

# **ALGEBRAIC TOPOLOGY NOTES, PART II: FUNDAMENTAL GROUP**

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## 1. BASIC DEFINITIONS

**Definition.** A space  $X$  is *locally path-connected* if every point  $x \in X$  has a neighbourhood basis of path-connected open sets.

The set of path components of  $X$  is denoted  $\pi_0(X)$ .

**Exercise 1.** Let  $X$  be connected and locally path-connected. Show that  $X$  is path-connected.

**Exercise 2.** Let  $W = \{(0, y) \mid |y| \leq 1\} \cup \{(x, \sin(1/x)) \mid 0 < x \leq \frac{2}{\pi}\}$  be the topologist's sine curve in  $\mathbb{R}^2$ . Show that  $W$  is connected, but not path-connected. If we adjoin an arc from  $(0, 1)$  to  $(\frac{2}{\pi}, 1)$  the resulting space is path-connected, but not locally path-connected.

**Definition.** A loop at  $x_0$  in  $X$  is a continuous map  $\sigma : [0, 1] \rightarrow X$  such that  $\sigma(0) = \sigma(1) = x_0$ , i.e., a path from  $x_0$  to  $x_0$ . Two loops at  $x_0$  are homotopic *rel*  $x_0$  if they are homotopic through loops at  $x_0$ .

We may identify the set of homotopy classes of loops at  $x_0$  in  $X$  with the homotopy set  $[[0, 1], \{0, 1\}; X, x_0]$ . The function  $e(t) = (\cos(2\pi t), \sin(2\pi t))$  maps  $[0, 1]$  onto  $S^1$ . Composition with  $e$  induces a bijection between maps from  $(S^1, 1)$  to  $(X, x_0)$  and loops at  $x_0$ , and (hence) between  $[S^1, 1; X, x_0]$  and  $[[0, 1], \{0, 1\}; X, x_0]$ . We shall use freely both versions.

**Definition.** Let  $\alpha, \beta : [0, 1] \rightarrow X$  be paths in  $X$  such that  $\alpha(1) = \beta(0)$ . The *concatenation* of  $\alpha$  and  $\beta$  is the path  $\alpha.\beta$  given by  $\alpha.\beta(t) = \alpha(2t)$  if  $t \leq \frac{1}{2}$  and  $= \beta(2t - 1)$  if  $t \geq \frac{1}{2}$ . Let  $\bar{\alpha}(t) = \alpha(1 - t)$ , for all  $0 \leq t \leq 1$ .

(Note. A minority of authors use the opposite convention for concatenation, as the fundamental groupoid defined below then has better properties.)

**Definition-Proposition-Exercise!** If  $\alpha$  and  $\beta$  are loops at  $x_0$  in  $X$  then so is  $\alpha.\beta$ . If  $\alpha \sim \alpha'$  and  $\beta \sim \beta'$  then  $\alpha.\beta \sim \alpha'.\beta'$ , so concatenation induces a "multiplication" on the set of homotopy classes of loops at  $x_0$ . Moreover this multiplication is associative, and if  $\epsilon_{x_0}$  is the constant loop at  $x_0$  then  $\alpha.\epsilon_{x_0} \sim \alpha \sim \epsilon_{x_0}.\alpha$  and  $\alpha.\bar{\alpha} \sim \epsilon_{x_0} \sim \bar{\alpha}.\alpha$ . The *fundamental group of  $X$  based at  $x_0$*  is the set of homotopy classes of loops at  $x_0$  in  $X$  with this multiplication, and is denoted  $\pi_1(X, x_0)$ .

Let  $f : X \rightarrow Y$  be a continuous map and let  $y_0 = f(x_0)$ . Then composition with  $f$  induces a function  $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ . This function is a homomorphism of groups. Moreover  $id_{X*} = id_{\pi_1(X, x_0)}$  and  $(fg)_* = f_*g_*$ , and so the fundamental group is functorial, for basepoint-preserving maps of pointed spaces.

The subscript 1 refers to the family of higher homotopy groups  $(\pi_q(X) = [S^q; X])$ , which we shall not consider in this course. We shall however consider the role of the basepoint later. Note that loops at  $x_0$  must have image in the path-component of  $X$  containing  $x_0$ , so we may usually assume  $X$  path-connected.

**Definition.** A space  $X$  with basepoint  $x_0$  is *simply-connected* (or *1-connected*) if it is path-connected and any loop at  $x_0$  is homotopic (*rel* basepoint) to the constant loop at  $x_0$ .

**Exercise 3.** Show that the following are equivalent

- (1)  $X$  is simply-connected;
- (2) any two paths  $\alpha, \beta : [0, 1] \rightarrow X$  with the same endpoints  $\alpha(0) = \beta(0)$  and  $\alpha(1) = \beta(1)$  are homotopic (relative to the endpoints);
- (3) any map  $f : S^1 \rightarrow X$  extends to a map from  $D^2$  to  $X$ .

Note in particular that whether  $X$  is simply-connected or not is independent of the choice of basepoint.

**Exercise 4.** Let  $X$  be a metric space, with metric  $\rho$  and basepoint  $x_0$ , and let  $\text{Map}_*(S^1, X)$  be the set of all maps  $f : S^1 \rightarrow X$  such that  $f(1) = x_0$ . Define a metric  $d$  on  $\text{Map}_*(S^1, X)$  by  $d(\alpha, \beta) = \max\{\rho(\alpha(s), \beta(s)) \mid s \in S^1\}$ . Check that this is a metric, and show that  $\pi_0(\text{Map}_*(S^1, X)) = \pi_1(X, x_0)$ . (Similarly,  $\pi_0(\text{Map}(S^q, X)) = \pi_q(X)$ ).

**Exercise 5.** Suppose that  $X = A \cup B$  where  $A$ ,  $B$  and  $A \cap B$  are nonempty, open and path-connected. Let  $*$  be a point in  $A \cap B$ . Show that  $\pi_1(X, *)$  is generated by the images of  $\pi_1(A, *)$  and  $\pi_1(B, *)$  (under the natural homomorphisms induced by the inclusions of these subsets into  $X$ ). In other words, show that every loop at  $*$  in  $X$  is homotopic to a product of finitely many loops at  $*$ , each of which lies either in  $A$  or in  $B$ .

[Hint: you need to use the compactness of the interval  $[0, 1]$  and the assumption that  $A \cap B$  is path-connected.]

**Exercise 6.** Use Exercise (5) to show that  $\pi_1(S^n, *) = 1$  if  $n > 1$ .

**Exercise 7.** Let  $X$  be a space with a basepoint  $*$  and such that there is a map  $\mu : X \times X \rightarrow X$  with  $\mu(x, *) = x = \mu(*, x)$  for all  $x \in X$ . Show that  $\pi = \pi_1(X, *)$  is abelian. (This applies in particular if  $X$  is a topological group).

[Hint: consider the induced homomorphism  $\mu_* : \pi \times \pi \rightarrow \pi$  and work on the algebraic level.]

**Exercise 8.** (a) Let  $G$  be a topological group. (Thus  $G$  is a group and has a topology such that multiplication  $G \times G \rightarrow G$  and inverse  $: G \rightarrow G$  are continuous). Show that  $G_e$ , the path component of the identity, is a normal subgroup and that  $\pi_0(G) \cong G/G_e$  is a group.

(b) Suppose that  $H$  is a **normal** subgroup of  $G$  which is **discrete** as a subspace. Show that if  $G$  is **path-connected**  $H$  is central in  $G$ , i.e., that  $gh = hg$  for all  $g \in G$  and  $h \in H$ .

(c) Give examples to show that each of the three bold-face assumptions is needed for part (b).

[Hint for (b): for each  $h \in H$  consider the function sending  $g \in G$  to  $ghg^{-1}$  and apply each of the bold-face assumptions to show that this function is constant.]

## 2. THE BASIC RESULT: $\pi_1(S^1, 1) \cong \mathbb{Z}$

This is the central calculation of the subject. We shall identify the 1-sphere  $S^1$  with the unit circle in the complex plane. Note that  $S^1$  is a topological group, and we shall take its identity element 1 as the basepoint.

Let  $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$  be the complex exponential, given by  $\exp(z) = e^{2\pi iz}$  for  $z \in \mathbb{C}$ . Then  $\exp$  restricts to a map  $\exp_{\mathbb{R}} : \mathbb{R} \rightarrow S^1$  which is an epimorphism of groups, with kernel the integers  $\mathbb{Z}$ . Thus  $\exp_{\mathbb{R}}$  induces an isomorphism (of groups)  $\mathbb{R}/\mathbb{Z} \cong S^1$ , which is easily seen to be a homeomorphism. (In fact  $\exp_{\mathbb{R}}$  maps each open interval of length  $< 1$  homeomorphically onto an open arc on the circle.)

A lift of a map  $f : Y \rightarrow S^1$  (through  $\exp$ ) is a map  $\tilde{f} : Y \rightarrow \mathbb{R}$  such that  $\exp \tilde{f} = f$ .

**Theorem 1.**  $\pi_1(S^1, 1) \cong \mathbb{Z}$ .

*Proof.* (Sketch.) We show that every loop  $\sigma$  at 1 in  $S^1$  has a unique lift to a path  $\tilde{\sigma} : [0, 1] \rightarrow \mathbb{R}$  starting at  $\sigma(0) = 0$ , that every homotopy of loops lifts to a homotopy of paths, and that  $\deg(\sigma) = \tilde{\sigma}(1)$  is an integer which depends only on the homotopy class of the loop  $\sigma$ . It is then fairly routine to show that  $\deg : \pi_1(S^1, 1) \rightarrow \mathbb{Z}$  is an isomorphism.  $\square$

The inclusion of  $S^1$  into  $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$  is a homotopy equivalence, and the degree homomorphism may be given by contour integration:

$$\deg(\sigma) = \frac{1}{2\pi i} \int_{\sigma} \frac{dz}{z}.$$

**Exercise 9.** Show that  $[X; S^1]$  is an abelian group with respect to the operation determined by multiplication of maps (i.e.,  $fg(x) = f(x)g(x)$  for all  $x \in X$ ).

**Exercise 10.** Show that forgetting the basepoint conditions determines an isomorphism from  $\pi_1(S^1, 1)$  to  $[S^1; S^1]$ .

**Exercise 11** (The fundamental theorem of algebra - proof by homotopy theory). Let  $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$  be a polynomial of degree  $n \geq 1$  with complex coefficients. Show that if  $r$  is large enough  $P_t(z) = (1-t)P(z) + tz^n$  has no zeroes on the circle  $|z| = r$  for any  $0 \leq t \leq 1$ . Hence the maps  $z \rightarrow z^n$  and  $z \rightarrow P(rz)/|P(rz)|$  are homotopic as maps from  $S^1$  to  $S^1$ . If  $P$  has no zeroes the latter map extends to a map from the unit disc  $D^2$  to  $S^1$ . CONTRADICTION. Why?

**Exercise 12.** Let  $f : D \rightarrow D$  be a continuous function, where  $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$  is the closed unit disc in the plane. Show that  $f$  has a fixed point, i.e., that  $f(z) = z$  for some  $z$  in  $D$ .

[Hint: suppose not. Then the line from  $f(z)$  through  $z$  is well defined. Let  $g(z)$  be the point of intersection of this line with  $S^1 = \partial D$  (so  $z$  lies between  $f(z)$  and  $g(z)$ ). Show that  $g : D \rightarrow S^1$  is continuous and  $g(z) = z$  for all  $z \in S^1$ . Obtain a contradiction by considering the induced homomorphisms of fundamental groups.]

### 3. COVERING MAPS

**Definition.** A map  $p : E \rightarrow X$  is a *covering map* (and  $E$  is a *covering space*) if every point  $x \in X$  has an open neighbourhood  $U$  such that  $p^{-1}(U)$  is a nonempty disjoint union of open sets  $V$  for which each restriction  $p|_V : V \rightarrow U$  is a homeomorphism.

Such an open subset  $U$  is said to be *evenly covered* (by  $p$ ) and the subsets  $V$  of  $E$  are called the *sheets* of the covering  $p$  over  $U$ . If  $U$  is *connected* the sheets of  $p$  above  $U$  are the components of  $p^{-1}(U)$ , but in general there may not be a canonical partition of  $p^{-1}(U)$  into sheets.

A covering map  $p$  is onto, by definition, and so  $p^{-1}(x)$  is nonempty, for all  $x \in X$ . This is a discrete subset of  $E$ , called the *fibre* over  $x$ . If  $X$  has a basepoint  $*$  we shall always choose the basepoint for  $E$  to lie in the fibre over  $*$ , so that  $p$  is a basepoint-preserving map.

**Examples.**

- (1)  $\exp : \mathbb{R} \rightarrow S^1$ , given by  $\exp(x) = e^{2\pi i x}$  for  $x \in \mathbb{R}$ .
- (2)  $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ , given by  $\exp(z) = e^{2\pi i z}$  for  $z \in \mathbb{C}$ .

(3) :  $z \mapsto z^n$  for  $z \in S^1$  and  $n \neq 0$ .

(4) The projection of  $X \times F$  onto  $X$ , where  $F$  is a discrete set.

However the restriction of  $exp$  to a map from  $(0, 2)$  to  $S^1$  is *not* a covering map, even though it is onto and is a local homeomorphism, for no neighbourhood of 1 in  $S^1$  is evenly covered by  $exp|_{(0,2)}$ .

**Exercise 13.** Let  $p : E \rightarrow X$  be a covering map, with  $X$  connected. Show that all the fibres  $p^{-1}(x)$  (for  $x \in X$ ) have the same cardinality.

**Definition.** The group of covering transformations of a covering map  $p : E \rightarrow X$  is  $Aut(p) = Aut(E/X) = \{f : E \rightarrow E \mid f \text{ is a homeomorphism and } pf = p\}$ .

(We shall see below that if the spaces involved are connected and locally path-connected the requirement that  $f$  be a homeomorphism is redundant).

Our goal is to show that the fundamental group of a “reasonable” space may be identified with the group of covering transformations of a certain covering of the space. This gives us a strong connection between two apparently quite different ideas – homotopy classes of loops in a space and groups acting as continuous permutations of a (covering) space. (Compare the identification of  $\pi_1(S^1, 1)$  with  $\mathbb{Z}$ , which used the description of  $S^1$  as the quotient of  $\mathbb{R}$  under the action of  $\mathbb{Z}$  by translation.)

A lift of a map  $f : Y \rightarrow X$  (through  $p$ ) is a map  $\tilde{f} : Y \rightarrow E$  such that  $p\tilde{f} = f$ .

In the following three lemmas we assume that  $p : E \rightarrow X$  is a covering map such that  $p(e) = x$ , where  $e$  and  $x$  are basepoints.

**Lemma 2** (The Uniqueness Lemma). Let  $f : Y \rightarrow X$  be a map with connected domain  $Y$ . If  $f_1, f_2 : Y \rightarrow E$  are two lifts of  $f$  which agree at at least one point then  $f_1 = f_2$ .

*Proof.* The subset  $A = \{y \in Y \mid f_1(y) = f_2(y)\}$  is non-empty, by hypothesis.

Let  $y \in Y$  and let  $U$  be an evenly covered open neighbourhood of  $f(y)$ . If  $y \in A$  let  $V$  be a sheet of the covering over  $U$  which contains  $f_1(y) = f_2(y)$ . Then  $f_1^{-1}(V) \cap f_2^{-1}(V)$  is open, and is contained in  $A$  since  $pf_1 = pf_2$  and  $p|_V$  is 1-1. Thus  $A$  is open.

If  $y \notin A$  then we may choose disjoint sheets  $V_1$  and  $V_2$  over  $U$  such that  $f_1(y) \in V_1$  and  $f_2(y) \in V_2$ . Then  $f_1^{-1}(V_1) \cap f_2^{-1}(V_2)$  is open, and is contained in  $Y \setminus A$ . Thus  $A$  is closed. Since  $Y$  is connected we must have  $A = Y$ , i.e.,  $f_1 = f_2$ .  $\square$

As the second lemma is an easy extension of the argument for  $X = S^1$ , we shall not give a proof.

**Lemma 3** (The Existence Lemma). Let  $\sigma : [0, 1] \rightarrow X$  be a path beginning at  $\sigma(0) = x$ . Then there is an unique lift  $\tilde{\sigma} : [0, 1] \rightarrow E$  such that  $\tilde{\sigma}(0) = e$ .  $\square$

We shall derive the third lemma from this one in the next section.

**Lemma 4** (The Covering Homotopy Lemma). Let  $F : Y \times [0, 1] \rightarrow X$  be a map such that  $f = F_0 : Y \rightarrow X$  has a lift  $\tilde{f} : Y \rightarrow E$ . Then  $F$  has an unique lift  $\tilde{F} : Y \times [0, 1] \rightarrow E$  such that  $\tilde{F}(y, 0) = \tilde{f}(y)$  for all  $y \in Y$ .

These three results are clearly generalizations of those we used in considering the case  $p = exp : \mathbb{R} \rightarrow S^1$ .

**Theorem 5.** The homomorphism  $p_* : \pi_1(E, e) \rightarrow \pi_1(X, x)$  is 1-1.

*Proof.* Let  $\sigma$  be a loop at  $e \in E$  such that  $p\sigma$  is homotopic to the constant loop, via a homotopy  $h_t$ . Then  $\tilde{h}_t$  is a homotopy from  $\sigma$  (the unique lift of  $p\sigma$  to a path starting at  $e$ ) to the constant loop at  $e$  (the unique lift of the constant loop at  $x$  to a path starting at  $e$ ). Since  $t \mapsto \tilde{h}_t(1)$  is a path in the fibre over  $x$  it is constant, and so  $\tilde{h}_t$  is a homotopy of loops.  $\square$

**Theorem 6** (The Lifting Criterion). *Let  $p : E \rightarrow X$  be a covering map and  $f : Y \rightarrow X$  a map, where  $Y$  is connected and locally path-connected. Suppose that  $e$  and  $y$  are points of  $E$  and  $Y$ , respectively, such that  $p(e) = f(y) = x$ . Then  $f$  has a lift  $\tilde{f} : Y \rightarrow E$  such that  $\tilde{f}(y) = e$  if and only if  $f_*(\pi_1(Y, y)) \subseteq p_*(\pi_1(E, e))$ .*

*Proof.* The condition is obviously necessary. For each point  $z \in Y$  choose a path  $\omega$  from  $y$  to  $z$  in  $Y$ . Then  $f\omega$  is a path from  $x$  to  $f(z)$  in  $X$ . Let  $\tilde{f}(z)$  be the endpoint of  $\tilde{f}\omega$ , the lift of  $f\omega$  to a path starting at  $e$  in  $E$ . We must check that this is well defined and continuous.

If  $\omega'$  is another path from  $y$  to  $z$  in  $Y$  then  $\omega.\overline{\omega'}$  is a loop at  $y$ . Hence  $f(\omega.\overline{\omega'})$  is a loop at  $x$  which by assumption lifts to a loop at  $e$ . Since  $f\omega$  is homotopic to  $f(\omega.\overline{\omega'}.\omega')$  (as paths from  $x$  to  $f(z)$ ) their lifts have the same endpoint, and so  $\tilde{f}\omega(1) = \tilde{f}(\omega.\overline{\omega'}.\omega')(1)$ . Thus  $\tilde{f}$  is well defined.

Let  $V$  be an open neighbourhood of  $\tilde{f}(z)$ . We may assume that  $V$  is a sheet above an evenly covered open subset  $U = p(V)$  of  $X$ . Let  $S$  be a path-connected open neighbourhood of  $z$  in  $f^{-1}(U)$ . Let  $s \in S$  and let  $\tau$  be a path from  $z$  to  $s$  in  $S$ . Then  $\omega.\tau$  is a path from  $y$  to  $s$  so  $\tilde{f}(s) = \tilde{f}(\omega.\tau)(1)$  which is the endpoint of the lift  $h$  of  $f\tau$  beginning at  $\tilde{f}(z)$  in  $V$ . But  $(p|_V)^{-1}f\tau$  is such a lift, and so  $h = (p|_V)^{-1}f\tau$ , by uniqueness of lifts. Therefore  $\tilde{f}(s) = h(1)$  is in  $V$ . Hence  $\tilde{f}(S) \subseteq V$  and so  $\tilde{f}$  is continuous at  $z$ . Since  $z$  was arbitrary this completes the proof of the Theorem.  $\square$

**Lemma 7.** *Let  $p : E \rightarrow X$  be a covering map, where  $E$  is simply-connected and locally path-connected, and let  $e, e' \in E$  be two points such that  $p(e) = p(e')$ . Then there is a unique map  $\phi : E \rightarrow E$  such that  $p\phi = p$  and  $\phi(e) = e'$ , and  $\phi$  is a homeomorphism.*

*Proof.* By the Lifting Criterion and the Uniqueness Lemma there are unique maps  $\phi, \psi : E \rightarrow E$  which lift  $p$  through itself and are such that  $\phi(e) = e'$  and  $\psi(e') = e$ . Then  $\phi\psi$  and  $\psi\phi$  are lifts of  $p$  through itself which agree with  $id_E$  at  $e'$  and  $e$ , respectively. Hence  $\phi\psi = \psi\phi = id_E$ , by the Uniqueness Lemma.  $\square$

This lemma justifies our claim above that a map  $f : E \rightarrow E$  such that  $pf = p$  is automatically a homeomorphism.

**Theorem 8.** *Let  $p : E \rightarrow X$  be a covering map, where  $E$  is simply-connected and locally path-connected. Then  $G = \text{Aut}(E/X)$  and  $\pi = \pi_1(X, x)$  are isomorphic.*

*Proof.* Define a function  $\rho : G \rightarrow \pi$  as follows. Given  $g \in G$ , choose a path  $\gamma_g$  from  $e$  to  $g(e)$  in  $E$ . Then  $p\gamma_g$  is a loop at  $x = p(e) = p(g(e))$ . Let  $\rho(g) = [p\gamma_g]$ . Since  $E$  is simply-connected any two such paths are homotopic, and a homotopy of such paths in  $E$  projects to a homotopy of loops in  $X$ . Therefore  $\rho$  is a well defined function.

Given  $g, h \in G$  and paths  $\gamma_g, \gamma_h$  from  $e$  to  $g(e)$  and  $h(e)$ , respectively, the concatenation  $\gamma_g.g(\gamma_h)$  is a path from  $e$  to  $g(h(e))$ . Therefore  $\rho(gh) = [p(\gamma_g.g(\gamma_h))] = [p\gamma_g]p\gamma_h = \rho(g)\rho(h)$ , and so  $\rho$  is a homomorphism.

If  $\rho(g) = 1$  then  $p\gamma_g$  is homotopic to the constant loop at  $x$ . We can lift such a homotopy to a homotopy  $h_t$  from  $\gamma_g$  to the constant path at  $e$ . Since the endpoint  $h_t(1)$  is a path in the fibre over  $x$  it is constant, and so  $g(e) (= \gamma_g(1)) = e = id_E(e)$ . Hence  $g = id_E$ , by the above Lemma, and so  $\rho$  is 1-1.

Suppose finally that  $\sigma$  is a loop at  $x$  in  $X$ . Let  $\tilde{\sigma}$  be the lift of  $\sigma$  to a path beginning at  $e$  (i.e.,  $\tilde{\sigma}(0) = e$ ). Let  $e' = \tilde{\sigma}(1)$ . By the above lemma there is a covering homeomorphism  $\phi \in G$  such that  $\phi(e) = e'$ . Since we may take  $\gamma_\phi = \tilde{\sigma}$  we have  $\rho(\phi) = [\sigma]$ . Thus  $\rho$  maps  $G$  onto  $\pi$ , and so is an isomorphism.  $\square$

**Exercise 14.** Suppose that  $X$  is connected and locally path-connected, and that  $*$  is a basepoint for  $X$ . Pointwise multiplication of functions induces a multiplication on the set  $[X, *; S^1, 1]$ . Show that the map  $\theta : [X, *; S^1, 1] \rightarrow Hom(\pi_1(X, *), \mathbb{Z})$  determined by  $\theta(f) = \pi_1(f)$  is a homomorphism of abelian groups. Using the Lifting Criterion, show that it is 1-1. (It is in fact an isomorphism for reasonable spaces  $X$ .)

**Exercise 15.** Let  $p : X \rightarrow Y$  be a covering map, where  $X$  is connected and locally path-connected and  $Y$  is simply-connected. Show that  $p$  is a homeomorphism.

**Exercise 16.** Let  $L = \mathbb{Z} \oplus \mathbb{Z}i$  be the standard lattice in  $\mathbb{C}$ . Let  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  be the extended complex plane. Show that every analytic function from  $\widehat{\mathbb{C}}$  to  $T = \mathbb{C}/L$  is constant.

There is however a rich theory of analytic functions from  $T$  to  $\widehat{\mathbb{C}}$ !

**Exercise 17.** Show that if  $f : S^2 \rightarrow \mathbb{R}^2$  is a function such that  $f(-x) = -f(x)$  for all  $x \in S^2$  then  $f(y) = 0$  for some  $y \in S^2$ .

[Hint. Suppose not. Let  $g(x) = f(x)/\|f(x)\|$ , for all  $x \in S^2$ . Show that there is a commuting diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & S^1 \\ \downarrow a & & \downarrow p \\ P^2(\mathbb{R}) & \xrightarrow{h} & S^1 \end{array}$$

where  $a(x) = a(-x)$ ,  $p(z) = z^2$  and  $h(a(x)) = g(x)^2$ . Apply the lifting lemma to get a function  $\tilde{h} : P^2(\mathbb{R}) \rightarrow S^1$  and deduce a contradiction.]

#### 4. THE COVERING HOMOTOPY LEMMA

The argument given in [Greenberg] is incomplete, although it suffices if we assume also that the space  $Y$  is locally connected. The full details of this argument (for the general case) may be found in [Spanier], pages 67-68.

Here follows an alternative argument, based on the special case  $Y = \{y\}$  (path lifting).

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\tilde{F}} & E \\ \downarrow a & & \downarrow p \\ Y \times [0, 1] & \xrightarrow{F} & X \end{array}$$

1. There is an unique function  $\tilde{F} : Y \times [0, 1] \rightarrow E$  such that

- a)  $p\tilde{F} = F$ ;
- b)  $\tilde{F}(y, 0) = \tilde{f}(y)$  for all  $y \in Y$ ; and
- c) the function  $\tilde{F}_y : [0, 1] \rightarrow E$  defined by  $\tilde{F}_y(t) = \tilde{F}(y, t)$  for all  $0 \leq t \leq 1$  is continuous, for all  $y \in Y$ .

Note that we are ignoring the topology of  $Y$  here, and so this assertion follows immediately from the existence and uniqueness of lifts of paths.

2. The function  $\tilde{F}$  defined above is continuous (i.e., as a function of two variables).

Let  $y \in Y$ . By the usual argument involving continuity of  $F$  and compactness of  $[0, 1]$  we may assume that there is an open neighbourhood  $N$  of  $y$  in  $Y$  and a partition  $0 = t_0 < t_1 < \dots < t_n = 1$  of  $[0, 1]$  such that  $F(N \times [t_{i-1}, t_i])$  is contained in an evenly covered open subset  $U_i$  (say) of  $X$ , for each  $1 \leq i \leq n$ .

Let  $V_i$  be a sheet of  $E$  above  $U_i$  which contains  $\tilde{F}(y, t_{i-1})$ . Since  $\tilde{f}$  is continuous there is an open neighbourhood  $N_1 \subseteq N$  of  $y$  with  $f(N_1) \subseteq V_1$ . Then  $(p|_{V_1})^{-1}F|_{N_1 \times [0, t_1]}$  is a map lifting  $F|_{N_1 \times [0, t_1]}$ . The uniqueness argument of part 1 implies that this map agrees with  $\tilde{F}$  on  $N_1 \times [0, t_1]$ . In particular,  $\tilde{F}$  is continuous on  $N_1 \times [0, t_1]$ .

We argue by induction. Suppose that  $y$  has an open neighbourhood  $N_k \subseteq N$  such that  $\tilde{F}$  is continuous on  $N_k \times [0, t_k]$ . Then there is an open neighbourhood  $N_{k+1} \subseteq N_k$  of  $y$  with  $\tilde{F}(y, t_k) \in V_{k+1}$  for all  $y \in N_{k+1}$ . The map  $(p|_{V_{k+1}})^{-1}F|_{N_{k+1} \times [t_k, t_{k+1}]}$  lifts  $F|_{N_{k+1} \times [t_k, t_{k+1}]}$ . Since it agrees with  $\tilde{F}$  on  $N_{k+1} \times \{t_k\}$  we find as before that  $\tilde{F}$  is continuous on  $N_{k+1} \times [t_k, t_{k+1}]$ . Hence  $\tilde{F}$  is continuous on  $N_{k+1} \times [0, t_{k+1}]$ . After  $n$  steps we see that  $y$  has a neighbourhood  $N_n$  such that  $\tilde{F}$  is continuous on  $N_n \times [0, 1]$ . Hence  $\tilde{F}$  is continuous everywhere.

## 5. GROUPS ACTING ON SPACES

Let  $S$  be a set and  $G$  a group. An action of  $G$  on  $S$  is a homomorphism from  $G$  to  $\text{Perm}(S)$ , the group of bijections from  $S$  to itself. (More precisely, this is a *left* action of  $G$  on  $S$ ). Equivalently, a (left) action of  $G$  on  $S$  is a function  $\mu : G \times S \rightarrow S$  such that  $\mu(1_G, s) = s$  and  $\mu(g, \mu(h, s)) = \mu(gh, s)$  for all  $g, h \in G$  and  $s \in S$ . We shall write  $gs = \mu(g, s)$  for simplicity. The *orbit* of  $G$  through a point  $s \in S$  is the subset  $Gs = \{gs \mid g \in G\}$ . The set  $G \backslash S$  of orbits is called the quotient of the action, and there is a natural map from  $S$  onto  $G \backslash S$ . (A *right* action is a function  $\nu : S \times G \rightarrow S$  such that  $\nu(s, 1_G) = s$  and  $\nu(\nu(s, g), h) = \nu(s, gh)$  for all  $g, h \in G$  and  $s \in S$ . Such an action corresponds to an *anti*-homomorphism from  $G$  to  $\text{Perm}(S)$ ). An action is *effective* if  $g.s = s$  for all  $s \in S$  implies  $g = 1$  in  $G$ , and it is *free* if it satisfies the stronger condition that  $g \neq 1$  implies that  $g.s \neq s$  for all  $s \in S$ .

If  $S$  is a topological space we require the action to be a homomorphism from  $G$  to  $\text{Homeo}(S)$ . Under suitable assumptions on the action there is a natural topology on the quotient set such that the projection  $p$  of  $S$  onto  $G \backslash S$  is continuous. In particular, if every point  $s \in S$  has an open neighbourhood  $V$  which meets none of its translates (i.e.,  $gV \cap hV$  is empty if  $g \neq h$ ) we may take the images  $p(V)$  of such open sets as a basis for the topology of  $G \backslash S$ . As  $p(V)$  is then an evenly covered open neighbourhood of  $p(s)$  the map  $p$  is a covering map. The action is said to be *free and properly discontinuous*. We shall consider only such group actions.



**Examples.**

- (1) Tori:  $\mathbb{Z}^n$  acting on  $\mathbb{R}^n$ .
- (2) Projective spaces:  $\{\pm 1\}$  acting on  $S^n$ .
- (3) Lens spaces:  $\mathbb{Z}/n\mathbb{Z} = \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$  acting on  $S^{2n+1}$ .
- (4) Figure eight:  $\mathbb{Z}^2$  acting on a rectangular lattice; the free group  $F(2)$  acting on a tree. (See §6 and §12 below).
- (5) Discrete subgroups of topological groups. (E.g.,  $SL(2, \mathbb{Z}) < SL(2, \mathbb{R})$ . The quotient space is homeomorphic to the complement of the trefoil knot!).

Given the simply-connected and locally path-connected covering space  $E$  and the group  $G = \text{Aut}(E/X)$  of covering automorphisms, we can recover  $X$  as the quotient space  $G \backslash E$ . For the Lifting Criterion implies that given any two points  $e_1, e_2$  of  $E$  with  $p(e_1) = p(e_2)$  there is a unique covering automorphism  $g$  such that  $g(e_1) = e_2$ . Thus the orbits of the action of  $G$  on  $E$  correspond bijectively to the points of  $X$ . Similarly, if we divide out by the action of a subgroup  $H \leq G$  we obtain intermediate coverings  $E \rightarrow H \backslash E$  and  $H \backslash E \rightarrow X$ .

## 6. THE INFINITE TV ANTENNA

The universal covering of  $W = S^1 \vee S^1$  is an infinite 4-valent tree  $\widetilde{W}$ . Thus it is a graph in which four edges meet at every vertex, and in which any two points are the endpoints of a unique “geodesic” path. If we give each edge the euclidean metric of the unit interval then the distance from the basepoint to any other vertex is a positive integer.

We shall construct  $\widetilde{W}$  as a subset of the unit disc in  $\mathbb{R}^2$ .

Fix  $\lambda \in (0, \frac{1}{2})$ .

**The vertices:** Let  $O = (0, 0)$  be the basepoint.

The vertices at distance  $n > 0$  from  $O$  (with respect to the natural graph-metric) are the points  $(a, b) = (\sum_{k=1}^n \varepsilon_k \lambda^k, \sum_{k=1}^n \eta_k \lambda^k)$ , where

- (1)  $\varepsilon_j, \eta_k \in \{-1, 0, 1\}$  for all  $j, k \leq n$ ;
- (2) if  $n > 0$  then exactly one of  $\varepsilon_1$  and  $\eta_1$  is 0;
- (3) if  $\varepsilon_j \neq 0$  and  $j < n$  then either  $\varepsilon_{j+1} = \varepsilon_j$  and  $\eta_{j+1} = 0$  or  $\varepsilon_{j+1} = 0$  and  $\eta_{j+1} = \pm 1$ ;
- (4) if  $\eta_k \neq 0$  and  $k < n$  then either  $\eta_{k+1} = \eta_k$  and  $\varepsilon_{k+1} = 0$  or  $\eta_{k+1} = 0$  and  $\varepsilon_{k+1} = \pm 1$ .

Note that exactly one of  $\varepsilon_k$  and  $\eta_k$  is 0, for each  $1 \leq k \leq n$ , and that the vertices all lie in the open disc of radius  $\sum_{k>0} \lambda^k = \lambda/(1 - \lambda)$  (which is  $< 1$ ).

**The edges:** Connect  $O$  to each of  $(\pm\lambda, 0)$  and  $(0, \pm\lambda)$ . Let  $\alpha$  and  $\beta$  be the edges from  $O$  to  $(\lambda, 0)$  and  $(0, \lambda)$ , respectively.

If  $n > 0$  and  $\varepsilon_n \neq 0$  connect  $(a, b)$  to each of  $(a + \varepsilon_n \lambda^{n+1}, b)$  and  $(a, b \pm \lambda^{n+1})$ . If  $n > 0$  and  $\varepsilon_n = 0$  connect  $(a, b)$  to each of  $(a \pm \lambda^{n+1}, b)$  and  $(a, b + \eta_n \lambda^{n+1})$ .

Note that if (say)  $\varepsilon_1 = 1$  then  $b < a$ , since  $\lambda^2/(1 - \lambda) < \lambda$ . (Hence) this graph embeds in the disc of radius  $\lambda/(1 - \lambda)$ . Note that the path-length metric on the graph is *not* the euclidean metric of the plane. (In fact  $\widetilde{W}$  does not embed isometrically in  $\mathbb{R}^2$ ).

**The group:** Let  $G$  be the subgroup of  $\text{Homeo}(\widetilde{W})$  generated by the “horizontal” and “vertical” self-homeomorphisms  $X$  and  $Y$  defined as follows:

If  $a + \lambda > 0$  let  $X(a, b) = \lambda(a + 1, b)$ . Otherwise let  $X(a, b) = \lambda^{-1}(a + \lambda, b)$ .

Similarly for  $Y$ .

Then  $G$  acts freely on  $\widetilde{W}$ , with a single orbit of vertices  $G.O$  and two orbits of edges  $G\alpha$  and  $G\beta$ . Moreover each vertex has a small neighbourhood disjoint from all of its translates, and the interior of each edge (i.e., the edge minus its endpoints) is disjoint from all of its translates. Hence  $G\backslash\widetilde{W} \cong S^1 \vee S^1$ , and the projection  $p\widetilde{W} \rightarrow G\backslash\widetilde{W}$  is a covering projection. Since  $\widetilde{W}$  is contractible,  $\pi_1(S^1 \vee S^1) \cong G$ .

[It is more natural to define  $X$  and  $Y$  as hyperbolic transformations of the open unit disc  $\mathbb{D}^2$ . Let  $G_k$  be the group of homeomorphisms of the extended complex plane generated by  $X_k(z) = \frac{kz+1}{z+k}$  and  $Y_k(z) = -iX_k(iz) = \frac{kz-i}{zi+k}$ , where  $1 < k < \infty$ . Then  $X_k$  preserves  $\mathbb{D}^2$  and  $\mathbb{R}$ , and  $X_k(0) = \frac{1}{k}$ , while  $Y_k$  preserves  $\mathbb{D}^2$  and the imaginary axis  $\mathbb{R}i$ , and  $Y_k(0) = -\frac{1}{k}i$ . Hence  $G_k$  preserves  $\mathbb{D}^2$ . Let  $TV$  be the union of the orbits of  $G_k$  through the real and purely imaginary diameters of  $\mathbb{D}^2$ .

Let  $Z_k = X_k Y_k X_k^{-1} Y_k^{-1}$ . Then  $Z_k(z) = \frac{\bar{\alpha}z + \bar{\beta}}{\beta z + \alpha}$ , where  $\alpha = k^4 - 2k^2 - 1 - 2k^2i$  and  $\beta = -2k(1 - i)$ . It is not hard to see that  $TV \cong \widetilde{W}$  if and only if  $|\arg(Z_k^n(1))| < \frac{\pi}{4}$ , for all  $n \geq 0$ . (Sketch the figure!) If so, then  $G_k$  acts freely on  $TV$ .

If  $|\arg(Z_k^n(1))| < \frac{\pi}{4}$ , for all  $n \geq 0$ , then the limit  $L = \lim Z_k^n(1)$  is well-defined, since  $Z_k$  is orientation preserving and maps  $S^1$  to itself, and  $Z_k(L) = L$ . Clearing denominators gives a quadratic equation

$$\beta L^2 + (\alpha - \bar{\alpha})L - \bar{\beta} = 0$$

for  $L$ . Hence

$$L = \frac{\bar{\alpha} - \alpha \pm \sqrt{(\alpha - \bar{\alpha})^2 + 4|\beta|^2}}{2\beta} = \frac{ki \pm \sqrt{2 - k^2}}{(i - 1)}.$$

These roots are in  $S^1$  if  $k \leq \sqrt{2}$ . (We want the root in the fourth quadrant.) If  $k = \sqrt{2}$  then  $\arg\left(\frac{ki \pm \sqrt{2 - k^2}}{(i - 1)}\right) = -\frac{\pi}{4}$ . Therefore if  $1 < k < \sqrt{2}$  then  $TV \cong \widetilde{W}$  and  $G_k$  acts freely on  $TV$ , with quotient  $W$ .]

## 7. $P^\infty(\mathbb{R})$

It can be shown that nontrivial finite groups do not act freely on finite-dimensional contractible spaces. However finite cyclic groups act freely on spheres, and by passing to an infinite union we may construct a contractible space on which all finite cyclic groups act freely.

In §4 of Part I we defined  $S^\infty = \varinjlim (S^n \rightarrow S^{n+1}) = \mathbb{I}S^n / \sim$  as the union of the spheres, with the ‘‘equatorial’’ identifications given by  $s \sim \iota_{n+1}(s)$  for all  $s \in S^n$ . This space is contractible. (See the unnumbered exercise in §12 of Part I.) The actions of  $S^1 \subset \mathbb{C}$  on the unit spheres  $S^{2n-1} \subset \mathbb{C}^n$  given by multiplying all the coordinates by  $z \in S^1$  are clearly compatible as  $n$  increases, and so  $S^1$  acts on  $S^\infty$ .

In particular, multiplication by  $-1$  generates a free action of  $Z/2Z$  on  $S^\infty$ . We may identify the orbit space  $Z/2Z \backslash S^\infty$  with  $P^\infty(\mathbb{R}) = \varinjlim (P^n(\mathbb{R}) \rightarrow P^{n+1}(\mathbb{R}))$ , which thus has fundamental group  $Z/2Z$  and contractible universal covering space.

Spaces with contractible universal covering spaces are said to be *aspherical*. The homotopy type of an aspherical space is determined by its fundamental group, and its homology can be calculated in purely algebraic terms. This realization led to the application of homological methods to group theory, and subsequently to their application in number theory and algebraic geometry.

## 8. INTERMEDIATE COVERINGS AND SUBGROUPS OF THE FUNDAMENTAL GROUP

**Theorem 9.** *Let  $p : E \rightarrow X$  and  $q : E' \rightarrow X$  be covering maps with  $E$  and  $E'$  connected and  $X$  locally path-connected. If  $\tilde{p} : E \rightarrow E'$  is a map such that  $p = q\tilde{p}$  then  $\tilde{p}$  is also a covering map.*

*Proof.* Note first that  $E$  and  $E'$  are also locally path-connected. Let  $U$  be a path-connected open subset of  $X$  which is evenly covered for both  $p$  and  $q$ . Then the sheets above  $U$  for  $p$  and  $q$  are the path components of the preimages:  $p^{-1}(U) = \coprod_{\alpha \in A} W_\alpha$  and  $q^{-1}(U) = \coprod_{\beta \in B} V_\beta$ , where  $p|_{W_\alpha}$  and  $q|_{V_\beta}$  are homeomorphisms. Since each  $W_\alpha$  is path connected and  $\tilde{p}$  is continuous  $\tilde{p}(W_\alpha) \subseteq V_{\rho(\alpha)}$ , for some  $\rho : A \rightarrow B$ . Since  $p|_{W_\alpha}$  and  $q|_{V_{\rho(\alpha)}}$  are homeomorphisms it follows that  $\tilde{p}|_{W_\alpha}$  is a homeomorphism onto  $V_{\rho(\alpha)}$ . It follows easily that each  $V_\beta$  is evenly covered for  $\tilde{p}$ .

In particular,  $\tilde{p}(E)$  is open in  $E'$ . Let  $z$  be a point of  $E'$  in the closure of  $\tilde{p}(E)$ , and let  $V$  be a neighbourhood of  $z$  which is evenly covered for  $\tilde{p}$ . Then  $V \cap \tilde{p}(E)$  is nonempty, and so  $V \subseteq \tilde{p}(E)$  (since it is evenly covered). Hence  $f \in \tilde{p}(E)$  and so  $\tilde{p}(E)$  is closed. Since  $E'$  is connected it follows that  $\tilde{p}(E) = E'$ , i.e.,  $\tilde{p}$  is onto. Hence  $\tilde{p}$  is a covering projection.  $\square$

**Exercise 18.** *Show that “ $p = qr$ ,  $p$  a covering,  $r$  onto” does not imply that  $q$  is a covering.*

**Exercise.** *Let  $q : E \rightarrow X$  be a covering map. Suppose that  $V$  and  $V'$  are connected open subsets of  $E$  such that  $q(V) = q(V')$  and  $q|_V$  and  $q|_{V'}$  are homeomorphisms. Then either  $V = V'$  or  $V \cap V' = \emptyset$ .*

**Exercise 19.** *Let  $S$  be the circle of radius  $\frac{1}{2}$  with centre  $(-\frac{1}{2}, 0)$  and let  $X = S \cup W = S \vee W$ , where  $W$  is the topologist’s sine curve. (See Exercise 2). Let  $E = \mathbb{R} \vee (\mathbb{Z} \times W)$ , where  $(n, (0, 0)) \in \mathbb{Z} \times W$  is identified with  $n \in \mathbb{R}$ , for all integers  $n \in \mathbb{Z}$ . Then  $\mathbb{Z}$  acts freely and properly discontinuously on  $E$  by translation, with orbit space  $X$ . Show that the sheets above a neighbourhood of  $w = (0, 1) \in W$  of diameter strictly less than 1 are not determined by their intersection with the fibre over  $w$ . (Note that  $W$  fails to be locally path-connected).*

It follows from the Lifting Criterion that if  $p : E \rightarrow X$  is a covering map with  $E$  simply-connected and locally path-connected covering then  $p$  is *universal* in the sense that it factors through any other covering map. For if  $q : E' \rightarrow X$  is a covering map such that  $q(e') = x$  for some basepoint  $e'$  then  $p$  lifts to a map  $p' : E \rightarrow E'$  such that  $p'(e) = e'$  and  $qp' = p$ , and  $p'$  is also a covering map, by Theorem 9. The argument sketched in the section on group actions implies that  $E' = H \backslash E$ , where  $\rho(H) = q_*(\pi_1(E', e'))$ .

If  $E'$  is also 1-connected then by the same argument there is a map  $q' : E' \rightarrow E$  such that  $q'(e') = e$  and  $q = pq'$ . It then follows by uniqueness of lifts that  $q'p' = id_E$  and  $p'q' = id_{E'}$ , and so  $E$  and  $E'$  are homeomorphic.

More generally, there is a bijective correspondence between isomorphism classes of connected covering maps  $q : E' \rightarrow X$  and subgroups of  $\pi_1(X, x)$  which sends the covering map  $q$  to  $q_*(\pi_1(E', e'))$  and the subgroup  $\kappa$  of  $\pi_1(X, x)$  to the covering map  $p_\kappa : \rho^{-1}(\kappa) \backslash E \rightarrow X$ . There is a close analogy with Galois Theory!

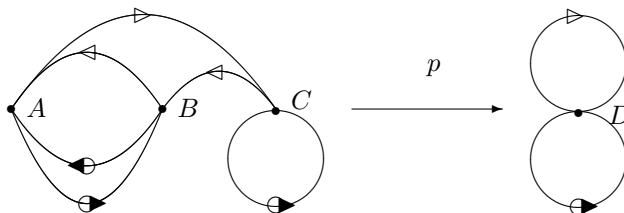
If we attempt to carry through the construction of the isomorphism  $\rho$  of Theorem 8 without the assumption that  $E$  be simply-connected we find the homotopy class of  $[p\gamma_g]$  depends on the choice of path  $\gamma_g$  from  $e$  to  $g(e)$  in  $E$ , and only the left coset  $p_*(\pi_1(E, e))[p\gamma_g]$  in  $p_*(\pi_1(E, e)) \backslash \pi_1(X, x)$  is well defined. Moreover not every

coset is realised. In fact  $[p\overline{\gamma_g}]$  is in the normalizer  $N_{\pi_1(X,x)}(p_*(\pi_1(E,e)))$ , since  $[p\overline{\gamma_g}]p_*\alpha[p\overline{\gamma_g}]^{-1} = [p(\gamma_g g(\alpha)\overline{\gamma_g})]$  is in  $p_*(\pi_1(E,e))$ , for any  $[\alpha]$  in  $\pi_1(E,e)$ . (In general, if  $H$  is a subgroup of  $G$  then  $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$  is the largest subgroup of  $G$  in which  $H$  is normal, and the coset space  $N_G(H)/H = H \backslash N_G(H)$  is a group). Thus we obtain a function

$$\rho_E : \text{Aut}(E/X) \rightarrow N_{\pi_1(X,x)}(p_*\pi_1(E,e))/p_*(\pi_1(E,e)).$$

**Exercise 20.** Check that  $\rho_E$  is an isomorphism.

**Exercise 21.** (a) Show that the map depicted below is a covering map, but that  $\text{Aut}(p)$  is trivial.



(b) Find a 2-fold covering of the domain of  $p$  such that the resulting 6-fold covering of the figure eight  $S^1 \vee S^1$  has automorphism group  $S_3$  (the symmetric group on 3 letters, i.e., the nonabelian group of order 6).

**Exercise 22.** Show that every connected covering of  $S^1$  is equivalent to either  $\text{exp} : \mathbb{R} \rightarrow S^1$  or to one of the  $n^{\text{th}}$  power maps  $p_n : S^1 \rightarrow S^1$  with  $p_n(z) = z^n$ , for some  $n \geq 1$ .

## 9. EXISTENCE OF UNIVERSAL COVERINGS

If  $X$  has a simply-connected and locally path-connected covering  $p : \tilde{X} \rightarrow X$  then  $X$  is connected (since  $p$  is onto) and locally path-connected (since  $p$  is a local homeomorphism). Moreover every point of  $X$  has an open neighbourhood  $U$  such that any loop in  $U$  is null homotopic in  $X$ . (For if  $U$  is evenly covered the inclusion of  $U$  into  $X$  factors through a map to  $\tilde{X}$ .)

Conversely, it can be shown that if  $X$  is connected, locally path-connected and every point of  $X$  has an open neighbourhood with the above property (“*semilocally 1-connected*”!) then it has a simply-connected covering (necessarily locally path-connected). We shall prove this under a somewhat more restrictive hypothesis.

**Theorem 10.** *Let  $X$  be a connected and locally path-connected space in which every point has a simply-connected open neighbourhood. Then  $X$  has a simply-connected covering.*

*Proof.* Choose a basepoint  $* \in X$ , and let  $P$  be the set of all paths  $\alpha : [0, 1] \rightarrow X$  starting at  $\alpha(0) = *$ . Define an equivalence relation  $\sim$  by  $\alpha \sim \beta$  if  $\alpha(1) = \beta(1)$  and  $\alpha$  and  $\beta$  are homotopic as paths between their common endpoints. Let  $\tilde{X} = P/\sim$  and let  $p : \tilde{X} \rightarrow X$  be the function defined by  $p([\alpha]) = \alpha(1)$ , where  $[\alpha]$  is the equivalence class of  $\alpha$ .

We define a topology on the set  $\tilde{X}$  as follows. For each path-connected open neighbourhood  $U$  of  $p([\alpha])$  let  $U_\alpha = \{[\alpha\beta] \mid \beta : [0, 1] \rightarrow U, \beta(0) = \alpha(1)\}$ . We take the collection of all such sets  $U_\alpha$  as a basis for the topology of  $\tilde{X}$ . Since

$p^{-1}(U) = \cup_{\gamma(1)=\alpha(1)} U_\gamma$  the map  $p$  is continuous, and since  $p(U_\alpha) = U$  (by the path-connectedness of  $U$ ) it is open.

If  $U$  is simply-connected then  $p : U_\alpha \rightarrow U$  is 1-1 (check this!) and onto. Hence  $p|_{U_\alpha}$  is a homeomorphism. Moreover if  $\alpha(1) = \alpha'(1)$  then either  $U_\alpha = U_{\alpha'}$  or  $U_\alpha \cap U_{\alpha'}$  is empty. Thus  $U$  is evenly covered and so  $p$  is a covering map.

Let  $c$  be the constant path at  $*$ . Given any other path  $\alpha$  starting at  $*$ , we may define a path  $\tilde{\alpha} = [\alpha_s]$  from  $[c]$  to  $[\alpha]$  by  $\tilde{\alpha}_s(t) = \alpha(st)$  for all  $0 \leq s, t \leq 1$ . (Note that  $\tilde{\alpha}$  lifts  $\alpha$ , i.e.,  $p([\alpha_s]) = \alpha(s)$ ). Thus  $\tilde{X}$  is path-connected.

If  $\tau$  is a loop in  $\tilde{X}$  at  $[c]$  and  $\alpha = p\tau$  then  $\tau$  lifts  $\alpha$  so  $\tau = \tilde{\alpha}$  so  $[\alpha] = \tilde{\alpha}(1) = \tau(1) = [c]$  so  $\alpha \sim c$ . A homotopy between these loops lifts to a homotopy from  $\tau$  to the constant loop at  $[c]$ . Thus  $\tilde{X}$  is simply-connected.  $\square$

**Exercise 23.** Check that the condition that  $U$  be simply-connected can be weakened to requiring only that every loop in  $U$  be homotopic to a constant loop in  $X$ .

The above argument is analogous to a classical construction by “analytic continuation” of the Riemann surface associated to a somewhere convergent power series.

**Example.** (The *Hawaiian earring*). Let  $C_n$  be the circle in  $\mathbb{R}^2$  with radius  $1/n$  and centre  $(1/n, 0)$ , and let  $X = \cup_{n \geq 1} C_n$ , with its natural topology as a subset of  $\mathbb{R}^2$ . Then  $X$  is not semilocally simply-connected at  $(0, 0)$ , as each  $C_n$  represents a nontrivial element of  $\pi_1(X, (0, 0))$ .

(Note that the obvious bijection from the semilocally simply-connected space  $\vee^{\mathbb{N}} S^1$  onto  $X$  is continuous, but is *not* a homeomorphism.)

**Exercise 24.** Show that  $(S^1)^{\mathbb{N}}$  (the product of countably many circles) does not have a universal covering space.

## 10. VARYING THE BASEPOINT

A path  $\omega : [0, 1] \rightarrow X$  from  $\omega(0) = x$  to  $\omega(1) = x'$  determines an isomorphism  $\omega_{\#} : \pi_1(X, x) \rightarrow \pi_1(X, x')$  by  $\omega_{\#}([\alpha]) = [\bar{\omega}.\alpha.\omega]$ . This isomorphism depends only on the homotopy class (*rel* endpoints) of the path. If  $\omega$  is a loop at  $x$  then  $\omega_{\#}$  is just conjugation by  $[\omega]$ . In particular, if  $X$  is path-connected the isomorphism class of  $\pi_1(X, x)$  is independent of the choice of base-point, and if  $\pi_1(X, x)$  is abelian for some (hence every) basepoint these groups are canonically isomorphic. (In the latter case we really can ignore the basepoint!)

The *fundamental groupoid* of  $X$  is the category whose set of objects is  $X$  itself (forgetting the topology) and whose morphisms are given by  $Hom(x, y) =$  the set of homotopy classes (*rel* endpoints) of paths from  $x$  to  $y$ . Note however that the rule for composing morphisms *reverses* the order of concatenation. In particular,  $Hom(x, x) \cong \pi_1(X, x)$ , for all  $x \in X$ , but the isomorphism sends  $[\alpha] \in Hom(x, x)$  to  $[\bar{\alpha}] \in \pi_1(X, x)$ , i.e., is inversion rather than the identity.

## 11. CARDINALITY OF $\pi_1$

A metric space  $X$  is *separable* if it has a countable dense subset, or equivalently if its topology has a countable basis. (The metric hypothesis below can be relaxed.)

**Theorem 11.** Let  $p : E \rightarrow X$  be a covering map, where  $E$  is connected and  $X$  is a separable, locally path-connected metric space. Then  $E$  is a separable metric space.

*Proof.* The space  $X$  has a countable basis  $\mathcal{B}$  consisting of evenly covered path-connected open sets, and each of these sets is separable. Let  $\mathcal{V}_E$  be the family of open subsets of  $E$  which are sheets over members of  $\mathcal{B}$ . Thus each  $V \in \mathcal{V}_E$  is a path-component of  $p^{-1}(U) = \amalg V_\alpha$ , for some  $U \in \mathcal{B}$ . For each such  $V \in \mathcal{V}_E$  and  $U \in \mathcal{B}$  the intersection  $V \cap p^{-1}(U) = \amalg (V \cap V_\alpha)$  is a disjoint union of open subsets of  $V$ . Since  $V$  is separable at most countably many of these subsets  $V \cap V_\alpha$  can be nonempty. Since  $\mathcal{B}$  is countable it follows that each  $V \in \mathcal{V}_E$  meets only countably many other members of this family.

Given  $W \subseteq E$  let  $V^0(W) = W$  and  $V(W) = \cup\{V \in \mathcal{V}_E \mid V \cap W \neq \emptyset\}$ . Then for each  $e \in E$  the set  $V^\infty(e) = \cup_{n \geq 0} V^n(\{e\})$  is open, and is separable, since it is a countable union of sets in  $\mathcal{V}_E$ . If  $V^\infty(e) \cap V^\infty(e')$  is nonempty then  $V^\infty(e) = V^\infty(e')$ . Hence these sets are also closed. Since  $E$  is connected these sets are all equal to  $E$ , and so  $E$  is separable.

Let  $n(x, y) = \min\{n \geq 0 \mid y \in V^n(\{x\})\}$ , and let  $d_E : E \times E \rightarrow [0, \infty)$  be the function defined by  $d_E(x, y) = d(p(x), p(y)) + n(x, y)$ . Then  $d_E$  is a metric on  $E$  which determines the given topology.  $\square$

**Corollary.** *If  $X$  is semilocally 1-connected then  $\pi_1(X, x)$  is countable.*

*Proof.* The fibre of the universal covering is a discrete subset of a separable metric space and so must be countable.  $\square$

The theorem and its corollary apply in particular if  $X$  is compact. Since a connected locally separable metric space is separable, it also applies to any metrizable manifold. We shall have more to say about fundamental groups of compact manifolds later.

Can one extend the idea of this theorem to bound the cardinality of  $\pi_1(X, *)$  when  $X$  is connected and locally separable, but not separable?

## 12. FREE GROUPS

**Theorem 12.** *Let  $X$  be a set. There is a group  $F(X)$  and a function  $j : X \rightarrow F(X)$  such that for every group  $H$  and function  $f : X \rightarrow H$  there is a unique homomorphism  $h : F(X) \rightarrow H$  such that  $f = hj$ .*

*The pair  $(F(X), j)$  is unique up to an unique isomorphism.*

*Proof.* (Sketch.) Let  $W$  be the set of words of finite length of the form  $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ , where  $x_i \in X$ ,  $\epsilon_i = \pm 1$  and  $n$  is a non-negative integer. (If  $n = 0$  write 1 for the “empty” word). Define a composition from  $W \times W$  to  $W$  by juxtaposition. Say two words  $v$  and  $w$  are equivalent,  $v \sim w$ , if they can each be reduced to the same word  $u$  by contracting subwords of the form  $x^\epsilon x^{-\epsilon}$  (e.g.,  $x_1 x_2^{-1} x_3^{-1} x_3 x_2 \sim x_1 x_2^{-1} x_2 \sim x_1 \sim x_4^{-1} x_4 x_1$ , etc.). Then the composition respects equivalence classes, and the set  $W/\sim$  with the induced composition is a group, which we shall call  $F(X)$ . There is an obvious 1-1 function  $j : X \rightarrow F(X)$ .

Given  $f : X \rightarrow H$  we may extend it to a function from  $W$  to  $H$  by sending  $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$  to  $f(x_1)^{\epsilon_1} \dots f(x_n)^{\epsilon_n}$  (where the product is formed in  $H$ ). It is not hard to check that equivalent words have the same image, and that we get a homomorphism  $h : F(X) \rightarrow H$  such that  $hj = f$ . It is clear that  $h$  is uniquely determined by  $f$ .

In particular, taking  $H = F(X)$  and  $f = j$  we see that if  $h$  is a self-homomorphism of  $F(X)$  such that  $hj = j$  then  $h = id_{F(X)}$ . If  $(\tilde{F}, \tilde{j})$  is any other such pair then

there are unique homomorphisms  $h : F(X) \rightarrow \tilde{F}$  and  $\tilde{h} : \tilde{F} \rightarrow F(X)$ . such that  $h\tilde{j} = \tilde{j}$  and  $\tilde{h}j = j$ . Hence  $\tilde{h}h = id_{F(X)}$  and  $h\tilde{h} = id_{\tilde{F}}$ , and so  $h$  and  $\tilde{h}$  are mutually inverse isomorphisms.  $\square$

**Addendum.** *The function  $j$  is 1-1 and the group  $F(X)$  is generated by  $j(X)$ .*  $\square$

In the language of categories, writing  $|H|$  for the *set* of elements of  $H$ , we have

$$\text{Hom}_{((Grp))}(F(X), H) = \text{Hom}_{((Set))}(X, |H|)$$

– the free group functor  $F$  from  $((Set))$  to  $((Grp))$  is left adjoint to the forgetful functor  $| - |$  from  $((Grp))$  to  $((Set))$ . (The final assertion of the theorem is typical of such adjoint pairs.) The group  $F(X)$  is called the *free group* with *basis*  $X$ .

Each element of  $F(X)$  has a unique *normal form*, obtained by contracting all possible subwords  $x^\epsilon x^{-\epsilon}$ . The length of an element  $w$  is the number  $\ell(w)$  of letters  $x^{\pm 1}$  in the normal form. (Note also that the notions of normal form and length are defined in terms of the basis  $X$ , and may not be preserved under isomorphisms.)

**Exercise 25.** *Show that  $F(X) \cong F(Y)$  if and only if  $X$  and  $Y$  have the same cardinality  $\#X = \#Y$ . (Note that if  $X$  is finite then  $\#\text{Hom}_{((Grp))}(F(X), Z/2Z) = \#\text{Hom}_{((Set))}(X, |Z/2Z|) = 2^{\#X}$ , while if  $X$  is infinite  $\#F(X) = \#X$ ).*

**Notation:** let  $F(r)$  be the free group with basis of cardinality  $r$ . It is easy to see that  $F(0) = F(\emptyset) = 1$  and that  $F(1) = F(\{x\}) \cong \mathbb{Z}$ .

Let  $\tilde{W}$  be the infinite TV antenna, and let  $G$  be the subgroup of  $\text{Homeo}(\tilde{W})$  generated by  $X$  and  $Y$ . There is an obvious epimorphism from  $F(2) = F(x, y)$  to  $G$ . Let  $w$  be a reduced word in the alphabet  $\{x, y\}$ , and let  $P$  be a vertex of  $\tilde{W}$ . Then  $w(P) = P$  implies that  $w = 1$ . (There is no back-tracking!). More generally, if  $u$  and  $v$  are reduced words and  $u(P) = v(P)$  then  $v^{-1}u(P) = P$ , so  $v^{-1}u$  is not reduced (unless  $u = v = 1$ ). Hence  $u$  and  $v$  have the same initial letter, say  $u = au_1$ ,  $v = bv_1$ . But then  $u_1(P) = v_1(P)$  and  $u_1, v_1$  are reduced. Induction on the length of  $u$  now gives  $u = v$ . In fact, if  $w$  is a reduced word of length  $n$  then  $d(w(O), O) = n$ , and the number of reduced words of length  $n$  is  $4 \cdot 3^{n-1}$ . if  $n \geq 1$ .

These observations imply that the epimorphism  $F(2) \rightarrow G$  is an isomorphism, and so  $\pi_1(S^1 \vee S^1) \cong G$  is free on the obvious generators. Moreover, words in  $\{x, y\}$  have a unique normal form: if  $u, v$  are reduced words and  $u \sim v$  then  $u = v$ .

### 13. COMBINATORIAL GROUP THEORY

The groups that arise in topology are often infinite, and so it is not practical to give their multiplication tables. Instead we use presentations, giving a set of generators (usually finite) and a list of relations between these generators that are sufficient to characterize the group. Combinatorial group theory is the study of groups in terms of such presentations. It has close connections with topology on the one hand and with the theory of formal languages on the other.

Every group is a quotient of a free group. Consider the identity function from  $|H|$  to  $H$ . By the characteristic property of free groups there is a homomorphism from  $F(|H|)$  to  $H$  extending this function, and this homomorphism is clearly onto. In general, this construction is grossly inefficient. We say that  $H$  is *finitely generated* if there is a finite set  $X$  and an epimorphism  $f : F(X) \rightarrow H$ . The set  $X$  may be viewed as a set of *generators* for  $H$ , as every element of  $H$  is the image of a word

in the  $x_i^{\pm 1}$ . If  $f$  is an isomorphism, we call  $X$  a *basis* for  $H$ . In general,  $\text{Ker}(f)$  represents the *relations* between the generators  $X$ .

Let  $F(X)$  be the free group with basis  $X$ , and let  $R$  be a subset of  $F(X)$ . Let  $\langle\langle R \rangle\rangle$  denote the smallest normal subgroup of  $F(X)$  containing  $R$ . Then the quotient  $F(X)/\langle\langle R \rangle\rangle$  is a group. Moreover any group is such a quotient, for if  $f : F(X) \rightarrow H$  is an epimorphism from a free group onto  $H$  then  $\text{Ker}(f)$  is a normal subgroup of  $F(X)$ , and there are many possible choices for the subset  $R$ . (For instance,  $R$  could be all of  $\text{Ker}(f)$ !). We say that  $H$  has a *presentation with generators  $X$  and relators  $R$* , and usually write  $H = \langle X \mid R \rangle$ , although strictly speaking we should also specify the epimorphism  $f$ . Any word in  $X$  which represents an element of  $\langle\langle R \rangle\rangle$  is said to be a *consequence* of  $R$ .

Each relator  $r$  corresponds to an equation  $f(r) = 1$  in  $H$ . It is often convenient to write the relators as *relations* between words, so that the relation  $r = s$  corresponds to the relator  $rs^{-1}$ . We shall do this without further comment below. (A relation  $u = v$  is a *consequence* of  $R$  if  $uv^{-1} \in \langle\langle R \rangle\rangle$ .)

### Examples.

- (1)  $\mathbb{Z} = \langle x \mid \emptyset \rangle$ .
- (2)  $F(x, y) = \langle x, y \mid \emptyset \rangle = \langle x, y, z \mid z \rangle$ .
- (3)  $F(r) = \langle x_1, \dots, x_r \mid \emptyset \rangle$ .
- (4)  $\mathbb{Z}^2$  is the quotient of  $F(x, y)$  via  $h(x) = (1, 0)$  and  $h(y) = (0, 1)$ . As  $\text{Ker}(h)$  is the smallest normal subgroup containing  $xyx^{-1}y^{-1}$  we have the presentation  $\mathbb{Z}^2 = \langle x, y \mid xyx^{-1}y^{-1} \rangle$ . (Equivalently,  $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$ .)

We say that the group  $H$  is *finitely presentable* if there is a presentation  $\langle X \mid R \rangle$  in which both  $X$  and  $R$  are finite sets. (The *normal* subgroup  $\langle\langle R \rangle\rangle$  is only finitely generated if the quotient  $F(X)/\langle\langle R \rangle\rangle$  is finite, or if  $R \subseteq \{1\}$ .)

Two presentations define isomorphic groups if and only if they can be related by a chain of “elementary Tietze transformations” of the following kinds:

- (1) introduce a new generator  $x$  and a new relation  $x = w$  where  $w$  is a word in the old generators;
- (2) adjoin a relation that is a consequence of the other relations; and their inverses.

**Example.** The symmetric group on three letters,  $S_3$ . Let  $r = a^2$ ,  $s = abab^{-2}$ ,  $t = aba^{-1}b^{-2}$ ,  $u = b^3$  and  $v = aba^{-1}b$  in  $F = F(a, b)$ . Then  $S_3 = \langle a, b \mid r, s \rangle$  (via  $f(a) = (12)(3)$  and  $f(b) = (123)$ ). (Equivalently,  $S_3 = \langle a, b \mid a^2, aba = b^2 \rangle$ .)

Now  $t = sb^2r^{-1}b^{-2}$  and  $s = tb^2rb^{-2}$ , so  $\langle\langle r, s \rangle\rangle = \langle\langle r, t \rangle\rangle$  in  $F$ . Hence  $S_3 = \langle a, b \mid r, t \rangle$ . Now  $u = (b^2t^{-1}b^{-2})t^{-1}(at^{-1}a^{-1})r(brb^{-1})$  is in  $\langle\langle r, t \rangle\rangle$ , and so  $S_3 = \langle a, b \mid r, t, u \rangle$ . Now  $v = tu$  and  $t = vu^{-1}$ , so  $\langle\langle r, t \rangle\rangle = \langle\langle r, t, u \rangle\rangle = \langle\langle r, u, v \rangle\rangle$ . Hence  $S_3 = \langle a, b \mid r, u, v \rangle = \langle a, b \mid a^2, b^3, aba^{-1}b \rangle$ . (Note: we cannot delete the relator  $b^3$  from the latter presentation, for the group with presentation  $\langle a, b \mid a^2, aba^{-1}b \rangle$  is isomorphic to the group of homeomorphisms of the real line  $\mathbb{R}$  generated by  $A(x) = -x$  and  $B(x) = x + 1$ , for all  $x \in \mathbb{R}$ , and so is infinite).

If some object or notion defined in terms of presentations for a group has the same value for Tietze-equivalent presentations it is an invariant for the group.

For instance, let  $P$  be a presentation with finitely many generators for a group  $G$  and let  $w_P(n)$  be the number of elements of  $G$  represented by words of length at most  $n$  in the given generators and their inverses. It can be shown that the asymptotic rate of growth of  $w_P$  depends only on  $G$ , and not on  $P$ . If  $G$  is a free group,



or is solvable but not virtually nilpotent then any such function has exponential growth; if  $G$  is abelian or (more generally) virtually nilpotent the growth rate is polynomial.

However there are logical difficulties in using presentations to study groups. For instance, it can be shown that it is impossible to construct a (universal) algorithm which will decide (in every case) whether a given presentation represents the trivial group. One must rely on one's ingenuity, experience and luck. (See exercises 28 and 29 below.)

**Exercise 26.** Find presentations for the groups  $\mathbb{Z} \oplus (Z/3Z)$  and  $\mathbb{Z}^3$ .

**Exercise 27.** Verify that the group with presentation  $\langle x, y \mid x^2 = y^2 = (xy)^2 \rangle$  is isomorphic to the “quaternion group”  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ .

[Hint: show that  $x^2 = (xy)^2$  implies that  $x^{-1}yx = y^{-1}$ , hence  $x^{-1}y^2x = y^{-2}$ , hence  $x^2 = x^{-2}$ . Thus the cyclic group generated by the image of  $x^2$  has order (at most) 2. Adding the extra relation  $x^2 = 1$  gives a presentation for the quotient group, which thus has order 4. Thus our group has order dividing 8. Show that there is a homomorphism from this group onto  $Q_8$ , i.e., that  $Q_8$  is generated by a pair of elements satisfying these relations.]

**Exercise 28.** Show that  $\langle x, y \mid xy^2x^{-1} = y^3, yx^2y^{-1} = x^3 \rangle$  is a presentation of the trivial group.

**Exercise 29.** Show that  $\langle x, y \mid xy^m x^{-1} = y^{m+1}, yx^n y^{-1} = x^{n+1} \rangle$  is a presentation of the trivial group.

**Exercise 30.** Let  $P, Q$  be presentations for groups  $G, H$ , respectively. Use these to give a presentation for the direct product  $G \times H$ .

**Exercise 31.** Show that  $\langle x, y \mid xyx^{-1} = y^3, yxy^{-1} = x^3 \rangle$  is a presentation of  $Q_8$ .

[Hint: compute the commutator  $xyx^{-1}y^{-1}$  in two ways, to see that  $x^2 = y^{-2}$ .]

#### 14. COPRODUCTS AND PUSHOUTS

By a similar procedure to that used in constructing free groups we may construct the *coproduct* of a family  $\{G_x \mid x \in X\}$  of groups indexed by a set  $X$ . (Use words in the elements of the disjoint union of sets  $|G_x| \setminus \{1\}$ , contracting subwords formed from adjacent elements from the same group in the obvious way). Then the coproduct  $*_{x \in X} G_x$  is generated by the images of the  $G_x$ , and any family  $\{f_x : G_x \rightarrow H \mid x \in X\}$  of homomorphisms gives rise to a unique homomorphism from  $*_{x \in X} G_x$  to  $H$ .

**Exercise 32.** Show that  $F(X) \cong *_{x \in X} Z_x$ , where  $Z_x = \mathbb{Z}$  for all  $x \in X$ .

There is a further extension of this construction which is useful in describing how the fundamental group of a union of two spaces with a common subspace is determined by the fundamental groups of the constituent spaces. Suppose given a group  $H$  and two homomorphisms  $\alpha_1 : H \rightarrow G_1$  and  $\alpha_2 : H \rightarrow G_2$ . There is an essentially unique “pushout” of this data: A group  $P$  with homomorphisms  $\omega_1 : G_1 \rightarrow P$  and  $\omega_2 : G_2 \rightarrow P$  with  $\omega_1 \alpha_1 = \omega_2 \alpha_2$  and such that given any other group  $Q$  and homomorphisms  $\psi_1 : G_1 \rightarrow Q$  and  $\psi_2 : G_2 \rightarrow Q$  with  $\psi_1 \alpha_1 = \psi_2 \alpha_2$  there is a unique homomorphism  $\Psi : P \rightarrow Q$  such that  $\Psi \omega_i = \psi_i$  for  $i = 1, 2$ .

The square with vertices  $H, G_1, G_2, P$  and edges  $\alpha_1, \alpha_2, \omega_1, \omega_2$  is called a “pushout square”; loosely speaking  $P$  is called the pushout of  $G_1$  and  $G_2$  over  $H$ . (Strictly speaking, the maps of  $G_1$  and  $G_2$  to  $P$  are part of the pushout data).

The pushout of a diagram may be constructed from the coproduct as follows. Let  $P = G_1 * G_2 / \langle \langle \alpha_1(h)\alpha_2(h)^{-1} \mid h \in H \rangle \rangle$ . (Thus  $P$  is the biggest quotient of  $G_1 * G_2$  in which both images of  $H$  become identified.) Let  $\omega_i : G_i \rightarrow G_1 * G_2 \rightarrow P$  be the obvious maps, for  $i = 1, 2$ .

If  $\alpha_1$  and  $\alpha_2$  are each injective we write  $P = G_1 *_H G_2$  and call this pushout the *generalized free product of  $G_1$  and  $G_2$  with amalgamation over  $H$* . The coproduct of  $G_1$  and  $G_2$  is just the pushout of these groups over the trivial group:  $G_1 * G_2 = G_1 *_{\{1\}} G_2$ .

**Exercise 33.** Show that  $(Z/4Z) *_{Z/2Z} (Z/4Z)$  has a composition series with quotients  $Z/2Z, \mathbb{Z}$  and  $Z/2Z$ .

**Exercise 34.** Describe the pushout of two homomorphisms  $\alpha_i : H \rightarrow G_i$  (for  $i = 1, 2$ ) in terms of amalgamated products of certain quotient groups.

The notions of product, coproduct and pushout square are all essentially categorical, as is the notion of a “free” object - an adjoint to the forgetful functor from  $((Grp))$  or  $((Mod_R))$  (etc.) to  $((Set))$ . See under the headings “limit” and “adjoint functors” in any reference on Categories.

## 15. THE VAN KAMPEN THEOREM

The Van Kampen Theorem is one of those theorems where it is very important to understand the result, but not at all necessary to know the techniques of proof to use efficiently. (I’ve survived thus for years.) The statement is essentially that “the fundamental group of a pushout of spaces is the pushout of their fundamental groups”. More precisely, suppose that  $X$  and  $Y$  are path-connected spaces with union  $Z$ , and that their intersection  $X \cap Y$  is also path-connected. Choose a point  $* \in X \cap Y$  to be the basepoint for all these spaces. Then  $\pi_1(Z, *)$  is the pushout of the homomorphisms determined by the inclusions of  $X \cap Y$  into  $X$  and  $Y$ . Thus it is generated by the images of  $\pi_2(X, *)$  and  $\pi_1(Y, *)$ , and the only relations between the generator of  $\pi_1(Z, *)$  are those which follow from relations in  $\pi_2(X, *)$  or  $\pi_1(Y, *)$  and from setting the images of  $\pi_1(X \cap Y, *)$  in  $\pi_2(X, *)$  and  $\pi_1(Y, *)$  to be equal. The statement is more complicated if one or more of the spaces involved is not connected. (One should use groupoids rather than groups. There is not yet a good analogue for the higher homotopy groups, which is one reason why they are hard to compute, but the Mayer-Vietoris Theorem gives a satisfactory analogue for homology.)

### Examples.

- (1) If  $Y$  is simply-connected then  $\pi_1(X \cup Y, *)$  is the largest quotient of  $\pi_1(X, *)$  in which the image of  $\pi_1(X \cap Y, *)$  is trivial.
- (2) Let  $S^1 \vee S^1$  be the figure eight and let  $*$  be the common point of the two circles. Then  $\pi_1(S^1 \vee S^1, *)$  is free on two generators (one for each circle).

If  $X$  is path connected then  $X \cup_f e^1 \simeq X \vee S^1$ , and hence  $\pi_1(X \cup_f e^1) \cong \pi_1(X) * \mathbb{Z}$ .

**Exercise 35.** Define isometries  $t, u$  of the plane  $\mathbb{R}^2$  by  $t(x, y) = (x, y + 1)$  and  $u(x, y) = (x + 1, -y)$  for all  $(x, y) \in \mathbb{R}^2$ . Verify that  $tut = u$  and that  $t$  and  $u^2$  commute. Let  $G$  be the group of homeomorphisms of  $\mathbb{R}^2$  generated by  $t$  and  $u$ .

(a) Show that the orbit of  $G$  through  $(x, y)$  is the set  $\{(x + m, (-1)^m y + n) \mid m, n \in \mathbb{Z}\}$ .

(b) Show that the projection of  $\mathbb{R}^2$  onto the orbit space  $G \backslash \mathbb{R}^2$  is a covering map. This orbit space is the Klein bottle  $Kb$ .

(c) Show that  $\langle t, u \mid tut = u \rangle$  is a presentation for  $\pi_1(Kb) \cong G$ .

**Exercise 36.** The Möbius band  $Mb$  may be constructed by identifying opposite sides of a rectangle with a half-twist. Explicitly, let

$$Mb = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, |y| \leq 1\} / \sim .$$

where  $(0, y) \sim (1, -y)$  for all  $-1 \leq y \leq 1$ . The image of the line segment  $\{(x, 0) \mid 0 \leq x \leq 1\}$  in  $Mb$  is a circle  $C$ , the centreline of  $Mb$ .

(a) Show that the inclusion of the centreline  $C$  is a homotopy equivalence, and that the inclusion of the boundary  $\partial Mb$  sends a generator of  $\pi_1(\partial Mb) \cong \mathbb{Z}$  to twice a generator of  $\pi_1(Mb) \cong \mathbb{Z}$ .

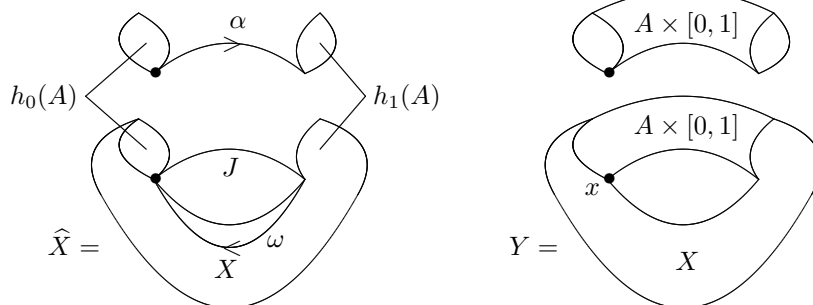
(b) The Klein bottle  $Kb$  may also be constructed by gluing two Möbius bands together along their boundaries. Use Van Kampen's Theorem with the observations of part (a) to obtain a presentation of  $\pi_1(Kb)$ .

(c) Show that the presentation obtained in this way is equivalent to that of Exercise 35.

We may reduce the study of the case with  $X \cap Y$  disconnected to the connected case as follows. Let  $A$  and  $X$  be path-connected spaces, and suppose that  $h_0, h_1 : A \rightarrow X$  are embeddings with disjoint images. Let  $Y = (X \amalg A \times [0, 1]) / \sim$ , where  $(a, 0) \sim h_0(a)$  and  $(a, 1) \sim h_1(a)$ , for all  $a \in A$ . (In other words, glue  $A \times [0, 1]$  to  $X$  along its ends). Fix a basepoint  $a \in A$  and take  $x = h_0(a)$  as the basepoint for each of  $X$  and  $Y$ . Since  $X$  is path-connected there is a path  $\omega : [0, 1] \rightarrow X$  from  $h_1(a)$  to  $h_0(a)$ .

We shall assume that  $\omega$  is an embedding, with image  $I$ . (This is OK for reasonable spaces, and can always be achieved after replacing  $X$  by a homotopy equivalent space, if necessary). Let  $\alpha(t) = (a, t)$  for  $0 \leq t \leq 1$  and let  $J$  be the image of  $\alpha$  in  $Y$ . Let  $\hat{X} = X \cup J \subseteq Y$ . Note that  $I \cup J \cong S^1$ . Then  $Y = \hat{X} \cup (A \times [0, 1])$  is a union of connected sets with connected intersection  $\hat{X} \cap (A \times [0, 1]) = h_0(A) \cup (\{a\} \times (0, 1)) \cup h_1(A)$ . Moreover  $\hat{X}$  is also such a union:  $\hat{X} = X \cup (I \cup J)$  where  $X \cap (I \cup J) = I$ . Thus we can use the above version of the Van Kampen Theorem to compute  $\pi_1(Y, x)$ .

$$\hat{X} \cap (A \times [0, 1]) =$$



For simplicity of notation let  $G = \pi_1(X, x)$  and  $H = \pi_1(A, a)$ . Let  $t$  be the loop  $\alpha.\omega$  in  $\widehat{X} \subseteq Y$ . Then  $\pi_1(\widehat{X}, x) \cong G * \mathbb{Z}$ , where the second factor is generated by the image of  $t$  and  $\pi_1(\widehat{X} \cap (A \times [0, 1]), x) \cong H * \bar{\alpha}_\# H$ . The geometry determines two homomorphisms from  $H$  to  $G$ :  $\theta_0 = h_{0*}$  and  $\theta_1 = \omega_\# h_{1*}$ . (We are being careful about basepoints!) In  $\pi_1(Y, *)$  these are related by  $t^{-1}h_{0*}(h)t = \omega_\# h_{1*}(h)$ , for all  $h \in H$ . Hence

$$\pi_1(Y, *) = \langle G, t \mid t^{-1}\theta_0(h)t = \theta_1(h), \forall h \in H \rangle.$$

If  $\theta_0$  and  $\theta_1$  are monomorphisms this construction is called the *HNN extension* with *base*  $G$ , *associated subgroups*  $H_0 = \theta_0(H)$  and  $H_1 = \theta_1(H)$ , and *defining isomorphism*  $\phi = \theta_1\theta_0^{-1}$ .

Notation:  $HNN(G; \phi : H_0 \cong H_1)$ ,  $G *_H \phi$  or just  $G *_\phi$ .

**Exercise 37.** Show how we may compute the fundamental group of a union  $X \cup Y$  where  $X \cap Y$  has finitely many components.

[Hint: constructing the union amounts to identifying certain connected subsets of  $X$  with subsets of  $Y$ . Identify one pair at a time.]

**Exercise 38.** Let  $G = \mathbb{Z} *_2$  be the HNN extension with base  $\mathbb{Z}$  and associated subgroups  $\mathbb{Z}$  and  $2\mathbb{Z}$ . Find a presentation for  $G$ , and show that  $G/G' \cong \mathbb{Z}$  and  $G' \cong \mathbb{Z}[\frac{1}{2}]$ , the additive group of dyadic rationals  $\{m/2^k \in \mathbb{Q} \mid m \in \mathbb{Z}, k \geq 0\}$ .

The following proof of the ‘‘connected’’ case of the Van Kampen Theorem is taken from *Elements de Topologie Algébrique*, by C.Godbillon. The proof applies only to spaces for which covering space theory works, but they are the most important. (I’ve no idea whether the Van Kampen Theorem is true in much greater generality.) The one basic fact that Godbillon uses is that if  $\pi_1(X, *) = H$  then for every set  $S$  with a left  $H$ -action there is an essentially unique covering  $p : E \rightarrow X$  such that  $p^{-1}(*) \cong S$  as  $H$ -sets.

Let  $X$  and  $Y$  be connected open sets of  $Z = X \cup Y$  with connected intersection  $X \cap Y$ , and let  $j_1 : \pi_1(X \cap Y, *) \rightarrow \pi_1(X, *)$ ,  $j_2 : \pi_1(X \cap Y, *) \rightarrow \pi_1(Y, *)$ ,  $k_1 : \pi_1(X, *) \rightarrow \pi_1(Z, *)$  and  $k_2 : \pi_1(Y, *) \rightarrow \pi_1(Z, *)$  be the homomorphisms induced by the inclusions. Suppose given a group  $G$  and homomorphisms  $h_1 : \pi_1(X, *) \rightarrow G$  and  $h_2 : \pi_1(Y, *) \rightarrow G$  such that  $h_1 j_1 = h_2 j_2$ . We shall show that there is a unique homomorphism  $h : \pi_1(Z, *) \rightarrow G$  such that  $h k_1 = h_1$  and  $h k_2 = h_2$ .

The groups  $\pi_1(X, *)$  and  $\pi_1(Y, *)$  act on  $G$  by left translations ( $g \mapsto h_1(x)g$ , etc.). Therefore there are regular coverings  $p_1 : E_1 \rightarrow X$  and  $p_2 : E_2 \rightarrow Y$  and isomorphisms  $r : G \rightarrow p_1^{-1}(*)$  and  $s : G \rightarrow p_2^{-1}(*)$  of sets with group actions. Let  $D_1 = p_1^{-1}(X \cap Y)$  and  $D_2 = p_2^{-1}(X \cap Y)$ . Since  $h_1 j_1 = h_2 j_2$  the induced coverings  $p|_{D_1} : D_1 \rightarrow X \cap Y$  and  $p|_{D_2} : D_2 \rightarrow X \cap Y$  are isomorphic, and there is an isomorphism  $t : D_2 \rightarrow D_1$  such that  $ts = r$ . Let  $E = E_1 \cup E_2$ . Then  $p_1$  and  $p_2$  together determine a covering map  $p : E \rightarrow Z$ . Let  $q : E_1 \amalg E_2 \rightarrow E$  be the natural map. Then  $q|_{E_1}$  and  $q|_{E_2}$  determine isomorphisms of  $E_1$  and  $E_2$  onto the induced covers  $p : p^{-1}(X) \rightarrow X$  and  $p : p^{-1}(Y) \rightarrow Y$ , respectively.

The covering  $p : E \rightarrow Z$  determines an action  $\theta$  of  $\pi_1(Z, *)$  on the fibre  $p^{-1}(*)$ . Taking into account the above isomorphisms we have  $\theta(k_1(x))qr(g) = qr(h_1(x)g)$  and  $\theta(k_2(x))qs(g) = qs(h_2(x)g)$ . Let  $u = qs = qts = qr$ . the function  $\hat{h}$  defined by  $\hat{h}(\gamma) = u^{-1}\theta(\gamma)u$  determines an action of  $\pi_1(Z, *)$  on  $G$  such that  $\hat{h}(k_1(x))(g) = h_1(x)g$  and  $\hat{h}(k_2(y))(g) = h_2(y)g$ . Since  $\pi_1(Z, *)$  is generated by the images of  $k_1$

and  $k_2$  the permutation  $\widehat{h}(\gamma)$  is translation by an element  $h(\gamma)$  of  $G$  and the map sending  $\gamma$  to  $h(\gamma)$  is a homomorphism of  $\pi_1(Z, *)$  to  $G$  such that  $h_1 = hk_1$  and  $h_2 = hk_2$ .

Since  $\pi_1(Z, *)$  is generated by the images of  $k_1$  and  $k_2$  the homomorphism  $h$  is unique. Hence  $\pi_1(Z, *)$  is the pushout.

P.J.Higgins has used groupoids to give a uniform treatment of the general case [Math. Proc. Cambridge Phil. Soc. 60 (1964), 7-20].

## 16. REALIZING GROUPS BY 2-COMPLEXES AND 4-MANIFOLDS

It follows from the Van Kampen Theorem that the fundamental group of a finite complex is finitely presentable. As it can be shown that every compact manifold is homotopy equivalent to a finite complex, compact manifolds also have finitely presentable fundamental groups.

Let  $P = \langle X \mid R \rangle$  be a finite presentation for a group  $G$ . This presentation determines a finite 2-dimensional cell complex  $C(P)$  with one 0-cell, a 1-cell for each generator and a 2-cell for each relator. Adjoining the 1-cells to the 0-cell (which we take as the basepoint) gives a wedge  $\vee^{|X|} S^1$ , with fundamental group  $F(|X|)$ . Each relator is a word in  $F(|X|)$ , and so corresponds to a homotopy class of loops in  $\vee^{|X|} S^1$ . We attach a 2-cell along a representative loop, for each relator. It then follows from Van Kampen's Theorem that  $\pi_1(C(P)) \cong G$ . Conversely, every connected 2-dimensional complex is homotopy equivalent to one with a single 0-cell, and every such complex arises in this way.

**Exercise 39.** A (finitely generated) group is free if and only if it is the fundamental group of a (finite) connected graph, i.e., a connected 1-dimensional cell complex. Use covering space theory to conclude that subgroups of free groups are free.

(This is quite delicate to prove algebraically!)

**Definition.** The *deficiency* of a finite presentation  $P = \langle X \mid R \rangle$  is

$$def(P) = |X| - |R| = 1 - \chi(C(P)).$$

The deficiency of a finitely presentable group is  $G$  is

$$\max\{def(P) \mid P \text{ presents } G\}.$$

**Example.** Suppose  $G$  has presentation  $\langle x, y \mid r \rangle$ . Let  $V = S^1 \vee S^1$ . Then  $\pi_1(V) \cong F(\{x, y\}) = F(2)$ . The word  $r$  determines a homotopy class of maps  $r : S^1 = \partial D^2 \rightarrow V$ . Let  $C = V \cup_r e^2 = (S^1 \vee S^1 \cup D^2) / \sim$ , where we identify points of  $\partial D^2$  with their images under  $r$ .

Let  $A = \text{int} \frac{2}{3} D^2$  and  $B = C - \frac{1}{2} D^2$ . Then  $A \cap B = S^1 \times (\frac{1}{2}, \frac{2}{3}) \simeq S^1$  and  $V \simeq B$ , so Van Kampen's Theorem gives  $\pi_1(C) \cong F(\{x, y\} / \langle\langle r \rangle\rangle) \cong G$ . The general case is similar.

We must work slightly harder to realize (finitely presentable) groups as fundamental groups of manifolds.

The circle  $S^1$  is the only compact connected 1-manifold without boundary, and  $\pi_1(S^1, 1) \cong \mathbb{Z}$  is the only 1-dimensional Poincaré duality group.

There is a complete list of closed surfaces (compact connected 2-manifolds without boundary), and their fundamental groups may be considered well understood. Each may be constructed by identifying the sides of a polygon in pairs. This leads to the presentations:

- (1) orientable surfaces:  $\langle a_i, b_i, 1 \leq i \leq g \mid \prod [a_i, b_i] = 1 \rangle$ . The case  $g = 1$  corresponds to the torus.
- (2) nonorientable surfaces:  $\langle v_j, 1 \leq j \leq c \mid \prod v_j^2 = 1 \rangle$ . The cases  $c = 1$  and  $c = 2$  correspond to the projective plane and Klein bottle, respectively.

Excepting only  $\pi_1(P^2(\mathbb{R})) = Z/2Z$ , they are all torsion free, and the torsion free surface groups have an intrinsic algebraic characterization as the 2-dimensional Poincaré duality groups. The fundamental group determines the surface up to homeomorphism.

The next exercise corresponds to the fact that  $T\#RP^2 \cong Kb\#RP^2 \cong \#^3RP^2$ , by the classification of surfaces.

**Exercise 40.** *Show that the following presentations are equivalent:*

- (1)  $\langle a, b, c \mid aba^{-1}b^{-1}c^2 = 1 \rangle$ ;  
 (2)  $\langle p, q, r \mid pqp^{-1}qr^2 = 1 \rangle$ ;  
 (3)  $\langle x, y, z \mid x^2y^2z^2 = 1 \rangle$ .

[Hint: for (1)  $\Leftrightarrow$  (3), note that  $c$  and  $xyz$  each represent the element of order 2 in the abelianization. The equivalence (2)  $\Leftrightarrow$  (3) follows easily from Exercise 36(c).]

Less is known about 3-manifolds and their groups. The indecomposable factors of a 3-manifold group are either  $\mathbb{Z}$ ,  $\mathbb{Z} \oplus (Z/2Z)$ , a finite subgroup of  $SO(4)$  which acts freely on  $S^3$  (one of a known infinite list of such subgroups) or a 3-dimensional Poincaré duality group. (It is not known whether the latter are always 3-manifold groups.) It is conceivable that 3-manifold groups are sufficiently special that geometric methods may provide algorithmic solutions to the standard decision problems of combinatorial group theory.

**Example.** If  $a, b, c, d$  be integers such that  $ad - bc = \pm 1$  the map  $f(w, z) = (w^a z^b, w^c, z^d)$  determines a self homeomorphism of the torus  $S^1 \times S^1$ . The union  $(S^1 \times D^2) \cup_f (S^1 \times D^2)$  has cyclic fundamental group.

$$\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3.$$

$$\pi_1(S^1 \times P^2(\mathbb{R})) \cong \mathbb{Z} \oplus (Z/2Z).$$

No other abelian group is the fundamental group of a closed 3-manifold.

On the other hand, every finitely presentable group is the fundamental group of some closed (orientable) 4-manifold. There are two natural approaches, each essentially “thickening” the above construction so that we may assume the attaching maps are embeddings.

(a) With a little more care, we may in fact assume that  $C(P)$  is a 2-dimensional polyhedron. Now every  $n$ -dimensional polyhedron can be embedded in  $R^{2n+1}$ . (This is a “general position” argument. The expected dimension of the intersection of linear subspaces  $P, Q \subseteq R^n$  is  $\dim(P) + \dim(Q) - n$ . Thus if  $C(P)$  is a 2-dimensional polyhedron it may be embedded as a union of polygon subsets of  $R^5$ . For  $\epsilon$  sufficiently small the set  $N_\epsilon = \{x \in R^5 \mid |x - C(P)| \leq \epsilon\}$  of all points within distance  $\epsilon$  of  $C(P)$  is a compact 5-manifold, and the inclusion  $C(P) \subset N_\epsilon$  is a homotopy equivalence. (Moreover any two such “regular neighbourhoods” are homeomorphic). Then  $\partial N_\epsilon$  is a closed orientable 4-manifold, and it can be shown that  $\pi_1(\partial N_\epsilon) \cong \pi_1(N_\epsilon) \cong G$ .

(b) We may mimic the construction of a cell complex by “attaching 1- and 2-handles” to the 4-disc, to obtain a compact, bounded 4-manifold homotopy equivalent to  $C(P)$ . We attach the 1-handles  $D^1 \times D^3$  by embedding  $|X|$  disjoint copies of  $S^0 \times D^3$  in  $\partial D^4$ , to get the boundary connected sum  $\natural^{|X|}(S^1 \times D^3) \simeq \vee^{|X|} S^1$ .

We then attach the 2-handles  $D^2 \times D^2$  by embedding  $|R|$  disjoint copies of  $S^1 \times D^2$  in  $\partial_1^{|X|}(S^1 \times D^3) = \#^{|X|}(S^1 \times S^2)$ . Note that  $\pi_1(\#^{|X|}(S^1 \times S^2)) \cong F(|X|)$ . Then  $M = D^4 \cup^{|X|}(D^1 \times D^3) \cup^{|R|}(D^2 \times D^2) \simeq C(P)$ , and  $\pi_1(\partial M)$  maps onto  $\pi_1(M)$ . The double  $D(M) = M \cup_{\partial M} M$  is then a closed 4-manifold and we again have  $\pi_1(M) \cong G$ .

## 17. THE HUREWICZ THEOREM

Let  $X$  be path-connected, and let  $*$  be a basepoint for  $X$ . For each  $q \geq 0$  let  $*_q : \Delta_q \rightarrow X$  be the constant map with value  $*$ , considered as a singular  $q$ -simplex.

The map  $p : \Delta_1 \rightarrow [0, 1]$  given by  $p(x, y) = y$  is a homeomorphism, and so a path  $\alpha : [0, 1] \rightarrow X$  determines a singular 1-simplex  $\widehat{\alpha} = \alpha p$ , with boundary  $\partial \widehat{\alpha} = \alpha(1) - \alpha(0)$ . Clearly every singular 1-simplex arises in this way. Note also that  $\widehat{\alpha}$  is a 1-cycle if and only if  $\alpha$  is a loop (i.e.,  $\alpha(1) = \alpha(0)$ ).

Let  $h$  be the function given by  $h(\alpha) = [\widehat{\alpha}]$  (the homology class of the cycle  $\widehat{\alpha}$ ) for all loops  $\alpha$  at  $*$  in  $X$ . The Hurewicz Theorem in degree 1 asserts that  $h$  induces an isomorphism  $\pi_1(X, *)^{ab} \cong H_1(X; \mathbb{Z})$ , where  $\pi^{ab} = \pi/\pi'$  is the abelianization of  $\pi$ .

Let  $\alpha$  be a loop at  $*$  in  $X$ . Define a singular 2-simplex  $\sigma : \Delta_2 \rightarrow X$  by  $\sigma(x, y, z) = \alpha(x + z)$ . Then  $\sigma F_2^0 = \widehat{\alpha}$ ,  $\sigma F_2^1 = *_1$  and  $\sigma F_2^2 = \widehat{\alpha}$ , so  $\partial \sigma = \widehat{\alpha} - *_1 + \widehat{\alpha}$ . Since  $\partial *_2 = *_1 - *_1 + *_1 = *_1$ , it follows that  $\widehat{\alpha} + \widehat{\alpha} = \partial(\sigma + *_2)$  and hence that  $h(\overline{\alpha}) = -h(\alpha)$ .

Suppose now that  $A : [0, 1]^2 \rightarrow X$  is a homotopy of loops from  $\alpha$  to  $\alpha'$ . Define singular 2-simplices by  $\sigma_I(x, y, z) = A(y, z)$  and  $\sigma_{II}(x, y, z) = A(1 - z, 1 - y)$ . Then  $\partial(\sigma_I - \sigma_{II} + 2*_2) = \widehat{\alpha} + \widehat{\alpha}'$ , so  $h(\alpha) = -h(\alpha') = h(\alpha')$ . Thus  $h$  gives rise to a function  $hwz$  from  $\pi_1(X, *)$  to  $H_1(X; \mathbb{Z})$ .

Let  $\beta$  be another loop at  $*$  in  $X$ . Define a singular 2-simplex  $\tau : \Delta_2 \rightarrow X$  by  $\tau(x, y, z) = \alpha(1 - x + z)$  if  $x \geq z$  and  $\tau(x, y, z) = \beta(z - x)$  if  $x \leq z$ . Then  $\tau F_2^0 = \widehat{\beta}$ ,  $\tau F_2^1 = \widehat{\alpha} \cdot \widehat{\beta}$  and  $\tau F_2^2 = \widehat{\alpha}$ , so  $\partial \tau = \widehat{\alpha} - \widehat{\alpha} \cdot \widehat{\beta} + \widehat{\beta}$ . In particular, if  $\alpha$  and  $\beta$  are loops at  $*$  then  $hwz(\alpha \cdot \beta) = hwz(\alpha) + hwz(\beta)$ . Thus  $hwz$  is a homomorphism, and hence  $\pi_1(X, *)' \leq \text{Ker}(hwz)$ , since homology groups are abelian. We shall show that  $hwz$  is onto and that  $\text{Ker}(hwz) = \pi_1(X, *)'$  (i.e., is no larger).

Choose a path  $\gamma_x$  from  $*$  to  $x$  for each  $x \in X$ . Then for any path  $\alpha$  in  $X$  the path  $\gamma_{\alpha(0)} \cdot \alpha \cdot \overline{\gamma_{\alpha(1)}}$  is a loop at  $*$ , and  $hwz(\gamma_{\alpha(0)} \cdot \alpha \cdot \overline{\gamma_{\alpha(1)}})$  is the homology class of  $\widehat{\gamma_{\alpha(0)}} + \widehat{\alpha} + \widehat{\overline{\gamma_{\alpha(1)}}}$  (which is a 1-cycle). Let  $\xi = \sum r_\alpha \widehat{\alpha}$  be a singular 1-cycle in  $X$ . (Here the sum is over a finite set of paths  $\alpha$  in  $X$  and the coefficients  $r_\alpha$  are integers). Then  $0 = \partial \xi = \sum r_\alpha \alpha(1) - \sum r_\alpha \alpha(0)$ , i.e., the endpoints of these paths match in pairs, with the same multiplicities. Therefore  $\sum r_\alpha (\widehat{\gamma_{\alpha(0)}} - \widehat{\overline{\gamma_{\alpha(1)}}}) = 0$  and so  $\xi = \sum r_\alpha (\widehat{\gamma_{\alpha(0)}} + \widehat{\alpha} - \widehat{\overline{\gamma_{\alpha(1)}}})$ , which has the same homology class as  $\sum r_\alpha (\widehat{\gamma_{\alpha(0)}} + \widehat{\alpha} + \widehat{\overline{\gamma_{\alpha(1)}}})$ . It follows that  $hwz$  is onto.

It remains to prove that if  $\alpha$  is a loop such that  $\widehat{\alpha} = \partial \zeta$  for some singular 2-chain  $\zeta$  in  $X$  then  $\alpha$  is in  $\pi_1(X, *)'$ , the commutator subgroup. Let  $\zeta = \sum_{i \in I} \varepsilon(i) \sigma_i$  where the  $\sigma_i$  are singular 2-simplices and  $\varepsilon(i) = \pm 1$ , and write  $\partial \sigma_i = \sigma_{i0} - \sigma_{i1} + \sigma_{i2}$ . Let  $I_+ = \{i \in I \mid \varepsilon(i) = 1\}$  and  $I_- = \{i \in I \mid \varepsilon(i) = -1\}$ . Let  $\Delta_1$  have the standard orientation (from  $(1, 0)$  to  $(0, 1)$ ), and orient (the boundary of)  $\Delta_2$  so that the face maps  $F_0$  and  $F_2 : \Delta_1 \rightarrow \Delta_2$  are orientation preserving. (Then  $F_1$  reverses the orientation). Note that this orientation corresponds to the sequence (cycle) of faces  $(2, 0, 1) = (0, 1, 2)$ . Reversing the orientation of the boundary of

$\Delta_2$  gives the cycle  $(2, 1, 0) = (1, 0, 2)$ . Let  $S = I \times \Delta_2$  and give each component of  $I_+ \times \Delta_2$  the standard orientation and each component of  $I_- \times \Delta_2$  the opposite orientation. Then  $S$  is a finite family of oriented affine 2-simplices. In the equation  $\hat{\alpha} = \sum_{i \in I} \varepsilon(i)(\sigma_{i0} - \sigma_{i1} + \sigma_{i2})$  the faces  $\sigma_{in}$  must match in pairs (with opposite signs), except for one equal to  $\hat{\alpha}$ .

CLAIM: if  $\sigma_{im} = \sigma_{jn}$  and cancel in this sum (i.e., have opposite signs) then exactly one of the two corresponding face maps  $F_m : \Delta_1 \rightarrow \{i\} \times \Delta_2$  and  $F_n : \Delta_1 \rightarrow \{j\} \times \Delta_2$  is orientation preserving. (Either  $\varepsilon(i) = \varepsilon(j)$  and  $\{m, n\} = \{0, 1\}$  or  $\{1, 2\}$ , or  $\varepsilon(i) \neq \varepsilon(j)$  and  $\{m, n\} \subseteq \{0, 2\}$  or  $m = n = 1$ .)

Identifying these two 2-simplices along this pair of edges gives a quadrilateral  $Q$  such that

- (1)  $\sigma_i \cup \sigma_j$  defines a continuous function from  $Q$  to  $X$ ;
- (2) the chosen orientations determine a consistent orientation for (the boundary of)  $Q$ ;
- (3) the CLAIM remains valid for other pairs of faces yet to be identified in this way.

After finitely many such pairwise gluings we obtain a finite set of polygons  $P = P_0, \dots, P_k$  with

- (a) oriented boundaries  $\partial P, \dots, \partial P_k$ ;
- (b) the edges making up the boundary of each polygon correspond to the as yet unused faces of  $\{\sigma_i \mid i \in I\}$ ;
- (c) these faces are matched in pairs (except for one equal to  $\hat{\alpha}$ , which corresponds to an edge of  $\partial P$ );
- (d) exactly one edge of each pair is oriented consistently with  $\partial P$ .
- (e) continuous functions from  $P_i$  to  $X$  extending the  $\sigma_i$ s.

We discard the components  $P_1, \dots, P_k$ . (These correspond to disjoint summands of  $\zeta$  which are 2-cycles, and thus have algebraic boundary 0). The boundary  $\partial P$  is a concatenation of paths  $\alpha.\beta_1 \dots \beta_{2k}$ , for some  $k \geq 1$ , and clearly represents the identity element of  $\pi_1(X, *)$  (since it extends to a map from  $P$  to  $X$ ). Let  $x(0) = *$  and  $x(i) = \beta_i(1)$ , for  $1 \leq i \leq 2k$ , and let  $\tilde{\beta}_i = \gamma_{x(i-1)}.\beta_i.\overline{\gamma_{x(i)}}$  for  $1 \leq i \leq 2k$ . Then  $\partial P$  is homotopic to  $\alpha.\widetilde{\beta}_1 \dots \widetilde{\beta}_{2k}$ , which is a product of loops at  $*$ . We thus get an equation  $1 = \alpha\Pi$ , where  $\Pi$  is the product of all the edges except  $\alpha$ . Clearly  $\Pi$  is in  $\pi_1(X, *)'$ , since each factor occurs twice with exponents 1 and  $-1$ , and so  $\alpha$  is also in  $\pi_1(X, *)'$ .

The Hurewicz homomorphism in degree 1 is basepoint-independent in the sense that if  $\omega$  is a path from  $x$  to  $x'$  and  $\alpha$  is a loop at  $x$  then  $hwz(\omega\widehat{\#}\alpha) = hwz(\widehat{\alpha})$ .