Stochastic quantisation of locally supersymmetric models

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Abstract

Stochastic quantisation normally involves the introduction of a fictitious extra time parameter, which is taken to infinity so that the system evolves to an equilibrium state.

In the case of a locally supersymmetric theory, an interesting new possibility arises due to the existence of a Nicolai map. In this case it turns out that no additional time parameter is required, as the existence of the Nicolai map ensures that the same job can be done by the existing time parameter after Euclideanisation. This provides the quantum theory with a natural probabilistic interpretation, without any reference to the concept of an inner product or a Hilbert space structure.

1 Introduction

In 1981 Parisi and Wu proposed a radical new way to obtain Green's functions in Euclidean quantum field theory [1]. Known as stochastic quantisation, this approach enjoys certain conceptual and practical advantages over more traditional ones. In this paper, we show how this approach arises naturally for locally supersymmetric models and provides them with a natural probabilistic interpretation of the wave function without reference to the concept of an inner product.

Stochastic quantisation normally requires the introduction of an extra parameter t, referred to as the fictitious time. Evolution with respect to t is assumed to be stochastic, governed by a type of Langevin equation which gradually forces the system into thermal equilibrium. The static probability distribution arising in the $t \to \infty$ limit can then be identified with the Euclidean path integral measure and used to calculate the Euclidean Green functions.

In the case of a supersymmetric theory, an interesting new possibility arises due to the existence of a Nicolai map. (This is a transformation which converts any supersymmetric theory into a non-interacting bosonic one [2], and which generally has the form of a stochastic differential equation in Euclidean theories.) In this case, there is no need to introduce a fictitious time parameter; the presence of the Nicolia map means that the same role can be performed quite satisfactorily by the physical time parameter from the original theory.

Regarding the Nicolai map as a stochastic process leads to an interpretation of certain components of the wave function (in an appropriate representation) as probability densities whose integral is conserved in Euclidean time. This provides a simple stochastic interpretation for supersymmetric quantum theories, without the need to define an inner product on the space of states or appealing to the concept of a Hilbert space. This is particularly useful in the context of quantum cosmology, where it is difficult to identify the inner product [3].

In general, finding Nicolai maps for supersymmetric theories is not an easy task. In fact, it may be easier to find the associated Fokker-Planck equation directly, thus by-passing the difficulties of explicitly constructing Nicolai maps.

This paper is organised as follows: A review of stochastic quantisation of a scalar field is given in §2. For a supersymmetric theory, the notion of Nicolai map is introduced in §3. In §4, the canonical formulation of a one-dimensional supersymmetric non-linear σ model is developed. Supersymmetry is always broken when boundaries are present. In §5 we will see that there is a boundary correction to the action that restores invariance under a sub-algebra of supersymmetry transformations. Derivation of Nicolai map and Fokker-Planck equation in case of a rigid supersymmetry is done in §6. In this section, it also has been discussed that the result can be obtained directly from canonical quantisation. Section §7 is devoted to quantisation of a locally supersymmetric model.

2 Stochastic quantisation of scalar field theories

The main idea in stochastic quantisation is to view Euclidean quantum field theory as the equilibrium limit of a statistical system coupled to a thermal reservoir. This system is assumed to evolve with respect to a fictitious time variable t and approaches the equilibrium limit as $t \to \infty$. The coupling to a heat reservoir is simulated by means of a stochastic noise field which causes the original Euclidean field to wander randomly on its manifold. In the equilibrium limit stochastic averages become identical to ordinary Euclidean vacuum expectation values.

In this section we outline the stochastic, quantisation of scalar field theory using the approach of Parisi and Wu[1]. For a more complete discussion, the reader is referred to the review article by Damgaard and Huffel [4].

The basic idea is as follows:

i) We imagine that each field $\phi(x)$ depends on an additional coordinate, the fictitious time t

$$\phi(x) \to \phi(x,t). \tag{1}$$

ii) We suppose that the evolution of $\phi(x,t)$ with respect to the fictitious time t is described by a stochastic differential equation that allows for relaxation to equilibrium. Specifically, one postulates that the evolution is governed by the Langevin equation

$$\frac{\partial\phi(x,t)}{\partial t} = -\frac{\delta S}{\delta\phi(x)} + \eta(x,t), \tag{2}$$

where $S[\phi]$ is the Euclidean action and $\eta(x,t)$ is a Gaussian white noise with correlations given by

$$\begin{aligned}\langle \eta(x,t)\rangle_\eta &= 0\\ \langle \eta(x_1,t_1)\eta(x_2,t_2)\rangle_\eta &= 2\delta^n(x_1-x_2)\delta(t_1-t_2)\end{aligned} \tag{3}$$

iii) Given some initial condition at $t=t_0$ and some realisation of $\eta(t)$,equation (2) has a unique solution $\phi_\eta(x,t)$. The correlation functions of ϕ_η are then obtained as Gaussian averages over all possible realisations of $\eta(t)$;

$$\langle \phi_\eta(x_1,t_1)\cdots\phi_\eta(x_k,t_k)\rangle_\eta = \frac{\int D\eta e^{-\frac{1}{4}\int d^nxdt\eta^2(x,t)}\phi_\eta(x_1,t_1)\cdots\phi_\eta(x_k,t_k)}{\int D\eta e^{-\frac{1}{4}\int d^nxdt\eta^2(x,t)}} \tag{4}$$

As $t\to 0$, equilibrium is reached, and the (equal time) correlation functions of ϕ_η tend to the corresponding quantum green functions

$$\lim_{t\to\infty}\langle \phi(x_1,t)\cdots\phi(x_k,t)\rangle_\eta = \langle\phi(x_1)\cdots\phi(x_k)\rangle \tag{5}$$

An alternative to the Langevin approach is to study the Fokker-Planck formulation. In this approach, the stochastic averages are represented as functional integrals

$$\langle \phi(x_1,t)\cdots\phi(x_k,t)\rangle_\eta = \int D\phi f(\phi,t)\phi(x_1)\cdots\phi(x_k) \tag{6}$$

where the probability density functional $f[\phi(x),t]$ is a solution to the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = \int d^nx \frac{\delta}{\delta\phi(x,t)}\Big(\frac{\delta S}{\delta\phi(x,t)} + \frac{\delta}{\delta\phi(x,t)}\Big)f. \tag{7}$$

The physical correlation functions are then calculated using the equilibrium probability density functional

$$f[\phi] = \lim_{t\to\infty} f[\phi,t] = f^{eq}(\phi) = \frac{e^{-S}}{\int D\phi^{-S}}. \tag{8}$$

If we define the operator

$$O = -\int d^nx \frac{\delta}{\delta\phi(x,t)}\Big(\frac{\delta S}{\delta\phi(x,t)} + \frac{\delta}{\delta\phi(x,t)}\Big), \tag{9}$$

then the Fokker-Planck equation (7) can be rewritten simply as

$$\frac{\partial f(x,t)}{\partial t} = -Of(x,t). \tag{10}$$

One can show that the results obtained from stochastic quantisation agree with those obtained by more conventional methods [1]. Now for a system with a single degree of freedom x, and a potential $V(x)$. The Fokker-Planck equation reads

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x} + \frac{\partial V}{\partial x}\Big) f(x,t) \tag{11}$$

3 Supersymmetric models and Nicolai maps

Nicolai has shown [2] that there is a map which transforms any supersymmetric theory to a free bosonic theory. The simplest example of Nicolai map is the Langevin equation, although in general for a typical supersymmetric action the Nicolai map will not have such a simple form [3].

The Nicolai's theorem essentially states the following: Take a supersymmetric theory and integrate out the fermion fields in the path integral. This contributes a non trivial determinant factor to the (bosonic) path integral. There now exists a transformation of the bosonic fields whose Jacobian determinant exactly cancels the determinant of the fermion integrations, and which simultaneously transforms the remaining bosonic part of the action to that of a non-interacting bosonic theory.

The transformation (Nicolai map) will be invertible if one impose an appropriate number of boundary conditions on the bosonic variables in the theory. However, imposition of such conditions break the supersymmetry since the supersymmetric variation of the Lagrangian produces a total divergence which yields a boundary term when integrated.

Nicolai's theorem will not be applicable unless the supersymmetry algebra has some graded sub-algebra whose bosonic generators preserve the boundary [5]. Note that in the framework of quantum theory, boundaries are important since one is generally calculating transition amplitudes between specified boundary data. Hence, boundary effects cannot be neglected.

In general, only the Euclidean version of a supersymmetric theory will admit a Nicolai map which can be interpreted as stochastic differential equation describing the evolution of the system in Euclidean time. This suggests that in quantum theory one can interpret the wave function (in an appropriate representation) as a probability density function whose integral is conserved in Euclidean time. In fact, we are more interested in the static state which is finally reached, since it represents the ground state of the Euclidean theory.

4 N=1 supersymmetry

Here we present the canonical formulation of a one-dimensional supersymmetric non-linear σ model describing a particle moving in a curved configuration space. This example will be used throughout the paper to illustrate an approach which can be applied to quite general supersymmetric models.

Suppose the position $\mathbf{q}(t)$ of the particle at time t is described by n coordinates $q^i(t)$. The action for the locally supersymmetric Euclidean model is $S = \int L dt$ with the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\left[N^{-1}g_{ij}\dot{q}^i\dot{q}^j + Ng^{ij}V_{,i}V_{,j} + \overline{\psi}_i(D\psi^i) - (D\overline{\psi}_i)\psi^i\right] \\ &\quad + NV_{;ji}\overline{\psi}^i\psi^j - \overline{\chi}\psi^i(g_{ij}\dot{q}^j + NV_{,i}) - \chi\overline{\psi}_i(\dot{q}^i - Ng^{ij}V_{,j}) \\ &\quad + N\overline{\chi}\chi\overline{\psi}_i\psi^i - \frac{1}{2}NR_{ijkl}\overline{\psi}^i\psi^j\overline{\psi}^k\psi^l. \end{aligned} \tag{12}$$

where $g_{ij}(\mathbf{q})$ is the metric of the configuration space in which the particle moves, $V(\mathbf{q})$ is a potential function defined on this space, and $\dot{q}^i \equiv dq^i/dt$. The covariant time derivatives of the fermion fields are defined as

$$D\psi^i = \dot{\psi}^i + \Gamma^i{}_{jk}\psi^j\dot{q}^k \tag{13}$$

$$D\overline{\psi}_i = \dot{\overline{\psi}}_i - \Gamma^j{}_{ik}\overline{\psi}_j\dot{q}^k \tag{14}$$

where $\Gamma^i{}_{jk}$ is the usual symmetric Christoffel connection on the configuration space, compatible with the metric g_{ij} [6]. The Riemann curvature tensor is

$$R^i{}_{jkl} = \partial_k\Gamma^i{}_{jl} - \partial_l\Gamma^i{}_{jk} + \Gamma^m{}_{jl}\Gamma^i{}_{mk} - \Gamma^m{}_{jk}\Gamma^i{}_{ml}. \tag{15}$$

In the Euclidean formulation described above, the lapse function N is a real-valued function of time t. The standard formulation can be obtained by making N imaginary, or equivalently by taking $N = i\tilde{N}$ with $\tilde{N}$ real. Then the Euclidean Lagrangian L also becomes imaginary and so it is natural to describe the theory in terms of the real function $\tilde{L} = iL$, which is the Lagrangian for the standard formulation. It follows that the momenta and Hamiltonian in the standard formulation are related to their Euclidean counterparts (defined below) by identities of the form $\Pi_{standard} = i\Pi_{Euclidean}$ and $H_{standard} = iH_{Euclidean}$. Note also that in the standard formulation the Grassman variables ψ^i and $\overline{\psi}^i$ must be related by complex conjugation to ensure unitarity in the quantum theory. (This is unnecessary in the Euclidean version, where unitarity is not a requirement.)

Having clarified its relationship with the standard formulation of the model, we henceforth consider only the Euclidean formulation described above.

There are two different ways to define the (Euclidean) momenta conjugate to the variables q^i. If we regard the Lagrangian (12) as a function of the variables $q^i, \psi^i, \overline{\psi}_i$ and velocities $\dot{q}^i, \dot{\psi}^i, \dot{\overline{\psi}}_i$, then differentiating with respect to $\dot{q}^i$ (while holding $\psi^i, \overline{\psi}_i, \dot{\psi}^i$ and $\dot{\overline{\psi}}_i$ fixed) gives

$$p_i \equiv \left.\frac{\partial L}{\partial \dot{q}^i}\right|_{\psi,\dot{\psi}} = N^{-1} g_{ij} \frac{dq^j}{dt} - \overline{\chi}\psi_i - \chi\overline{\psi}_i + \Gamma^j{}_{ki}\overline{\psi}_j\psi^k. \tag{16}$$

However, if we regard L as a function of $q^i, \psi^i, \overline{\psi}_i, \dot{q}^i$, and the *covariant* velocities $D\psi^i$ and $D\overline{\psi}_i$ and differentiate with respect to $\dot{q}^i$ (now holding $\psi^i, \overline{\psi}_i, D\psi^i$ and $D\overline{\psi}_i$ fixed) we instead obtain a set of covariantly defined momenta

$$\Pi_i \equiv \left.\frac{\partial L}{\partial q^i}\right|_{\psi,D\psi} = p_i - \Gamma^j{}_{ki}\overline{\psi}_j\psi^k. \tag{17}$$

These prove to be more useful for our purposes.

From (12), we find that the momenta conjugate to χ, $\overline{\chi}$ and N are not independent quantities, but are subject to the primary constraints

$$\Pi_\chi \approx \Pi_{\bar{\chi}} \approx \Pi_N \approx 0. \tag{18}$$

Since Π_χ, $\Pi_{\bar{\chi}}$ and Π_N vanish at all times, their time derivatives must also vanish. After applying the equations of motion, this requirement gives rise to the secondary constraints [7]

$$Q \approx \bar{Q} \approx \mathcal{H} \approx 0 \tag{19}$$

where we have defined the quantities[1]

$$Q \equiv i\psi^i(\Pi_i + V_{,i}), \qquad \overline{Q} \equiv i(\Pi_i - V_{,i})\overline{\psi}^i \tag{20}$$

[1] the ordering of the terms is immaterial here, but becomes more significant in the quantum theory. Different orderings may be used, and correspond to different choices of measure on the configuration space.

and

$$\mathcal{H} \equiv -\frac{1}{2} g^{ij}(\Pi_i \Pi_j - V_{,i} V_{,j}) + V_{;ij} \overline{\psi}^i \psi^j - \frac{1}{2} \overline{R}_{ijkl} \overline{\psi}^i \psi^j \overline{\psi}^k \psi^l. \tag{21}$$

The constraint functions $Q, \bar{Q}$ and $\mathcal{H}$ are first class, and therefore generate of gauge symmetries. In fact, Q and $\bar{Q}$ are the generators of supersymmetry transformations, while $\mathcal{H}$ is the generator of reparametrisations. The factors of i in the definitions (20) and the initial minus sign in the definition (21) are included so that Q, $\bar{Q}$ and $\mathcal{H}$ are identical to the constraints appearing in the conventional formulation of the model.

The total Hamiltonian is then found to be just a linear combination of these constraints, and has the form

$$H = -N(\mathcal{H} + i\chi\overline{Q} + i\overline{\chi}Q) \tag{22}$$

where N, χ, and $\overline{\chi}$ can be regarded now as Lagrange multipliers. It vanishes when the constraints (19) are imposed.

There are also second-class constraints relating ψ^i and $\bar{\psi}_i$ to their conjugate momenta. These constraints are interpreted as strong equalities and can be used to eliminate these particular momenta from the theory. Of course, the presence of second-class constraints requires us to employ Dirac brackets rather than the more familiar Poisson brackets.

Following [8], the elementary Dirac brackets are found to be simply

$$\{q^i, q^j\} = 0, \qquad \{q^i, \Pi_j\} = \delta^i{}_j, \qquad \{\Pi_i, \Pi_j\} = \overline{\psi}_k \psi^l R^k{}_{lij} \tag{23}$$

$$\{\psi^i, \psi^j\} = 0, \qquad \{\overline{\psi}_i, \overline{\psi}_j\} = 0, \qquad \{\psi^i, \overline{\psi}_j\} = \delta^i{}_j \tag{24}$$

$$\{q^i, \psi^j\} = 0, \quad \{q^i, \overline{\psi}_j\} = 0, \quad \{\Pi_i, \psi^j\} = \Gamma^j{}_{ki}\psi^k, \quad \{\Pi_i, \overline{\psi}_j\} = -\Gamma^k{}_{ji}\overline{\psi}_k \tag{25}$$

from which we obtain

$$\{Q, \mathcal{H}\} = 0, \qquad \{\overline{Q}, \mathcal{H}\} = 0. \tag{26}$$

and

$$\{Q, \overline{Q}\} = 2\mathcal{H}. \tag{27}$$

These equations are the hallmark of supersymmetry.

Defining the fermion number $F \equiv \overline{\psi}_i \psi^i$, we also have

$$\{F, \overline{\psi}_j\} = \overline{\psi}_j \qquad \{F, \psi_j\} = -\psi_j \tag{28}$$

while

$$\{F, \overline{Q}\} = \overline{Q} \qquad \{F, Q\} = -Q, \qquad \{F, \mathcal{H}\} = 0 \tag{29}$$

showing that supersymmetry transformations do not mix fermion number, and that fermion number is unaffected by reparametrisations. Moreover, fermion number is a constant of the motion, since

$$\{F, H\} \approx 0. \tag{30}$$

We conclude this section by remarking that in the case of rigid supersymmetry, $\overline{\chi}(t), \chi(t)$ and $N(t)$ are specified functions rather than Lagrange multipliers. Hence the supersymmetry and reparametrisation generators $Q, \overline{Q}$ and $\mathcal{H}$ are not constrained to vanish, and nor is the total Hamiltonian (22).

5 Boundary corrections to the action

Supersymmetry is always broken when boundaries are present, since the supersymmetric variation of the Lagrangian is a total divergence which yields a boundary term when integrated. In this section we will see that there is a boundary correction to the action that restores invariance under a subalgebra of supersymmetry transformations [9].

The classical trajectories are invariant under the following infinitesimal supersymmetry transformations:

$$\delta q^i = \overline{\epsilon}\psi^i + \epsilon\overline{\psi}^i \tag{31}$$

$$\delta \psi^i = -\epsilon g^{ij}(\Pi_i - V_{,i}) - \Gamma^i{}_{jk}\psi^j \delta q^k \tag{32}$$

$$\delta \overline{\psi}_i = -\overline{\epsilon}(\Pi_i + V_{,i}) + \Gamma^j{}_{ik}\overline{\psi}_j \delta q^k \tag{33}$$

$$\delta\chi = N^{-1}\dot{\epsilon} + 2\epsilon\overline{\chi}\chi \tag{34}$$

$$\delta\overline{\chi} = N^{-1}\dot{\overline{\epsilon}} + 2\overline{\epsilon}\chi\overline{\chi} \tag{35}$$

$$\delta N = -2(\overline{\epsilon}\chi + \epsilon\overline{\chi}) \tag{36}$$

where ϵ, $\overline{\epsilon}$ are anticommuting Grassmann variables. The variation of the action is given by the boundary term

$$\delta S = \frac{1}{2}\Big[\epsilon\overline{\psi}_i(N^{-1}\dot{q}^i + g^{ij}V_{,j} - \overline{\chi}\psi^i) + \overline{\epsilon}\psi^i(N^{-1}g_{ij}\dot{q}^i - V_{,i} - \chi\overline{\psi}_i)\Big]_{t_1}^{t_2}. \tag{37}$$

While it is customary to disregard boundary terms, we cannot afford to do so here. The supersymmetry transformation is broken by the boundary terms

(37), preventing us from using Nicolai's theorem. In order to be able to construct a Nicolai map, we must find a way to restore the invariance of the action under a sub-algebras of the supersymmetry generators [5].

Exact invariance of the action under a sub-algebra of supersymmetry generators can be restored if the Lagrangian is augmented or diminished by the total derivative

$$L_B = \frac{d}{dt}(V + \frac{1}{2}\overline{\psi}_i \psi^i). \tag{38}$$

Indeed, under the infinitesimal supersymmetry transformations (31-36) the variation of the integral

$$I \equiv \int_{t_1}^{t_2} L_B \, dt \; = \; \Big[V + \frac{1}{2}\overline{\psi}_i \psi^i\Big]_{t_1}^{t_2}$$

is found to be

$$\delta I = \frac{1}{2}\Big[\epsilon\overline{\psi}_i(N^{-1}\dot{q}^i + g^{ij}V_{,j} - \overline{\chi}\psi^i) - \overline{\epsilon}\psi^i(N^{-1}g_{ij}\dot{q}^i - V_{,i} - \chi\overline{\psi}_i)\Big]_{t_1}^{t_2}. \tag{39}$$

Consequently, the supersymmetric variations of the modified actions

$$S_\pm = S \pm I = \int_{t_1}^{t_2} (L \pm L_B) dt \tag{40}$$

are

$$\delta S_+ = \Big[\epsilon\overline{\psi}_i(N^{-1}\dot{q}^i + g^{ij}V_{,j} - \overline{\chi}\psi^i)\Big]_{t_1}^{t_2} \tag{41}$$

and

$$\delta S_- = \Big[\overline{\epsilon}\psi^i(N^{-1}g_{ij}\dot{q}^i - V_{,i} - \chi\overline{\psi}_i)\Big]_{t_1}^{t_2} \tag{42}$$

If we specify that $\epsilon(t_1) = \epsilon(t_2) = 0$, then δS_+ will vanish exactly. In other words, the modified action S_+ is exactly invariant under the subalgebra of infinitesimal supersymmetry transformations obtained by imposing Dirichlet boundary conditions on ϵ. We will refer to this as the *left-handed subalgebra*.

Similarly, the modified action S_- is exactly invariant under the subalgebra of infinitesimal supersymmetry transformations obtained by imposing Dirichlet boundary conditions on $\overline{\epsilon}$. We will refer to this as the *right-handed subalgebra.*

Because the modified actions differ from the original only by boundary terms, the classical equations of motion are unaffected. (In fact, by adding a boundary correction to the action, we have really just performed a canonical transformation.) However, the different versions of the action give rise to different expressions for the momenta. If we use the modified action $S_+ = \int(L + L_B)$ (which is invariant under the left-handed subalgebra of supersymmetry generators), then the covariant momentum conjugate to q^i is

$$\Pi_i^+ \equiv \left.\frac{\partial(L + L_B)}{\partial \dot{q}^i}\right|_{\psi,\bar{\psi},D\psi,D\bar{\psi}} = \Pi_i + V_{,i}$$

and we can write the supersymmetry generators as

$$Q = i\psi^i \Pi_i^+, \quad \overline{Q} = i(\Pi_i^+ - 2V_{,i})\overline{\psi}^i. \tag{43}$$

On the other hand, if we use the modified action $S_- = \int(L - L_B)$ (which is invariant under the right-handed subalgebra of supersymmetry generators), then the covariant momentum conjugate to q^i is

$$\Pi_i^- \equiv \left.\frac{\partial(L - L_B)}{\partial \dot{q}^i}\right|_{\psi,\bar{\psi},D\psi,D\bar{\psi}} = \Pi_i - V_{,i}$$

and we can write the supersymmetry generators as

$$Q = i\psi^i(\Pi_i^- + 2V_{,i}), \quad \overline{Q} = i\Pi_i^- \overline{\psi}^i. \tag{44}$$

Quantisation of the theory then involves representing the canonical variables as operators and interpreting t. However before doing this let us briefly consider the special case of rigid supersymmetry.

6 Rigid Supersymmetry and Nicolai Maps

The local symmetry described above reduces to rigid supersymmetry if one fixes the values of the (non-dynamical) Lagrange multipliers as (for example)

$$N = 1, \qquad \chi = 0, \qquad \bar{\chi} = 0. \tag{45}$$

In order that these values are preserved under supersymmetry transformations, the transformation parameters must be constant:

$$\dot{\epsilon} = 0, \qquad \dot{\bar{\epsilon}} = 0. \tag{46}$$

Because $N, \chi, \bar{\chi}$ have fixed values and are not allowed to vary, they no longer act as Lagrange multipliers enforcing first-class constraints. Hence, when the supersymmetry is rigid, the quantities $Q, \overline{Q}$ and $\mathcal{H}$ defined in equations (20,21) are no longer required to vanish. As a consequence, the Hamiltonian may also be non-vanishing.

The two invariant forms of the action now reduce to

$$S_+ = \int_{t_1}^{t_2} \left\{ \tfrac{1}{2} g_{ij} (\dot{q}^i + V^{,i})(\dot{q}^j + V^{,j}) + \overline{\psi}_i D\psi^i + V_{;ij} \overline{\psi}^i \psi^j - \tfrac{1}{2} R_{ijkl} \overline{\psi}^i \psi^j \overline{\psi}^k \psi^l \right\} dt \tag{47}$$

and

$$S_- = \int_{t_1}^{t_2} \left\{ \tfrac{1}{2} g_{ij} (\dot{q}^i - V^{,i})(\dot{q}^j - V^{,j}) - (D\overline{\psi}_i)\psi^i + V_{;ij} \overline{\psi}^i \psi^j - \tfrac{1}{2} R_{ijkl} \overline{\psi}^i \psi^j \overline{\psi}^k \psi^l \right\} dt \tag{48}$$

According to Nicolai's theorem [2], the invariance of each of these actions under a subalgebra of supersymmetry generators ensures that the theory can be transformed into a free bosonic theory by integrating out the fermions. The transformation will generally have the form of a first-order differential equation, and so will only be invertible if initial conditions are imposed on the bosonic variables $q^i(t)$ in the interacting theory. However, for Nicolai's theorem to work, any initial conditions must be invariant under the same supersymmetry subalgebra as the action [5]. For the action S_+, appropriate initial conditions are

$$q^i(t_1) = q^i_1, \qquad \psi^i(t_1) = 0 \tag{49}$$

since these (like S_+ itself) are invariant under the left-handed subalgebra. For S_- one must use instead the initial conditions

$$q^i(t_1) = q^i_1, \qquad \bar{\psi}_i(t_1) = 0. \tag{50}$$

Integrating out the fermions (subject to the appropriate initial conditions), one finds that the weight of a particular path $q(t)$ in the ensemble of possible paths is given by [10]

$$\begin{aligned} \mathcal{P}_\pm[q(t)] &= \int [\psi, \bar{\psi}] \exp\left(-\frac{1}{\hbar} S_\pm[q, \psi, \bar{\psi}] \right) \\ &= J_\pm[q] \exp\left\{ -\frac{1}{\hbar} \int_{t_1}^{t_2} \tfrac{1}{2} g_{ij} (\dot{q}^i \pm V^{,i})(\dot{q}^j \pm V^{,j}) \right\} \end{aligned} \tag{51}$$

where

$$J_\pm[q] \equiv \exp\left\{ \pm \frac{1}{2\hbar} \int_{t_1}^{t_2} [g^{ij} V_{;ij} - \tfrac{1}{4} R] \right\} dt \tag{52}$$

and $R(\mathbf{q})$ denotes the curvature scalar of the configuration space at the point $\mathbf{q}(t)$. In fact the functional $J_\pm[q]$ turns out to be the Jacobian for the transformation $\xi^a(t) \mapsto q^i(t)$ defined by the differential equation

$$\frac{\Delta q^i}{dt} = \mp g^{ij} V_{,i} \ + e^i{}_a \cdot \xi^a \tag{53}$$

where $e^i{}_a$ is a vielbein field on the configuration space with the property that

$$g_{ij} e^i{}_a e^j{}_b = \delta_{ab} \tag{54}$$

and the differentials $\Delta q^i = dq^i + g^{jk}\Gamma^i{}_{jk} dt$ and $e^i{}_a \cdot \xi^a dt$ are defined so that they transform covariantly in the Itô calculus. It follows by a change of variable that the weight for a given history $\xi^a(t)$ of the new variable is

$$\mathcal{P}[\xi(t)] = \exp\left\{ -\frac{1}{\hbar} \int_{t_1}^{t_2} \frac{1}{2} \delta_{ab} \xi^a \xi^b dt \right\} \tag{55}$$

and so $\xi^a(t)$ can be interpreted as a white noise process with auto-correlation

$$\langle \xi^a(t) \xi^b(t') \rangle = \frac{\hbar}{2} \delta(t - t') \tag{56}$$

that drives the motion of the particle in configuration space via the Langevin equation (53). The probability density function $f_\pm(t, \mathbf{q})$ for the particle's position $\mathbf{q}$ at time t will then evolve according to the associated Fokker-Planck equation

$$\frac{\partial f_\pm}{\partial t} = \frac{\partial}{\partial q^i} \left[g^{1/2} g^{ij} \left(\frac{\hbar}{2} \frac{\partial f_\pm}{\partial q^j} \ \pm V_{,j} f_\pm \right) \right] \tag{57}$$

where g denotes the determinant of the matrix of components g_{ij} of the configuration space metric.

In fact this result can be obtained directly from canonical quantisation. The evolution of the quantum state vector $|\Psi(t)\rangle$ is governed by the Schrödinger equation, which in this case can be written

$$\frac{d}{dt}|\Psi(t)\rangle = \frac{1}{\hbar} H |\Psi(t)\rangle = -\frac{1}{2\hbar^2} Q\overline{Q}|\Psi(t)\rangle \ - \frac{1}{2\hbar^2} \overline{Q}Q|\Psi(t)\rangle \tag{58}$$

on account of the operator identity $Q\overline{Q} + \overline{Q}Q = -2\hbar H$ that follows from the form of the Dirac brackets and our choices of values for the Lagrange multipliers.

As remarked above, Nicolai's theorem is applicable only if the action and initial conditions are invariant under one of the supersymmetry subalgebras. Requiring invariance under the left-handed subalgebra generated by Q means that the initial state $|\psi(t_1)\rangle$ must be annihilated by the operator Q; and since this operator commutes with the Hamiltonian it follows that $Q|\psi(t)\rangle = 0$ for all t and hence

$$\frac{d}{dt}|\Psi(t)\rangle = -\frac{1}{2\hbar^2}Q\overline{Q}|\Psi(t)\rangle = \frac{1}{2\hbar^2}\psi^i\Pi_i^+(\Pi_j^+ - 2V_{,j})\overline{\psi}^j\,|\Psi(t)\rangle \tag{59}$$

where we have used (43) to write Q and $\overline{Q}$ in terms of the covariant momenta Π_i^+ associated with the Q-invariant action S_+. In fact these momenta are naturally represented by $-\hbar\nabla_i$, where ∇_i denotes the covariant derivative with respect to q^i.

In general the wave function will have a number of components, corresponding to solutions with different fermion numbers. Each such component is naturally represented as a p-form on the configuration space, where $n-p$ is the fermion number [11, 12]. The fermion annihilation and creation operators ψ^i and $\overline{\psi}_i$ then act on a form ω according to

$$\psi^i\omega = dq^i \wedge \omega, \qquad \overline{\psi}_j\omega = \hbar i_j\omega \tag{60}$$

where $i_j\omega$ denotes the contraction of the form ω with the vector field $\partial/\partial q^j$. With the momenta Π_i^- represented in the manner described above, it follows that

$$\psi^i\Pi_i^+\omega = -\hbar d\omega, \qquad \Pi_i^+\overline{\psi}^i\omega = \hbar^2\delta\omega \tag{61}$$

where d is the exterior derivative and δ is it adjoint, the coderivative. So if the state $|\Psi(t)\rangle$ is represented by the form ω, then (59) implies

$$\frac{\partial\omega}{\partial t} = -\tfrac{1}{2}d(\hbar\delta\omega - 2i_V\omega) \tag{62}$$

where i_V denotes the contraction with the vector field $g^{ij}V_{,j}\partial/\partial q^i$. The Q-invariant initial condition (49) implies $\psi^i|\Psi(t_1)\rangle = 0$ and hence $F|\Psi(t)\rangle = 0$ for all t (since the fermion number $F = \overline{\psi}_i\psi^i$ is conserved). Consequently, ω will be an n-form, and at each point in the configuration space must be proportional to the elementary n-form:

$$\omega = f_+(\mathbf{q})dq^1 \wedge dq^2 \wedge \ldots \wedge dq^n. \tag{63}$$

It is then easily verified that (62) agrees precisely with equation (57) for $f_+(\mathbf{q})$.

Alternatively, if one starts with the $\overline{Q}$-invariant action S_- and initial conditions (50) then the Schrödinger equation (58) reduces to

$$\frac{d}{dt}|\Psi(t)\rangle = -\frac{1}{2\hbar^2}\overline{Q}Q|\Psi(t)\rangle = \frac{1}{2\hbar^2}\Pi_i^-\overline{\psi}^i\psi^j(\Pi_j^- + 2V_{,j})\,|\Psi(t)\rangle \qquad (64)$$

and so if the state is represented by a differential form ω then

$$\frac{\partial\omega}{\partial t} = -\tfrac{1}{2}\delta(\hbar d\omega - 2dV\wedge\omega). \qquad (65)$$

(Note that in this representation it is the momenta Π_i^-, rather than that are Π_i^+, that are represented by the operator $-\hbar\nabla_i$; hence $\psi^i\Pi_i^-\omega = -\hbar d\omega$ and $\Pi_i^-\overline{\psi}^i\omega = \hbar^2\delta\omega$.) In this case, the initial conditions $\overline{\psi}^i|\Psi(t_1)\rangle = 0$ imply that ω is a 0-form; say $\omega = f_-(\mathbf{q})$ and so this equation reduces to (57) with the minus sign chosen. The evolution equation (62) is then seen to be equivalent to (57) for $f_-(\mathbf{q})$.

Which of the two representations is more appropriate depends on the form of the configuration space potential $V(\mathbf{q})$. For definiteness, we suppose that the potential grows as $|\mathbf{q}|$ increases such that $\lim_{|\mathbf{q}|\to\infty} V(\mathbf{q})/\ln|\mathbf{q}| > 2\hbar n$. In this case, the Fokker-Planck equation (57) admits a static normalisable solution

$$f_+(\mathbf{q}) = A\exp\left(-\frac{V(\mathbf{q})}{2\hbar}\right) \qquad (66)$$

that represents the limiting $t\to\infty$ probability density function of a Brownian particle driven by the Langevin equation

$$\frac{\Delta q^i}{dt} = -g^{ij}V_{,i} \; + e^i{}_a\cdot\xi^a. \qquad (67)$$

7 Quantisation with Local Supersymmetry

Our goal in this section is to quantise the locally supersymmetric model, and to interpret the resulting theory. However we start by reviewing the method described above for rigid supersymmetry (albeit in slightly different notation).

If the model has only *rigid* supersymmetry, quantisation means finding solutions of the Schrödinger equation

$$\frac{d}{dt}|\Psi(t)\rangle = \frac{1}{\hbar} H\,|\Psi(t)\rangle \tag{68}$$

with $H = -N(\mathcal{H} + i\chi\overline{Q} + i\overline{\chi}Q)$ and fixed values of $\bar{\chi}$, χ and N. For the Euclidean theory we choose $N = 1$ and $\bar{\chi} = \chi = 0$, and so the general solution has the form

$$|\Psi(t)\rangle = e^{-t\mathcal{H}/\hbar}|\Psi_0\rangle \tag{69}$$

with $|\Psi_0\rangle$ an arbitrary initial state.

Assuming that $V(\mathbf{q})$ grows as $|\mathbf{q}|$ increases, we choose the initial state $|\Psi_0\rangle$ to be invariant under left-handed supersymmetry; $Q|\Psi_0\rangle = 0$. Because $[\mathcal{H}, Q] = 0$, it follows that

$$Q|\Psi(t)\rangle = 0 \qquad \forall t \geq 0. \tag{70}$$

As discussed in the last section, in this case (68) can then be represented as a Fokker-Planck equation describing the evolution of a conserved probability distribution.

The static probability distribution can be obtained by taking the $t \to \infty$ limit. (This limit must exist, due to the assumed form of the potential $V(\mathbf{q})$.) In the current notation, this is represented by the state

$$|\Psi_\infty\rangle = \lim_{t\to\infty} |\Psi(t)\rangle.$$

The time-independence of this state implies that it has zero energy;

$$\mathcal{H}|\Psi_\infty\rangle = 0.$$

(Note that in this respect the Euclidean theory considered here differs from the conventional theory, which admits a ground state with a positive energy and hence an oscillating phase. However, because the Euclidean evolution operator $e^{-t\mathcal{H}/\hbar}$ is hermitian rather than anti-hermitian, only a zero-energy state can have a constant norm.)

Thanks to (70), this limiting state is also Q-invariant:

$$Q|\Psi_\infty\rangle = 0. \tag{71}$$

Moreover, since $0 = \langle\Psi_\infty|2\hbar\mathcal{H}|\Psi_\infty\rangle = \langle\Psi_\infty|Q\overline{Q}|\Psi_\infty\rangle + \langle\Psi_\infty|\overline{Q}Q|\Psi_\infty\rangle = 0 + ||\,\overline{Q}|\Psi_\infty\rangle||^2$, this state must also be $\overline{Q}$-invariant:

$$\overline{Q}|\Psi_\infty\rangle = 0. \tag{72}$$

This is readily verified in the Fokker-Planck representation considered earlier; one has only to show that the function $f_+(\mathbf{q})$ given by (66) is annihilated by the operator $\overline{Q}$ defined in (43).

The state $|\Psi_\infty\rangle$ is thus annihilated by all the first-class constraints of the *locally* supersymmetric theory and thus represents an acceptable quantum state for this theory also.

Let us summarise the argument so far. Thanks to Nicolai's theorem the supersymmetry of the model ensures that there exists a representation of the quantum theory in which the Euclidean Schrödinger equation takes the form of a Fokker-Planck equation governing the evolution of a conserved probability density function. By taking the (Euclidean) long-time limit one obtains a time-independent state satisfying all the constraints of the locally supersymmetric theory and represented by a static probability density function. This procedure is readily adapted (at least in principle) to quite general theories with local supersymmetry, and is naturally regarded as a type of stochastic quantisation.

Apart from giving a procedure for finding quantum state, this approach also provides a natural probabilistic interpretation of this state space without reference to any particular inner product. Indeed, by reviewing the derivation of the probability density function (66), the reader can confirm that the inner product has not been used at all. This is potentially of importance for theories such as supergravity, in which the choice of the inner product is itself problematic. By using the method outlined above, one can (in principle) obtain not only the quantum state that satisfying all the constraints of the theory but also the corresponding probability density function on the configuration space.

8 Discussion

In this paper, a type of stochastic quantisation of a locally supersymmetric model was performed without introducing an extra fictitious time variable as is normally done in stochastic quantisation.

In the context of supersymmetric models,as indicated Nicolai maps exist for certain components of the wave function. This in turn suggested an interpretation of the wave function as a probability density whose integrals is conserved in Euclidean time. In the quantum theory, the wave function satisfied a type of Fokker-Planck equation and thus was interpreted stochastically. However, since our supersymmetric theory was local one could by-pass the difficulties of explicitly constructing Nicolai maps by finding the associated Fokker-Planck equation directly via canonical quantisation with a stochastic interpretation.

In conventional quantisation of a classical theory one has to construct the Hilbert space and define an inner product in the Hilbert space. However, in the stochastic quantisation of locally supersymmetric theory, Since the solution of the Fokker-Planck equation is a probability density function, thus there is a ready-made probabilistic interpretation of the quantum theory without any reference to the concept of inner product as one see in the usual Hilbert space formulation of quantum theory. This should be a big advantage in cosmological models with various types of extended supersymmetries (including full supergravity) where the choice of satisfactory inner product has long been viewed as one of the most fundamental problems.

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