+1}$.

$\therefore \sigma = s\, \theta_{i+1} s_{i+1} \dots s_r.$

Lemma: $\sigma \notin W^J$ if $\exists s$ st

$\sigma s \in W^J \Rightarrow \sigma s < \sigma.$

Proof Sup $\sigma s > \sigma$ $\sigma s \in W^J, \sigma \notin W^J.$

$\tau \in W^J.$

Let $\tau = \sigma s > \sigma.$, $\tau s = \sigma < \tau.$

$\tau s < \tau.$

Then $\tau \in W^J$ $s \in D_R(\tau)$

but $\underset{\sigma.}{\overset{\shortparallel}{\tau s}} \notin W^J \Rightarrow \tau s > \tau.$ —X— $\square$

Back to example $\ell(\tau, \sigma) = 1.$

$\sigma = s_1 \dots s_r$, $\tau = s_1 \dots \hat{s}_i \dots s_r$ No D's!

$U1 \dots U1 U0 U1 \dots U1.$

sup $\theta = s_1 \dots, s_{i-1}$ then $\theta \in W^J.$

(by def. requirement of $\mathcal{D}^J$) in Deodhar paper!

if $\theta s_i \notin W^J$ then $\exists p > i$ st

$s_1 \dots s_i \dots s_p \in W^J$ and $\forall p' > i$ $p' < p$

$s_1 \dots s_i \dots s_{p'} \notin W^J$. Of course such p exists since $\sigma \in W^J$!!

let $\theta_p = s_1 \dots s_{p-1} \notin W^J.$ $\theta_p s_p \in W^J$

since (s_i) is a rex of σ $\theta_p s_p > \theta_p.$

but this contradicts the lemma above.

$\therefore R^J_{\tau,\sigma} = (q-1).$ $n_2 = 0$ $m = 0.$

$n_1 = 1$

e.g. $\ell(\tau, \sigma) = 2$

$\sigma = s_1 \dots s_r$ $\tau = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_r$

$U1 \dots U0 U1 \dots U0 \dots U1.$

$111 \dots 1011 \dots 101 \dots 1$

One case may happen here?

Certainly $\theta s_i \notin W^J$ cannot happen! ✓ $\theta = s_1 \dots s_{i-1}$

what if $\tilde{\theta} s_j \notin W^J$ for $\tilde{\theta} = s_1 \dots \hat{s}_i \dots s_{j-1}$.

can happen!

e.g. $\tilde{A}_2$ $\sigma = us$ $\tilde{\theta} = e$

$\tau = \hat{u}\hat{s} = e$ $\tilde{\theta} s = s \notin W^J$.

$$R^J_{\tau,\sigma} = \begin{cases} q(q-1) & \text{if } \tilde{\theta} s_j \notin W^J \\ (q-1)^2 & \text{o/w}. \end{cases}$$

e.g. Using the recursion.

$R^J_{\tau,\sigma}$ when $\ell(\tau, \sigma) = 1.$ $\sigma = s_1 \dots s_r,$ $\tau = s_1 \dots \hat{s}_i \dots s_r.$

$ws > w$

$R^J_{\tau,\sigma} = R^J_{s_1 \dots s_{i-1}, s_1 \dots s_i} = R^J_{w, ws}$

$= (q-1) R^J_{w,w} + q \cancel{R^J_{ws,w}} \nearrow 0.$

Spherical $\boxed{w=-1}$

no downs!

$U1 \cdots U0 \cdots U0 \cdots U1$

e.g $\ell(\tau,\sigma)=2$. $\sigma = s_1 \cdots s_i \cdots s_j \cdots s_r$

$\tau = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$.

U

$R^J_{\tau,\sigma} = R^J_{\underbrace{s_1 \cdots \hat{s}_i \cdots s_{j-1}}_{\ell=j-2},\ \underbrace{s_1 \cdots s_{j-1} s_j}_{\ell=j}}$ $\ell(\theta)=j-1$ $\ell(w)=j-2$.

surely $s_{j-1} \neq s_j$ rex

$w = s_1 \cdots \hat{s}_i \cdots s_{j-1}$

$\theta = s_1 \cdots s_i \cdots s_{j-1}$

$R^J_{\tau,\sigma} = R^J_{w,\theta s_j}$ $v = \theta s_j$ $\theta < v$.

$= R^J_{w,v}$. $v s_j = \theta < v$.

$R^J_{w,v} = ?$ can happen that $ws_j < w$?.

sup $s_1 \cdots \hat{s}_i \cdots s_{j-1} s_j < w$.

$\Rightarrow \underset{ws_j}{\underset{\|}{s_1 \cdots \hat{s}_i \cdots s_{j-1} s_j}} = s_1 \cdots \hat{s}_i \cdots \hat{s}_k \cdots s_{j-1}$.

~~[illegible]~~

In that case $R^J_{\tau,\sigma} = R^J_{ws_j,\theta}$. → So we can pick minimal length case.

by minimality. $\ell(ws_j)=j-1$

So supp $ws_j > w$. then if $ws_j \in W^J$

$R^J_{\tau,\sigma} = R^J_{w,v} = (q-1)\underbrace{R^J_{ws_j,v}}_{(q-1)\text{ by previous case.}} + q R^J_{ws_j,vs_j}$

$\ell(w)=j-2$

$\ell(ws_j)=j-1$.

$\ell(v)=j$

$vs_j=\theta<v$

$\ell(vs_j)=j-1$.

sup

$ws_j \le vs_j \Rightarrow ws_j = vs_j$

$\Rightarrow w = v$ —×— since $w<v$

$\Rightarrow \Rightarrow ws_j \not\le vs_j$

$\therefore R_{ws_j,vs_j}=0$

$\therefore R^J_{\tau,\sigma} = (q-1)^2$.

$ws_j = s_1 \cdots \hat{s}_i \cdots s_j$

$v = s_1 \cdots \cdots s_j$

$ws_j < v$

$\ell(ws_j) = \ell(v)-1$.

previous case applies since $\ell(ws_j,v)=1$.

$j=5$.

$we = abde$

$abd = w$. ~~wba~~

$ve = abcd$.

$abcde = v$

$we = ve \Rightarrow de = cd$.

$abcde$

$= abd\cancel{e}\cancel{e} = abd = w$.

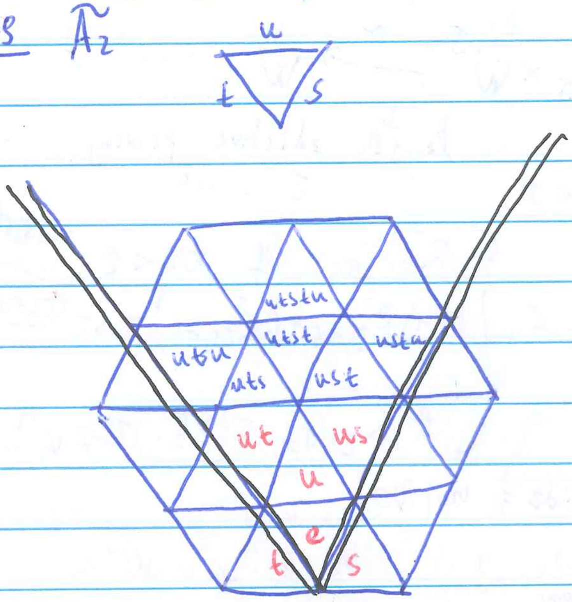
$\overset{\theta_{j-1}}{\overset{\|}{}}$

other case $ws_j > w$ and $ws_j \notin W^J$.

$R^J_{\tau,\sigma} = R^J_{w,v} = q\underbrace{R_{ws_j,v}}_{q-1\text{ previous case}} = q(q-1)$

---

e.g $\tilde{A}_2$



$R^J_{e,u} = (q-1)R_{e,e} + q\cancel{R_{s,e}}^{0} = q-1$

$\tau s < \sigma$

$$R^J_{\tau,\sigma} = \begin{cases} R^J_{\tau s,\sigma s} & \text{if } \tau s < \tau \\ (q-1)R^J_{\tau,\sigma s} + q R^J_{\tau s,\sigma s} & \text{if } \tau s > \tau\ \ \tau s \in W^J \\ q R^J_{\tau,\sigma s} & \text{if } \tau s \notin W^J. \end{cases}$$

$R^J_{u,ut} = (q-1)R^J_{u,u} + q\cancel{R_{ut,u}}^{0} = q-1$

$R^J_{u,us} = (q-1)$

$R^J_{e,ut} = q R^J_{e,u} = q(q-1)$

$\|$

$R^J_{e,us}$

$n_1 = 1$ ✓.

ut UO UO. $n_2 = 1$.

OO

$ut \in W^J$

utsu UUUU $s \notin W^J$,

1100 1100.

ut $t \notin W^J$.

$R^J_{e,us} = R^J_{e,ut} = q(q-1)$.

$R^J_{ut,uts} = \cancel{q}\,1(q-1)+0$.

uts UUU $(q-1)$

110 110

ut

$\tilde{A}_2$

$R^J_{e,e}=1,\ R^J_{e,u}=(q-1),\ R_{e,ut}=R_{e,us}=q(q-1)$

$R^J_{ut,utsu} = q(q-1) = R_{us,ustu}.$

utsu utsu $\in W^J$ but
1100
UUUU. wtu $\notin W^J$

$R^J_{u,uts} = (q-1)^2 = R^J_{u,ust}$

uts
100
UUU

$R^J_{ut,utst} = (q-1)^2 = R^J_{u,uts}.$

$R^J_{uts,utsut} = (q-1)^2 = R^J_{ust,ustus}.$

$(q-1)^2.$
$\|$
$R_{uts,utstu} = R_{ust,ustsu} = (q-1)^2.$

utsut
11100.

utstu utst.
11100. utsu.

uts

$utsu \in W^J$
$utst \in W^J$

$\ell(\tau,\sigma)=3.$

$R^J_{e,uts} = q^2(q-1) = R^J_{e,ust}.$

uts
000.
UUU

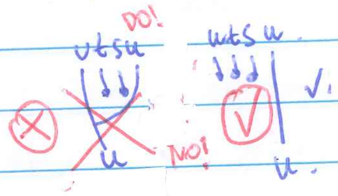
$R_{e,uts} = (q-1)^3$

e

$R^J_{e,uts} = q\overbrace{R^J_{e,ut}}^{q(q-1).} = q^2(q-1) = R^J_{e,ust}$

$R^J_{u,utsu} = q^2(q-1) = R^J_{ustu}.$

$\# UO \begin{cases} n_1 \\ n_2 \end{cases}$



UOUOUOOU1
√ × ×
~~[illegible]~~

√.

#~~DP[illegible]~~

#D1 = m_0.

$(n_1+n_2)=\#UO$

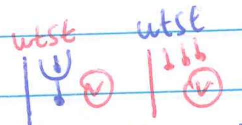
$R^J_{u,utsu} = R^J_{e,uts} = q^2(q-1)$

$R^J_{u,utst} = (q-1)^3 + q(q-1)$

utst $= (q-1)R^J_{u,uts} + qR^J_{ut,uts}.$
1000 $= (q-1)^3 + q(q-1)$ ~~[illegible]~~.
UUUU. ✓.

$ut \in W^J$.

utst
1101



U1U1UO(D1) | U1UOUOUO
$q(q-1)$ | $(q-1)^3$

$\tilde{R}_{e,sts}(q) = q^{3/2}[\mu^3+\mu]$ $\tilde{R}_{e,sts} = q^3+q.$

$R^J_{e,ut} = (q-1)^2 = q(q^{1/2}-q^{-1/2})^2,\ \tilde{R}^J_{e,ut} = q^2$

$R^J_{e,u} = (q-1) = q^{1/2}(\ \mu\ ),\ \tilde{R}_{e,u} = q.$

$\boxed{\ell(\tau,\sigma)=2}$

In $\tilde{R}$, $J=\phi$ $\tilde{R}_{\tau,\sigma} = q^2$ always, but

$R^J_{ut,utsu} = q(q-1) = q^{3/2}(q^{1/2}-q^{-1/2})$.

Bjorner & Brenti

$\alpha = \sum a_i \alpha_{s_i}$

$\beta = \sum b_i \alpha_{s_i}$

R-poly §5.2 & §5.3 (Continuation)

Proof (Prop 5.2.1 (Cont)) $\alpha < \beta \in \Phi^+$.

$a, b > 0$ st $a\alpha + b\beta \in \Phi^+$.

$$\underbrace{\lambda(a\alpha+b\beta)}_{\lambda'}(a\alpha+b\beta) = c(\lambda_\alpha \alpha) + (1-c)\lambda_\beta \beta.$$

$c \in (0,1)$.

$\lambda_\alpha \alpha <_{lex} \lambda_\beta \beta \rightsquigarrow$ ~~λ~~ $(\lambda_\alpha a_i) <_{lex} (\lambda_\beta b_i)$

$\alpha_{s_i} < \alpha_{s_j}$ if $i<j$.

$(a_i') < (b_i')$

$$c(\lambda_\alpha \alpha) + (1-c)(\lambda_\beta \beta)$$

$$= c\lambda_\alpha \sum a_i \alpha_{s_i} + (1-c)\lambda_\beta \sum b_i \alpha_{s_i}$$

$\boxed{2 \text{ cases}}$ $$= c\sum a_i' \alpha_{s_i} + (1-c)\sum b_i' \alpha_{s_i}.$$

def

$\lambda_\alpha \alpha = \sum a_i' \alpha_{s_i}$, $\lambda_\beta \beta = \sum b_i' \alpha_{s_i}$.

$(a_i') <_{lex} (b_i')$. $\exists j$ st. (j → minimal)

$a_j' < b_j'$ for that j.

$c \in (0,1)$.

$$a_j' < c a_j' + (1-c) b_j' < b_j'$$

if $j' < j$

$a_{j'}' = b_{j'}'$.

$\therefore (a_j') <_{lex} (ca_j' + (1-c)b_j') <_{lex} b_j'$

$\therefore \alpha < a\alpha + b\beta < \beta$ □

Let $<$ be a reflection ordering, and $s \in S$. We define a total ordering $<^s$ on Φ^+ as follows. For $\beta, \gamma \in \Phi^+$, we set $\beta <^s \gamma$ iff either one of the following (mutually exclusive) conditions apply:

(i) $\beta, \gamma < \alpha_s$ and $\beta < \gamma$,

(ii) $\beta, \gamma > \alpha_s$ and $s(\beta) < s(\gamma)$,

(iii) $\beta < \alpha_s < \gamma$

(iv) $\gamma = \alpha_s$.

We call $<^s$ the upper s_i conjugate of $<$

Prop 5.2.2 Let $<$ r.o., $s \in S$, $\beta \in \Phi^+ \setminus \{\alpha_s\}$.

Then, $\beta < \alpha_s$ iff $S(\beta) < \alpha_s$. (in the second part)

Def of $<_s$ is the same but changing $s(\beta)$ by β in (ii) and the same with α.

Repeat the same in (i).

And (iv)' $\beta = \alpha_s$

Note that $(<_s)^s = <^s$. $\forall s \in S$ $\forall <$.

Prop 52.3 $<^s, <_s$ are r.o.

Proof ($<^s$)

$\alpha < \beta$, $\alpha < a\alpha + b\beta < \beta$

Sup $\beta = \alpha_s$. $\therefore \gamma <^s \alpha_s$ $\forall \gamma \in \Phi^+$.

$\alpha < a\alpha + b\beta < \beta = \alpha_s \Rightarrow \alpha <^s a\alpha + b\beta$. ✓

Sup $\alpha = \alpha_s$. $\alpha_s < \beta$.

$$s(\beta) = +\frac{1}{b}(-a\alpha_s + b s(\beta)) + \frac{a}{b}\alpha_s$$

$$= \frac{1}{b} \underbrace{s(\underbrace{a\alpha_s + b\beta}_{\in \Phi^+})}_{\in \Phi^+} + \frac{a}{b}\alpha_s$$

→ for lemma 4.4.3

Lemma 4.4.3 s permutes the set $\Phi^+ \setminus \{\alpha_s\}$

Proof: $\gamma \in \Phi^+ \setminus \{\alpha_s\} \Rightarrow \gamma \neq c\alpha_s$.

$\therefore \langle \alpha_{s'}^* | \gamma \rangle > 0$ for some $s' \neq s$.

$\langle \alpha_{s'}^* | s(\gamma) \rangle = \langle s(\alpha_{s'}^*) | \gamma \rangle = \langle \alpha_{s'}^* | \gamma \rangle > 0$.

Prop 4.2.3 If $\rho \in V^*$ $\langle w(\rho), \beta \rangle = \langle \rho, w^{-1}(\beta) \rangle$

Since $w(\alpha_s) \in \Phi^+ \Leftrightarrow ws > w$

then $ss' > s \Rightarrow s(\alpha_{s'}^*) \in \Phi^+$, $\langle \alpha_{s'}^* | s(\gamma) \rangle > 0$ ↗ $s(\gamma) \in \Phi^+$

$\therefore s(\gamma) \in \Phi^+$. □

$\alpha_s < s(\beta)$ (iff $\alpha_s < \beta$)

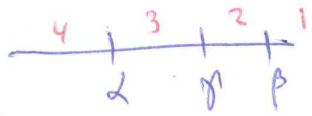
$\alpha_s < S(\beta)$.

$\alpha_s < a\alpha + b\beta < \beta$ PdI $s(a\alpha + b\beta) > s(\beta)$

$\alpha_s < s(a\alpha + b\beta)$ but $s(\beta) = \underset{>0}{\lambda}\gamma + \underset{>0}{\mu}\alpha_s$.

$\alpha_s < S(\beta)$

$\alpha_s < s(\beta) < \gamma = s(a\alpha + b\beta)$.



$$\alpha_s = \alpha < \overbrace{a\alpha_s + b\beta}^{\gamma} < \beta$$

~~$s(\alpha_s)$~~ $\quad s(a\alpha_s + b\beta) > s(\beta)$

$\alpha_s < a\alpha_s + b\beta \Rightarrow \alpha_s < s(\text{---})$

$\alpha_s < \beta \quad \Rightarrow \alpha_s < s(\beta)$

$\therefore\ \beta \overset{s}{<} a\alpha_s + b\beta \quad \overset{s}{<} \alpha_s$

Sup $\alpha, \beta \neq \alpha_s$. 4 cases.

(1) $\beta < \alpha_s$ ✓ obvio.

(2) $\gamma < \alpha_s < \beta$, $\alpha < \gamma < \alpha_s < \beta$

$\Rightarrow \alpha \overset{s}{<} \gamma < \beta$ ✓.

(3) $\alpha < \alpha_s < \gamma < \beta$

$s(\alpha) < \alpha_s < s(\beta)$ $\quad \alpha_s < s(\gamma)$

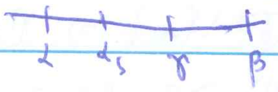
$\alpha_s < s(\beta)$

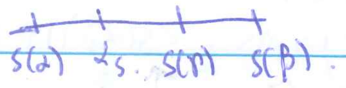
$\Rightarrow s(\alpha) < s(\beta)$

$$s(\alpha) < \underset{\substack{\shortparallel \\ s(\gamma)}}{a s(\alpha) + b(s(\beta))} < s(\beta)$$

$\alpha_s < s(\gamma)$

$s(\alpha) < \alpha_s$.





$\alpha \overset{s}{<} \gamma \overset{s}{<} \beta$

(4) $\alpha_s < \alpha$. by def of $<$

$s(\alpha) < s(\gamma) < s(\beta)$

or $\quad > \quad >$ ✓.

one of the ? follows □.

No write here!

$<$ total ordering on T

Lemma 5.?.3 Let $u, v \in W$ s.t. $u \to v$, and $s \in S \setminus \{u^{-1}v\}$. Then, $us \to vs$.

Bruhat paths $\rightsquigarrow$ interpretation of $\widetilde{R}$ polynomials.

$\Delta = (a_0, \text{—}, a_r)$ a path. $\quad <$ refl ordering on Φ^+ (or on T)

let

$E(\Delta) \overset{\text{def}}{=} \{a_{i-1}^{-1} a_i \mid i = 1 \dots r\} \subseteq T$

&

$D(\Delta; <) \overset{\text{def}}{=} \{i \in [r-1] : a_{i-1}^{-1} a_i > a_i^{-1} a_{i+1}\}$

$E(\Delta) :=$ edge set of Δ

$D(\Delta; <) :=$ descent set w.r.t. $<$.

Define $R_<$ in the incidence algebra $I(W, <)$ of W by $\quad$ W ordered by Bruhat order.

$$R_<(u,v) \overset{\text{def}}{=} \sum_{\{\Delta \in B(u,v) \,:\, D(\Delta, <) = \phi\}} q^{\ell(\Delta)}$$

where $B(u,v)$ is the set of all directed paths in $\Omega_{(W,S)}$ from u to v.

Thm (Dyer 93) Let $<$ be a r.o. $u < v$. Then,

$$\widetilde{R}_{u,v}(q) = R_<(u,v)$$

Rmk:

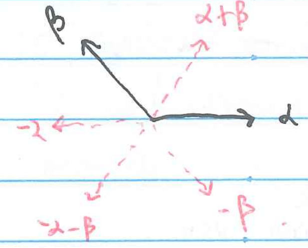
$R_<$ does not depend on the choice of $<$!!

Rmk $\Phi^+ \xrightarrow{\sim} T$

$w(\alpha_s) = \gamma \longmapsto wsw^{-1}$.

Examples of Dyer's formula for $\tilde{R}$.

$\boxed{A_2}$



$\{\alpha, \beta, \alpha+\beta\} = \Phi^+$

2 possible r.o. $\alpha < \alpha+\beta < \beta$.

$\beta < \alpha+\beta < \alpha$.

$\beta = \alpha_t \rightsquigarrow t$

$\alpha = \alpha_s \rightsquigarrow s$ t

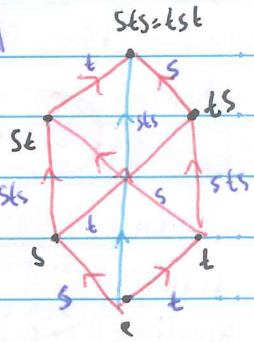
$\alpha_s+\alpha_t$ ~~β~~ $\rightsquigarrow tst$

$t(\alpha_s) = \alpha_s - \underbrace{\langle \alpha_t^\vee, \alpha_s \rangle}_{-1} \alpha_t = \alpha_s + \alpha_t$

$\alpha_s + \alpha_t \rightsquigarrow tst$ $s < sts < t$ or

$t < sts < s$.

$\boxed{\Omega(W,S)}$



$\tilde{R}_{x,x} = q^0 = 1$.

$B(x_0,x_0) = \{\{x_0\}\}$, $E = \phi$.

$\ell(\{x\}) = 0$.

$\ell(y,w) = 1$.

$\tilde{R}_{y,w} = q$ if $[y,w] = \{y,w\}$.

$R = q^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = q-1$.

$\tilde{R}_{y,w} = q^2$ if $\ell(y,w) = 2$. ?

~~e,w~~ $\ell(w) = 2$ ~~$q/\ell(y,w)$~~

$e \rightsquigarrow st$ $R = q(q^{1/2} - q^{-1/2})^2$

$\{s, sts\}$

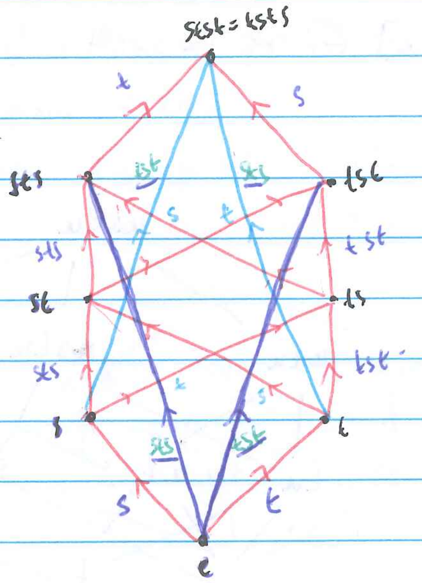
$\{t, s\}$

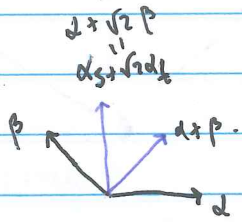
$\tilde{R}_{e,sts} = q^3 + q$

$R = q^{3/2}[(q^{1/2}-q^{-1/2})^3 + (q^{1/2}-q^{-1/2})]$

$R_{e,sts} = (q-1)^3 + q(q-1)$

D_8.





$\alpha < \alpha+\beta < \alpha+\sqrt{2}\beta < \beta$ or

$\alpha > \alpha+\beta > \alpha+\sqrt{2}\beta > \beta$

$t(\alpha_s) = \alpha_s + \sqrt{2}\alpha_t$. $s < sts < tst < t$

$\alpha = \alpha_s \longrightarrow s$

$\beta = \alpha_t \longrightarrow t$

$\alpha+\beta \longrightarrow sts$

$\alpha+\sqrt{2}\beta \longrightarrow tst$

Def (r.o. Dyer) $<$ is r.o. iff for any dihedral refl subgroup W' of W either

$r < rsr < rsrsr < \cdots < srsrs < srs < s$

or $r > rsr > rsrsr > \cdots > srsrs > srs > s$

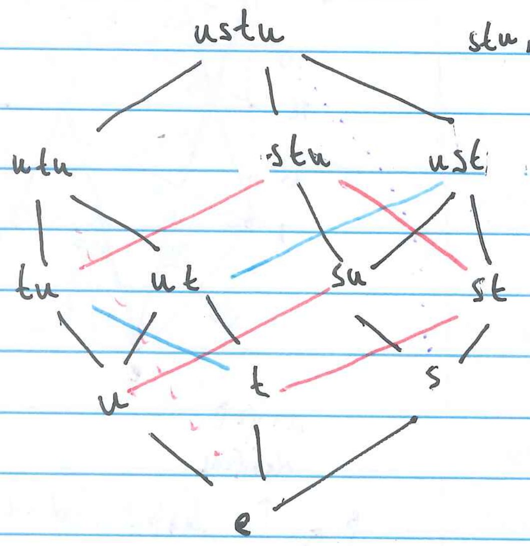
where $\chi(W') = \{t \in W \mid N(t) \cap W' = \{t\}$.

Motivation of Dy 91.

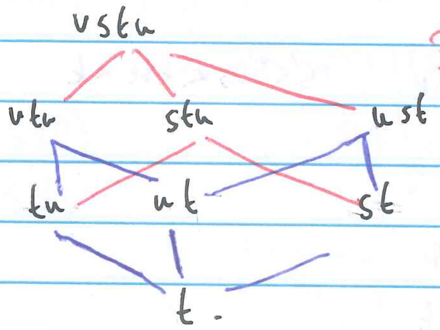
4132 = (142) = stut = sutu = ustu   type in A_3

Idea:

[s, ustu] ⊛ try to get the info that $s \longrightarrow ustu$ only using the Bruhat order!



stu, utu, $\underset{\overset{\shortparallel}{s}}{\overset{\times}{usu}}$, vst.



3-Crown!.

$\Rightarrow \overset{tu}{\overset{\sim}{\boxed{tustu}}} \notin T$.

Same number of letters as the reflection!

$tsutu$
$= tstut$.

~~tustut~~! $\boxed{utstu}$.

$= ustsu = sutus = \boxed{stuts}$

tsutu!

tstut!.

ststut!.

The Bruhat graph of a Coxeter group

References:

S.A. M. Dyer 1990 "Reflection subgroups of Coxeter systems"
Comp. " 1991 "On the "Bruhat graph" of " "
Comp. " 1993 "Hecke algebras and shellings of Bruhat intervals"

Plan:

① Motivation
② Basic results
③ Proof of Miracle 3.

§1. Motivation

Let (W,S) be a finite Coxeter system

$$T := \bigcup_{w\in W} wSw^{-1} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{positive}\\ \text{roots}\end{matrix}\right\}.$$

↗ set of reflections

Let $\ell : W \to \mathbb{N}$ be the length function

Def The Bruhat graph $\Omega_{(W,S)}$ is the directed graph with.

Vertex set : W

Edge set : $E_{(W,S)} := \left\{ (tw, w) \;\middle|\; \begin{matrix} w\in W \\ t\in N(w)\end{matrix} \right\}$

where

$$N(w) := \{ t\in T \mid \ell(tw) < \ell(w) \}.$$

If $(x,y)\in E_{(W,S)}$ we sometimes will label the edge with $xy^{-1}\in T$.

(✻) Some questions about Coxeter groups can be solved only in terms of the Bruhat order.

Def A path Δ of $\Omega_{(W,S)}$ from x to y is a sequence $(x_0=x, x_1, x_2, \dots, x_n=y)\in W^n$ s.t. $(x_i, x_{i+1})\in E_{(W,S)}$.

The Bruhat order $\leq$ of W is defined by $x\leq y$ iff $\exists$ path from x to y in Ω.

If $X\subset W$ denote by $\Omega_{(W,S)}(X)$ to be the full subgraph of $\Omega_{(W,S)}$ containing X as vertex set.

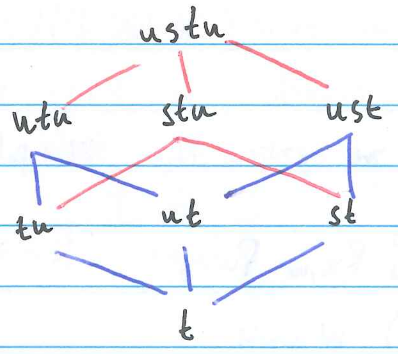
Miracle 1 One can determine, whether $xy^{-1}\in T$ using only $\leq$. $s^2=t^2=u^2=e$

e.g $S_4 = \langle s,t,u \mid sts=tst,\ utu=tut,\ su=us \rangle$

Clearly $utstu\in T$.

Does $tustu\in T$? (i.e. $(t, ustu)\in E_{(W,S)}$)

Let's see the poset structure of $[t, ustu]$



3-crown!

By Dyer 91 (Miracle 2) a 3-crown cannot produce the relation $xy^{-1}\in T$. interval $[x,y]$

Miracle 2

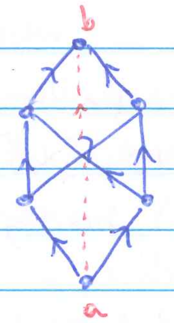
Prop (Dy 91) $I\subset W$ interval.

Can construct $\Omega_{(W,S)}(I)$ using only $(I,\leq)$ in the following way.

① Let $E_1 = \{ (x,y)\in I\times I \mid x \triangleleft y \}$.

Let $B\supset E_1$ the minimal set containing E s.t

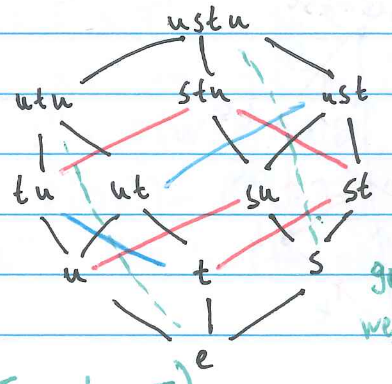
If



$\in B$ then $(a,b)\in B$.

Then $B = \Omega_{(W,S)}(I)$

e.g S_4. Bruhat subgraph. $\Omega_{(W,S)}([e, ustu])$



Therefore $utu\in T$. $sustu\in T$.

green dotted lines were added using $(ab\in T \Leftrightarrow ba\in T)$

Let (W,S) be any Coxeter system.

Conjecture (Combinatorial invariance Conjecture
Kazhdan-Lusztig 1979)

If $[x,y] \cong [\tilde{x},\tilde{y}]$ then
$$P_{x,y} = P_{\tilde{x},\tilde{y}}.$$

(**Rmk** Open problem even in finite type ... (e.g type A))

R-polynomials are easier than KL-polys.

$$q^{\ell(x,y)} P_{x,y} = \sum R_{x,w} R_{w,y} \qquad \tilde{R} \leftrightarrow R.$$

Thm (Dyer 1993) W finite.

$$\tilde{R}_{x,y} = \sum_{\substack{\Delta \text{ "nice" paths} \\ \text{from } x \text{ to } y \\ \text{in } \Omega([x,y])}} q^{|\Delta|}$$

("nice" means descendent set is ϕ for a choice of a total order in Φ^+)

This theorem and Miracle 2 give support to the conjecture. But the description still involves T (or Φ^+).

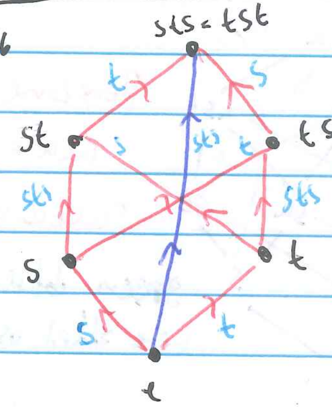
Q. How many iso types of Bruhat intervals are possible?

Miracle 3

Let $n \in \mathbb{N}$ fixed. There are finitely many iso types (as posets) of intervals $[x,y]$ of length n. (i.e. $\ell(y)-\ell(x)=n$) occurring in finite ~~class~~ type Coxeter systems.

§2. Basic results

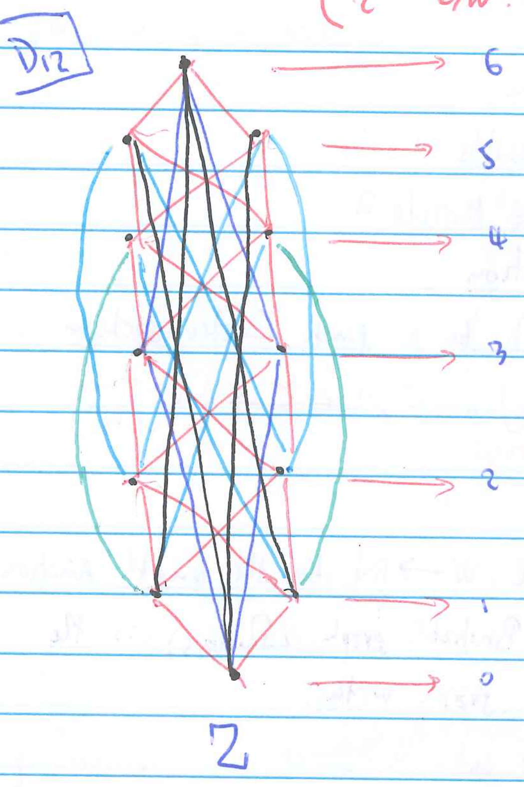
$A_2 = D_6$



2) W dihedral gp. $I = [x,y] \subset W$ s.t. $\ell(y)-\ell(x) = n$.

Let $I \xrightarrow{f} \{0,1,\dots,n\}$

st $\#f^{-1}(i) = \begin{cases} 1 & \text{if } i=1,n \\ 2 & \text{o/w}. \end{cases}$



$(x,y) \in E(\Omega(I))$ if
$f(y)-f(x)$ odd
and $f(x) < f(y)$

Def (Dyer 90) A subgroup W' of W is a reflection subgroup if
$W' = \langle W' \cap T \rangle$.

Thm (Dy 90) Let $W' \subset W$ be a ref subgroup
$S' = \{t \in T \mid N(t) \cap W' = \{t\}\} =: \chi(W')$
then (W',S') is a Coxeter system.

Prop (Dy 90)
$W' \subset W$ ref. sub. $S' = \chi(W')$.

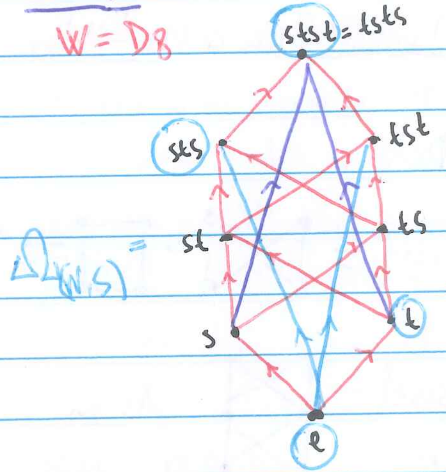
(i) $$\Omega_{(W',S')} = \Omega_{(W,S)}(W')$$

(ii) For $x \in W$, let x_0 be the minimal length representative of the right coset $W'x$. Then the map $w \longmapsto wx_0$ induces a directed graph isomorphism:

$$\Omega_{(W',S')} \xrightarrow{\sim} \Omega_{(W,S)}^{(W'x)}$$

Example

$W = D8$



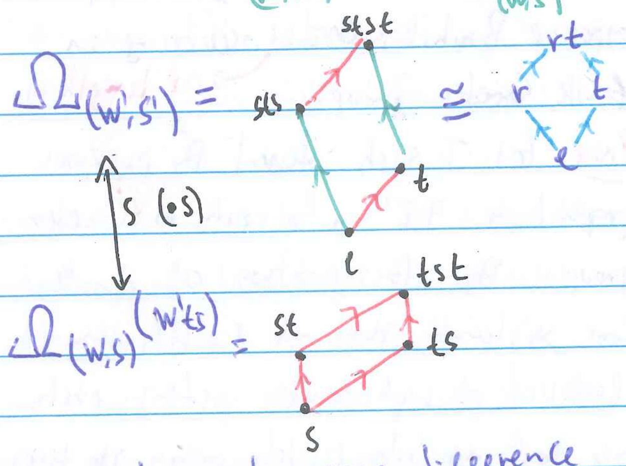
$W' = \langle sts, t \rangle = \{e, t, sts, stst\}$.

$x = ts$

$W'x = W'ts = \{ts, s, tst, st\}$.

$x_0 = s \to$ minimal length representative of $W'ts$

The proposition says the map $w \longmapsto ws$ induces iso of directed graphs between $\Omega_{(W',S')}$ and $\Omega_{(W,S)}^{(W'ts)}$



The map does not preserve difference of lengths!

§3. Proof of Miracle 3

Prop (Dy §1)

Let (W,S) be a finite Coxeter group $I = [x,y]$, $n := \ell(y) - \ell(x)$. Then $W' := \{uv^{-1} \mid u, v \in I\}$ is a reflection subgroup of rank $\leq n$. (i.e. $\# \chi(W') \leq n$). Moreover, there exist an interval I' in W' such that

$$I \cong I'$$

poset isomorphism.

Proof:

$\underline{n=0|}$, $I = \{x\}$, $W' = \{e\} = I'$
$\chi(W') = \phi$.

$\underline{n=1|}$ $I = [tw, w] = \{tw, w\}$
$W' = \{e, t\} = [e, t] = I' \cong I$.

$\underline{n \geq 2|}$

Let $(x_0, x_1, \ldots, x_n)$ be a path in $\Omega_{(W,S)}$ from x to y (i.e. $x_0 = x$, $x_n = y$).

Define $W'' = \langle \{x_{i-1} x_i^{-1}\} \mid i \in \{1, \ldots, n\} \rangle$

So W'' is a reflection subgroup.

Consider $W''x$ and let z be the minimal length representative of $W''x$.

By the proposition before the map $w \longmapsto wz$ induces an iso of directed graphs

$$\Omega_{(W'',S'')} \xrightarrow{\sim} \Omega(W''x)$$

Let $(x_0 z^{-1}, x_1 z^{-1}, \ldots, x_n z^{-1})$ is a path from xz^{-1} to yz^{-1} in $\Omega_{(W'',S'')}$ where $S'' = \chi(W'')$. Then,

$$\ell''(yz^{-1}) - \ell''(xz^{-1}) \geq n$$

where ℓ'' is the length of W''.

let $(w_0, \_\_, w_m)$ any path from xz^{-1} to yz^{-1} in W'', then $(w_i z)_{i=0}^{m}$ is a path from x to y.

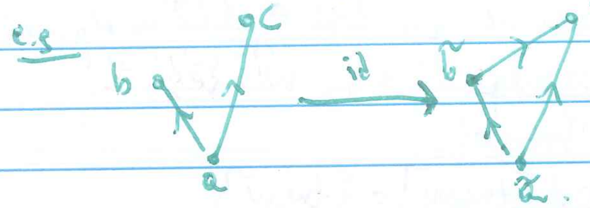
$\therefore \ell''(yz^{-1}) - \ell''(xz^{-1}) \leq n$.

Define $I' = [xz^{-1}, yz^{-1}] \subset W''$, and

$$\begin{tikzcd} I' \arrow[r, "f"] & I \\ w \arrow[r, mapsto] & wz \end{tikzcd}$$

this map must be injective and order preserving. But is still not clear that is an isomorphism of posets.

e.g



the map "id" is order preserving bijection but the target set has the extra relation $\tilde{b} < \tilde{c}$. However, b and c are not comparable.

By a theorem of Bjorner and Wachs, 1982 if $[x,y]$ is a length $n^{\geq 3}$ interval then

$[x,y] \setminus \{x,y\}$, viewed as an abstract simplicial complex is a combinatorial $n-2$ sphere.

If $n=2$, our f is a poset isomorphism.

Suppose $n \geq 3$. f maps injectively

$$\underbrace{[xz^{-1}, yz^{-1}] \setminus \{xz^{-1}, yz^{-1}\}}_{S^{n-2} \text{ comb. sphere}} \text{ in } \underbrace{[x,y] \setminus \{x,y\}}_{S^{n-2} \text{ comb sph.}}.$$

the only way to have an inclusion between two $(n-2)$-homogeneous boundaryless connected manifolds is via an isomorphism.

Therefore $f: I' \to I$ is a posed isomorphism.

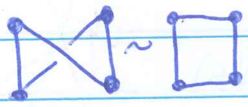
By a theorem of Dyer, the rank of a refl. subgroup is bounded by the number of generators, then

$\# \chi(W'') \leq n$.

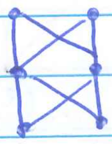
Now $W'' \subset W'$. Also $W' \subseteq W''$ since for $u, v \in I'$ we have

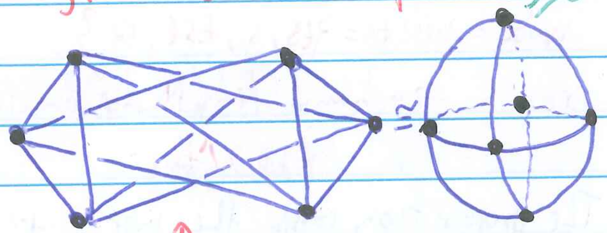
$f(u) f(v)^{-1} = uv^{-1}$. Hence $W' = W''$ $\square$.

e.g $W = S_3 = D_6$,

$[e, sts] \setminus \{e, sts\} =$  $\cong S^1$

$W = D8$

$[e, stst] \setminus \{e, stst\} =$  As an abstract simplicial complex every chain of length 2 is a 2-simplex. $\cong S^2$



8 triangles

Corollary (Miracle 3) For each $n \in \mathbb{N}$, there are only finitely many isomorphism types of Bruhat intervals of length n occurring in finite Coxeter groups.

Proof let I such interval. By previous proposition $\exists I'$ in a rank n finite Coxeter group. By classification of finite Cox. systems there are finitely many families of such Coxeter systems, each one contains only finitely many iso types of Bruhat intervals $\square$. THE END.

from
(Thomas Gobet notes)

Reflection groups and Coxeter groups

Let V be a f.d. vsp $/\mathbb{R}$. An element $s \in GL(V)$ is a reflection if $H_s := \operatorname{Ker}(s - id_V)$ is a hyperplane and $s^2 = 1$. ($\Rightarrow s$ has a unique eigenvalue not equal to 1, which has to be -1). A refl s has the form ($s \in GL(V)$)

$$s(x) = x - \check{\alpha}_s(x)\,\alpha_s \text{ where}$$

- $\check{\alpha}_s \in V^*$ is a linear form w/kernel H_s. $\check{\alpha}_s : V \longrightarrow \mathbb{R}$. (linear, $\ker(\check{\alpha}_s) = H_s$)
- $\alpha_s \in V$ is an eigenvector of $s : V \longrightarrow V$ with e.value -1. s.t. $\check{\alpha}_s(\alpha_s) = 2$.

Let $W \subset GL(V)$ be a subgroup, $\operatorname{Ref}(W) := \{s \in W \mid s \text{ is a reflection}\}$. Then W is a real reflection group if $W = \langle \operatorname{Ref}(W) \rangle$ (⚠ depends on V!)

Note: reflections are stable by W-conjugacy. ($w \in W$, $r \in \operatorname{Ref}(W)$ then $wrw^{-1} \in \operatorname{Ref}(W)$.)

We call α_s (resp. $\check{\alpha}_s$) a root (resp. coroot) attached to s. A reflection group is irreducible if V is an irreducible rep of W. If W is finite: V is a direct sum of irreps. (and W is the direct product of the corresp. subgps.)

Let $W \subseteq GL(V)$, $V = \mathbb{R}^n$ be a fin. refl. grp. Let $\mathcal{A}_W := \{H_s\}_{s \in W}$ be the corresponding (W-invariant) set of reflecting hyperplanes.

Obs For $H \in \mathcal{A}_W$, $\exists!$ reflection $s_H \in W$ s.t. $H_{s_H} = H$.

Chamber geometry Let $\mathcal{A}$ be a finite collection of hyperplanes. The connected components of $V - \bigcup_{H \in \mathcal{A}} H$ are called the chambers (or alcoves) of the arrangement $\mathcal{A}$. The walls of a chamber C are those $H \in \mathcal{A}$ s.t. $H \cap \overline{C}$ contains a non-empty open set of H.

Thm Let W be a finite refl gp in a f.d vs V. Let C be a chamber with set of walls $\mathcal{M}$, let $S := \{s_H \mid H \in \mathcal{M}\}$. Then (W, S) is a Coxeter system.

Lemma: Let $G \subseteq GL(V)$ be a finite subgp, V f.d v.s. There exists a G-invariant scalar product on V. (In particular, reflections in G are orthogonal refl^n wrt this scalar product, i.e., the (-1)-eigenspace of s is $H_s^\perp$)

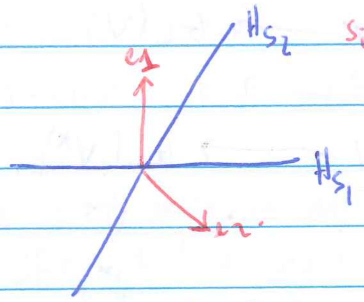
Thm Finite reflection groups are Coxeter groups.

Q: Can we realize every (finite) Coxeter group as a reflection group?

(,)

Let s_1, s_2 orthogonal refl^n w.r.t standard scalar product in $\mathbb{R}^2$. Choose $e_1 \perp H_{s_1}$, $e_2 \perp H_{s_2}$ s.t. $(e_1, e_1) = 1 = (e_2, e_2)$, and define $\check{e}_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $\check{e}_i(v) = 2(v, e_i)$. We got

$$s_i(v) = v - \check{e}_i(v)\,e_i.$$



Tits representation Let (W, S) be any (not nec. finite) Coxeter system. Let $V = \bigoplus_{s \in S} \mathbb{R} e_s$. Def a symmetric bilinear form

$$B : V \times V \longrightarrow \mathbb{R}$$
$$(e_s, e_t) \longmapsto -\cos\left(\frac{\pi}{m_{s,t}}\right) \quad \left(\frac{\pi}{m_{s,t}} = 0 \text{ if } m_{s,t} = \infty\right)$$

Prop The map

$$s \longmapsto \{\sigma_s : v \longmapsto v - 2B(v, e_s)e_s\}$$

defines an action of W on V, for which $B(\_,\_)$ is W-invariant. ($w(v,w) = (wv, ww)$)

Thm $s \longmapsto \sigma_s$ is faithful rep.

About the proof: One need a notion of "chambers", s.t. if $w=1$, then $\sigma_w(C)\cap C = \phi$.

The correct setting to do this is not V, because in general when W is infinite the reflecting hyperplanes do not distinguish the reflections. Hence we replaces V by the contragredient rep. V^*

$$(sf^*)(v) = f^*(sv) \quad \forall s\in S.$$

Choosing the dual basis $\{e_s^*\}_{s\in S}$ to $\{e_s\}_{s\in S}$ and setting

$$C := \{x^*\in V^* \mid x^*(e_s)>0 \quad \forall s\in S\}.$$

it can be shown that if $w\neq 1$, then

$\sigma_w(C)\cap C = \phi$ □.

e.g. $\tilde{A}_1$ type $W = \langle s,t \mid s^2=t^2=e\rangle$

$S=\{s,t\}$

$$V = \bigoplus_{s\in S} \mathbb{R}e_s = \mathbb{R}e_s \oplus \mathbb{R}e_t.$$

choose e_s^* in V^*.

$$\rho : W \longrightarrow GL(V).$$

$$s \longmapsto \sigma_s : v \longmapsto v - 2B(v,e_s)e_s$$

$$\rho^* : W \longrightarrow GL(V^*)$$

$$s \longmapsto \sigma_s^* : V^* \longrightarrow V^*$$

$$f^* \longmapsto sf^*.$$

$$sf^*(v) = f^*(sv)$$

$\Big(B(\alpha_s,\alpha_s) = 1,\ B(\alpha_s,\alpha_{s'})\leq 0$ if $s\neq s'\Big)$

$$\langle f, \lambda\rangle = \langle w(f), w(\lambda)\rangle,\ w\in W,\ f\in V^*,\ \lambda\in V$$

$Z_s := \{f\in V^* \mid \langle f,\alpha_s\rangle = 0\}$,

$A_s := \{f\in V^* \mid \langle f,\alpha_s\rangle > 0\}$

$A_s' := \{f\in V^* \mid \langle f,\alpha_s\rangle < 0\} = s(A_s$.

$$C = \bigcap_{s\in S} A_s.$$

$$V = \oplus\, \mathbb{R}\alpha_s$$

Notation;

$\{f_i\}_{i=1}^{n} \subset V^*$ dual basis $n=|S|$

$s(f_i)(\alpha_s) = f_i(s(\alpha_s))$, $s\neq s_i$

$= 0 = f_i(\alpha_s) = f_i(-\alpha_s)$.

$\boxed{s=s_i}$

$s(f_i)(\alpha_s) = f_i(-\alpha_s) = -f_i(\alpha_s)$.

$f_i(\alpha_s)$

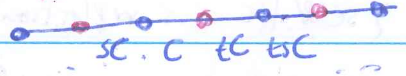
Brown book in buildings

§2 Coxeter groups

§2.2.2 The infinite Dihedral group

D_∞. $W = \{\langle s,t \mid s^2=t^2=e\rangle$

$L = \{y=1\}$ an "affine line"

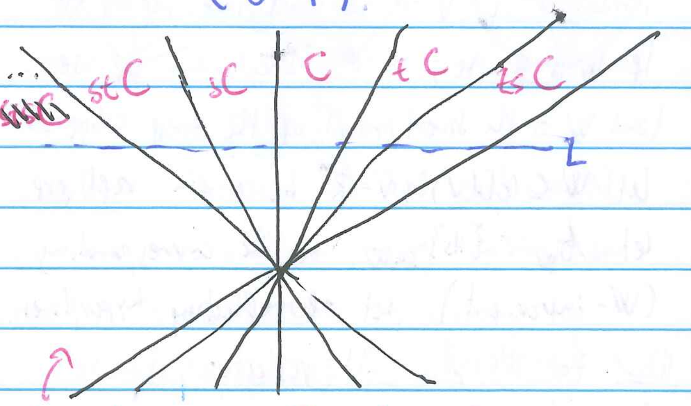
$L =$ 

$L\subset \mathbb{R}^2 = V$. The affine action of W on L extends to a linear action of W on V

$s(x,1) = (-x,1)$ can act

$$s \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$t \longmapsto \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$



the Chambers of D_∞; linear version.

Def V real v.s (not nec. endowed w/ inner product, then a linear reflection on V is a linear map that fix an hyperplane H and acts by (-1) on some complement H' of H (i.e. $V = H\oplus H'$.

Before (W finite) and (V euclidean) a reflection was orthogonal in the sense $H' = H^\perp$ w.r.t. std inner product.

(⚠ A linear reflection as just defined is not uniquely determined by its hyperplane H of fixed points)

Still in D_∞ ✓ true the fact that W is generated by linear reflections whose associated hyperplanes are two walls of a "fundamental chamber" C. And is still true that $\overline{C}$ is a strict fundamental domain for the action of W on $\bigcup_{w \in W} w\overline{C}$. But this union is not the whole vector space V. It is, rather, the convex cone consisting of the upper half plane together with the origin.

<u>Chamber geometry</u>: Since we have no natural inner product on V, we introduce the dual space V^* and use inner-product notation for the canonical pairing

$V^* \times V \longrightarrow \mathbb{R}$, i.e.

$\langle f, v \rangle := f(v)$ for $f \in V^*$, $v \in V$.

Define $e_s, e_t \in V^*$ by $(V = \mathbb{R}^2)$

$\langle e_s, (x,y) \rangle := x$ $W \curvearrowright V$

$\langle e_t, (x,y) \rangle := y - x$.

<u>Rk</u>: The fixed hyperplane of s is given by $\langle e_s, \_ \rangle = 0$.

Fundamental chamber $C := \{\langle e_s, - \rangle > 0\} \cap \{\langle e_t, - \rangle > 0\}$

$M = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}$ Coxeter matrix.

Define $B : V^* \times V^* \longrightarrow \mathbb{R}$ by

$B(e_t, e_t) = B(e_s, e_s) = 1 = -\cos(\frac{\pi}{1})$

$B(e_t, e_s) = B(e_s, e_t) = -\cos\frac{\pi}{\infty} = -1$.

$s'(f) = f - 2B(e_s, f)e_s$ → they fix

$t'(f) = f - 2B(e_t, f)e_t$ ↗ an hyperplane in V^*

The fixed hyperplane $A_s' = A_t' = \mathbb{R}(e_s + e_t)$

$s'(e_s) = e_s - 2e_s = -e_s$, $t'(e_t) = -e_t$

$s'(e_t) = e_t + 2e_s$, $t'(e_s) = e_s + 2e_t$.

$W \curvearrowright V$ $W \curvearrowright V^*$; Contragredient action

$f \in V^*$, $(s \cdot f)(v) := f(s \cdot v)$ $(s.f) \in V^*$

i.e. $\langle sf, v \rangle = \langle f, sv \rangle$.

$s^*(e_s) = ?$ $v = (x,y) \in V = \mathbb{R}^2$.

$s(e_s)(v) = \langle e_s, s(v) \rangle = \langle e_s, (-x, y) \rangle = -x$

$s'(e_s) = -e_s \Rightarrow \langle -e_s, (x,y) \rangle = -x$.

$\therefore s^*(e_s) = s'(e_s)$, (1)

$s^*(e_t) = ?$

$s^*(e_t)(x,y) = \langle e_t, (-x,y) \rangle = y - (-x) = y + x$.

$s'(e_t) = e_t + 2e_s$

$s'(e_t)(x,y) = \langle e_t, (x,y) \rangle + \langle 2e_s, (x,y) \rangle$

$= y - x + 2x = y + x$.

$\therefore s^*(e_t) = s'(e_t)$ (2) (1)+(2) $\Rightarrow s^* = s'$.

$t^*(e_t)(x,y) = \langle e_t, (2y - x, y) \rangle = y - 2y + x = x - y$.

$t'(e_t)(x,y) = -e_t(x,y) = -(y - x) = x - y$ ✓.

$t^*(e_s)(x,y) = \langle e_s, (2y - x, y) \rangle = 2y - x$.

$t'(e_s)(x,y) = (e_s + 2e_t)(x,y) = x + 2(y - x)$

$= 2y - 2x + x = 2y - x$.

$\therefore t^* = t$ (Rmk $s(x,y) = (-x,y)$, $t(x,y) = (2y - x, y)$)

In summary, start with an abstract v.s. $V^* = \mathbb{R}e_s \oplus \mathbb{R}e_t$ and define an action of W on $V = (\mathbb{R}e_s \oplus \mathbb{R}e_t)^*$ by copying formulas of the finite case using Coxeter matrix.

<u>Construction of Tits rep</u> <u>& dual rep</u>

$V = \bigoplus_{s \in S} e_s \mathbb{R}$ $B(e_s, e_t) = -\cos\frac{\pi}{m_{s,t}}$

If $\alpha \in V$ s.t. $B(\alpha, \alpha) \neq 0$ then $V = \mathbb{R}\alpha \oplus \alpha^\perp$.

$\alpha^\perp := \{x \mid B(\alpha, x) = 0\}$: $\exists$ linear reflection σ on V s.t. $\sigma(\alpha) = -\alpha$ and fixes $\alpha^\perp$. It is clear that σ is orthogonal reflection w.r.t. B.

(i.e. $B(\sigma(x), \sigma(y)) = B(x,y)$) Moreover

$\sigma(x) = x - 2B(\alpha, x)\alpha$

$V = \bigoplus e_s \mathbb{R}$

$\langle e_s^\vee, x \rangle := 2B(e_s, x)$.

$\sigma_s(v) = v - \langle e_s^\vee, x \rangle e_s \quad , \quad e_s^\vee \in V^*$

$\{e_s^\vee\} \subseteq V^*$ for $\xi \in V^*$ the action of s in ξ is given by (contragredient action)

$$\langle s.\xi , x \rangle = \langle \xi, s(x) \rangle \text{ for } x \in V.$$

$$\langle \xi, s(x) \rangle = \langle \xi, x - \langle e_s^\vee, x \rangle e_s \rangle$$
$$= \langle \xi, x \rangle - \langle \xi, e_s \rangle \langle e_s^\vee, x \rangle.$$

NOT RIGHT.

~~$\| \; s \cdot \xi(x) = \xi(x) - 2B(\xi^*, e_s)\, e_s^\vee(x) \; \| \mathbb{R}$~~

~~$\therefore \; \| \; s \cdot \xi = \xi - 2B(\xi^*, e_s)\, e_s^\vee \; \|$~~

$\to \xi^*$?

$$s \cdot \xi = \xi - \langle \xi, e_s \rangle e_s^\vee \quad . \; e_s^\vee \in V^*$$

better!

$\therefore W \circlearrowright V^*$.

eg $\quad W = D_\infty = \langle s, t \mid s^2 = t^2 = e \rangle$.

abstract vector space. $V = \mathbb{R} e_s \oplus \mathbb{R} e_t \quad , \; B(e_s, e_{s'}) = \begin{cases} 1 & s = s' \\ -1 & s \neq s' \end{cases}$.

$v \in V$, $\quad s(v) = v - \langle e_s^\vee, v \rangle e_s$.

$\langle e_s^\vee, e_s \rangle = 2$.

$s(e_s) = -e_s \quad , \quad s(e_t) = e_t + 2e_s$.

$t(e_s) = e_s + 2e_t \quad , \quad t(e_t) = -e_t$

$H_s = (e_s + e_t)\mathbb{R} = \operatorname{span}_{\mathbb{R}}(e_s + e_t)$.

$B(e_s, e_s + e_t) = 0$.

$H_s = H_t \qquad e_s^\perp = e_t^\perp$. $\quad \downarrow$ the form is degenerate.

V^*, $\qquad f \in V^*$, $\underbrace{\qquad}$?.

$s(e_s^\vee) = e_s^\vee - \langle e_s^\vee, e_s \rangle e_s^\vee$
$= -e_s^\vee$. $\qquad t(e_t^\vee) = -e_t^\vee$

$s(e_t^\vee) = e_t^\vee + 2e_s^\vee$

$t(e_s^\vee) = e_s^\vee + 2e_t^\vee$

$V^* = e_s^* \mathbb{R} \oplus e_t^* \mathbb{R} \qquad e_s^*$ algebraic dual of e_s

$$e_s^*(e_{s'}) = \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{if } s \neq s' \end{cases}$$

$V^* = e_s^* \mathbb{R} \oplus e_t^* \mathbb{R}$.

$v = \alpha e_s + \beta e_t$.

$e_s^\vee(v) = 2\alpha + -2\beta = 2(\alpha - \beta)$

$e_t^\vee(v) = -2\alpha + 2\beta = 2(\beta - \alpha)$

$e_s^\vee = -e_t^\vee$. $\qquad e_s^\vee = 2e_s^* - 2e_t^*$

$e_t^\vee = 2e_t^* - 2e_s^*$.

$H_s = \{\langle - , e_s \rangle = 0\}$

$s \cdot \xi = \xi - \langle \xi, e_s \rangle e_s^\vee$

s fixes in V^* the space

$H_s^* = \{x^* \mid \langle x^*, e_s \rangle = 0\}$

$s(e_t^\vee) = e_t^\vee + 2e_s^\vee = e_s^\vee = -e_t^\vee$

$t(e_s^\vee) = e_t^\vee = -e_s^\vee$.

$e_t^\vee = -e_s^\vee$, $\quad e_s^\vee = 2(e_s^* - e_t^*)$

$e_t^\vee = 2(e_t^* - e_s^*)$.

$V^* = \mathbb{R}^2 \ni (x, y)$. $\quad V^* = f_1 \mathbb{R} + f_2 \mathbb{R}$.

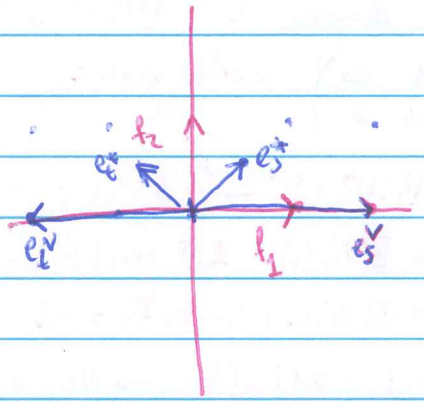
$(1,0) = ?\, f_1 \quad , \quad (0,1) = f_2$.

$f_1 + f_2 = 2e_s^*$.

$f_2 - f_1 = 2e_t^*$.

$f_2 = e_s^* + e_t^*$

$f_1 = e_s^* - e_t^* = \frac{1}{2} e_s^\vee = -\frac{1}{2} e_t^\vee$.



$H_s^* = \{f \mid \langle f, e_s \rangle = 0\}$.

$0 \neq e_s^\vee + e_t^\vee \in H_s^*$.

$e_s^\vee = 2e_s^* - 2e_t^*$
$e_t^\vee = -e_s^\vee.$

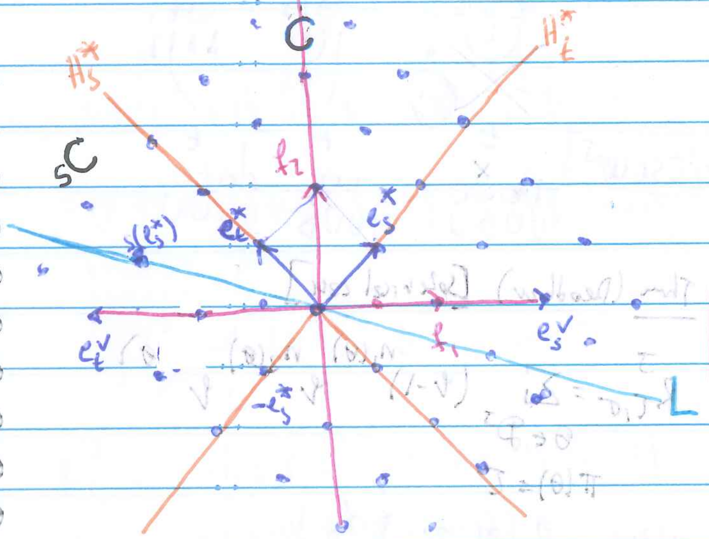
$s(e_s^*) = ?$
$\langle e_s^*, x\rangle = ?$ for $x \in V.$

$x = \alpha e_s + \beta e_t.$

$\langle e_s^*, x\rangle = \alpha$, $\langle e_t^*, x\rangle = \beta.$

$H_s^* = \{x \in V^* \mid \langle x, e_s\rangle = 0\}.$

$H_s^* = e_t^*\mathbb{R}$, H_t^*



$s(\xi) = \xi - \langle \xi, e_s\rangle e_s^\vee.$

$H_t^* = \{x \in V^* \mid \langle x, e_t\rangle = 0\}.$

$f_1 = e_s^* - e_t^* = \frac{1}{2} e_s^\vee = -\frac{1}{2} e_t^\vee.$
$f_2 = e_s^* + e_t^*$

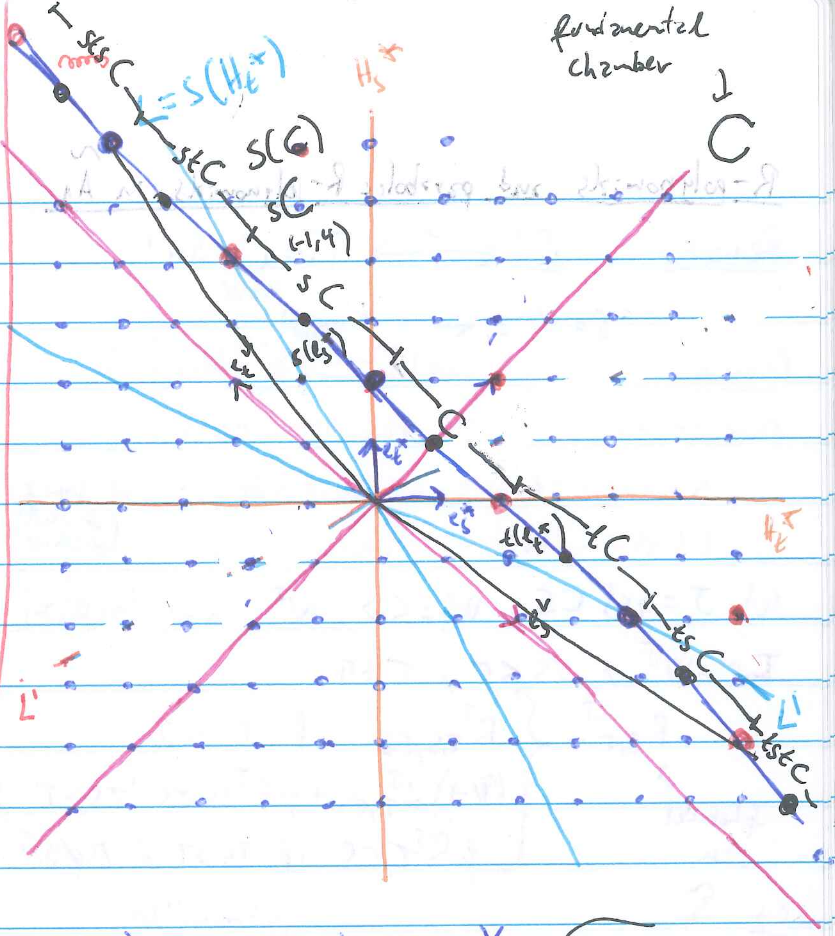
$L = s(H_t^*)$

s fixes H_s^*.

$$s(e_s^*) = e_s^* - \overset{1}{\langle e_s^*, e_s\rangle} e_s^\vee.$$
$$= e_s^* - e_s^\vee.$$
$$= e_s^* - 2e_s^* + 2e_t^*$$
$$= -e_s^* + 2e_t^*.$$

$s(e_t^*) = e_t^* - \cancel{\langle e_t^*, e_s\rangle}^{0} e_s^\vee$
$t(e_s^*) = e_s^*.$
$t(e_t^*) = e_t^* - \langle e_t^*, e_t\rangle e_t^\vee$
$= e_t^* - e_t^\vee$
$= e_t^* + e_s^\vee.$

fundamental chamber



$t(e_t^*) = e_t^* - \langle e_t^*, e_t\rangle e_t^\vee.$
$= e_t^* - e_t^\vee = e_t^* + e_s^\vee.$

$e_s^\vee = 2e_s^* - 2e_t^*.$
$e_t^\vee = 2e_t^* - 2e_s^*$

$t(e_t^*) = e_t^* - \langle e_t^*, e_t\rangle e_t^\vee$
$= e_t^* - e_t^\vee.]$
$= e_t^* + e_s^\vee = e_t^* + 2e_s^* - 2e_t^*.$
$= 2e_s^* - e_t^*$

$$ts = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

$$\underset{s}{\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}}\underset{t}{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

$(-1,4) \in sC.$

$t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\boxed{t = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_s^*$ $t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_t^*$

$s\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\boxed{s = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.}$

$s\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$ $t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

R-polynomials and parabolic R-polynomials in $\tilde{\hat{A}}_1$

Remark: $\Phi^+ \overset{\sim}{\longleftrightarrow} T := \bigcup_{w\in W} wSw^{-1}$.

$\gamma = w(\alpha_s) \longleftrightarrow wsw^{-1}$

Remark 2: In $U_2 = \langle s,t \mid s^2=t^2=e\rangle$ (type $\tilde{A}_1$)

There are only 2 possible reflection orderings:

$s < sts < ststs < \cdots < tstst < tst < t$ and take x

$t < tst < tstst < \cdots < ststs < sts < s$. (minimal length rep.)

Let $J = \{s\} \subset S$. $W_J = \langle s\rangle$, $W^J = W_J\backslash W = \{W_J x\}$

$\tau,\sigma \in W^J$, $\sigma s < \sigma$, $\tau \le \sigma$.

$$R^J_{\tau,\sigma} = \begin{cases} R^J_{\tau s,\sigma s} & \text{if } \tau s < \tau \\ (q-1)R^J_{\tau,\sigma s} + q R^J_{\tau s,\sigma s} & \text{if } \tau s > \tau \ \boxed{\&\ \tau s \in W^J} \\ q R^J_{\tau,\sigma s} & \text{if } \tau s > \tau \ \&\ \tau s \notin W^J. \end{cases}$$

↗ spherical case

eg $\hat{A}_1$.

"set minus" bar.

$W^J = \{e, t, ts, tst, \ldots\}$, $W \backslash W^J = \{s, st, sts, \ldots\}$.

$R_{x,x} = 1$, $\tau,\sigma \in W^J$.

$\ell(\tau,\sigma) = 1$, $R^J_{e,t} = (q-1)\overset{1}{\cancel{R^J_{e,e}}} + q\overset{0}{\cancel{R^J_{t,e}}} = (q-1)$.

$(\tau = e, \sigma = t, \sigma t < \sigma, \tau t > \tau, \tau t \in W^J)$

$R^J_{t,ts} = (q-1)R^J_{t,t} = (q-1)$.

$R^J_{ts,tst} = (q-1)$. $\boxed{R^J_{\tau,\sigma} = (q-1)}$ $\ell(\tau,\sigma) = 1$.

$\ell(\tau,\sigma) = 2$, $R^J_{e,ts} = q R^J_{e,t} = q(q-1)$.

$R^J_{t,tst} = R^J_{e,ts} = q(q-1)$. $\ell(\tau,\sigma) = 2$

$R^J_{ts,tsts} = R^J_{t,tst} = q(q-1)$. $\boxed{R^J_{\tau,\sigma} = q(q-1)}$

$\ell(\tau,\sigma) < 3$. $R^J_{e,tst} = (q-1)R^J_{e,ts} + q\underbrace{R^J_{t,ts}}_{q-1}$

$q^2(q-1) = q(q-1)^2 + q(q-1)$.

$R^J_{t,tsts} = (q-1)R^J_{t,tst} + qR^J_{ts,tst}$

$= (q-1)^2 q + q(q-1) = q^2(q-1)$

$R^J_{ts,tstst} = (q-1)R^J_{ts,tsts} + qR^J_{tst,tsts}$

$= (q-1)q(q-1) + q(q-1) = q^2(q-1)$

$\boxed{R^J_{\tau,\sigma} = q^2(q-1)}$ $\ell(\tau,\sigma) = 3$

$R^J_{e,tsts} = qR^J_{e,tst}$.

$= q \cdot q^2(q-1) = q^3(q-1)$

$R^J_{t,tstst} = R^J_{e,tsts}$ $\boxed{R^J_{\tau,\sigma} = q^3(q-1)}$ $\ell(\tau,\sigma) = 4$

$R^J_{e,tstst} = (q-1)R^J_{e,tsts} + qR_{t,tsts}$

$= (q-1)q^3(q-1) + q\,q^2(q-1)$

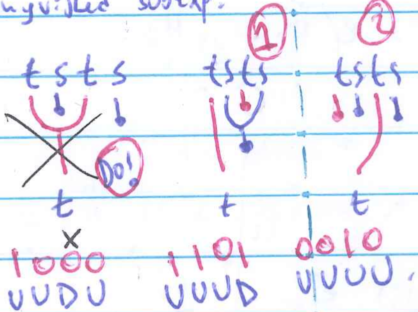
$= q^3(q-1)(q-1+1)$

$= q^4(q-1)$.

Claim $R^J_{\tau,\sigma} = q^{\ell(\tau,\sigma)-1}(q-1)$

Compare to Deodhar's formula!

J-distinguished subexp.



Thm (Deodhar) [spherical case]

$$R^J_{\tau,\sigma} = \sum_{\substack{\theta\in\mathcal{D}^J \\ \pi(\theta)=\tau}} (q-1)^{n_1(\theta)} q^{n_2(\theta)} q^{m(\theta)}$$

where θ has no DO's!

$n_1(\theta) = \#0\ \&\ \theta_{i+1}s_i \in W^J$

$n_2(\theta) = \#0\ \&\ \theta_{i+1}s_i \notin W^J$

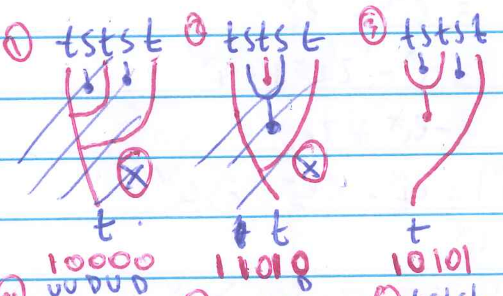
$m(\theta) = \#D1$

$R^J_{t,tsts} = (q-1)q + (q-1)^2 q \neq q^2(q-1)$ ✓)

① $n_1(\theta) = 1$ ② $n_1(\theta) = 2$

$n_2(\theta) = 0$ $n_2(\theta) = 1$

$m(\theta) = 1$ $m(\theta) = 0$



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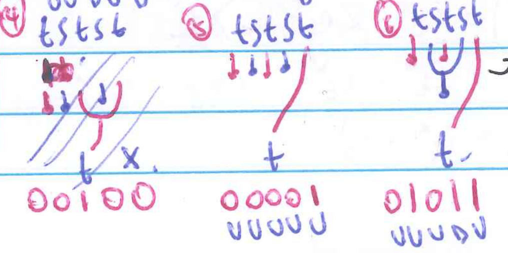


→ NO J-distinguished!

No J-dist!

↓

③ $n_1 = 1$ ⑤ $n_1 = 2$ ⑥ $n_1 = 1$

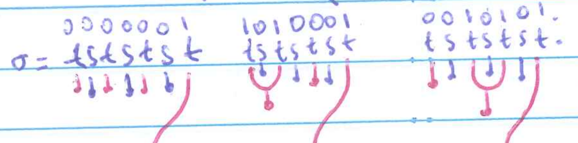
$n_2 = 1$ $n_2 = 2$ $n_2 = "1"$

$m = 1$ $m = 0$ $m = 1$

$$R^J_{t,tstst} = (q-1)q^2 + (q-1)^2 q^2 + \cancel{(q-1)q^2}$$

No!

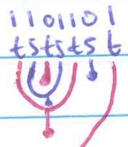
$$= (q-1)q^2(1 + q-1) = q^3(q-1).$$



3 terms!.

$$R^J_{t,\sigma} = (q-1)^3 q^3 + (q-1)^2 q^2 q + (q-1)^2 q^2 q + \bigcirc$$

$$= (q-1)^2 q^3 (q-1 + \overbrace{1+1}^{+2}) = (q-1)^2 q^3 (q+1)$$



$$\rightarrow (q-1) q q^2$$

$$R^J_{t,tststst} = (q-1)q^3\left((q-1)^2 + 2(q-1) + 1\right)$$

$$= (q-1) q^3 (q-1+1)^2$$

$$= (q-1) q^5 \text{ !!}$$

$$R^J_{t,tststst} = R^J_{e,tststs}$$

$$= q R^J_{e,tstst} = q^5 (q-1)$$

(in this case)
Recursion is much faster than Deodhar's formula

(if you know previous cases?).

No parabolic:

$$R_{t,tstst} = (q-1)^2 q + (q-1)^4 + (q-1)^2 q$$

$$= q^2\left[(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2 + (q^{\frac{1}{2}}-q^{-\frac{1}{2}})^4 + (q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2\right]$$

$$= (q-1)^4 + 2(q-1)^2 q.$$

$$\tilde{R}_{t,tstst} = q^4 + 2q^2$$

$$R^J_{t,tstst} = q^2\left[q(q-1)\right] = q^2\left[q^{\frac{1}{2}} q (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\right]$$

$$= q^2\left[(q-1)^2 + (q-1)\right] =$$

$$= q^2\left[(q^{\frac{1}{2}})^2 (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 + q^{\frac{1}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\right]$$