Littlewood–Richardson polynomials

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Abstract

We introduce a family of rings of symmetric functions depending on an infinite sequence of parameters. A distinguished basis of such a ring is comprised by analogues of the Schur functions. The corresponding structure coefficients are polynomials in the parameters which we call the Littlewood–Richardson polynomials. We give a combinatorial rule for their calculation by modifying an earlier result of B. Sagan and the author. The new rule provides a formula for these polynomials which is manifestly positive in the sense of W. Graham. We apply this formula for the calculation of the product of equivariant Schubert classes on Grassmannians which implies a stability property of the structure coefficients. The first manifestly positive formula for such an expansion was given by A. Knutson and T. Tao by using combinatorics of puzzles, and the stability property can also be derived from the puzzle rule. As another application, we use the Littlewood–Richardson polynomials to describe the multiplication rule in the algebra of the (virtual) Casimir elements for the general linear Lie algebra in the basis of the (virtual) quantum immanants constructed by A. Okounkov and G. Olshanski.

1 Introduction

Let $a = (a_i)$, $i \in \mathbb{Z}$ be a sequence of variables. Consider the ring of polynomials $\mathbb{Z}[a]$ in the variables a_i with integer coefficients. Introduce another infinite set of variables $x = (x_1, x_2, \dots)$ and for each nonnegative integer n denote by Λ_n the ring of symmetric polynomials in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}[a]$. The ring Λ_n is filtered by the usual degrees of polynomials in $x_1, \dots, x_n$. The evaluation map

$$\varphi_n : \Lambda_n \to \Lambda_{n-1}, \qquad P(x_1, \dots, x_n) \mapsto P(x_1, \dots, x_{n-1}, a_n) \tag{1.1}$$

is a homomorphism of filtered rings so that we can define the inverse limit ring Λ by

$$\Lambda = \varprojlim \Lambda_n, \qquad n \to \infty, \tag{1.2}$$

where the limit is taken with respect to the homomorphisms (1.1) in the category of filtered rings. When a is specialized to the sequence of zeros, this reduces to the usual definition of the ring of symmetric functions; see e.g. Macdonald [13]. In that case, a distinguished basis of Λ is comprised by the Schur functions $s_\lambda(x)$ parameterized by all partitions λ. The respective analogues of the $s_\lambda(x)$ in the general case are the *double Schur functions* $s_\lambda(x\,\|\,a)$ which form a basis of Λ over $\mathbb{Z}[a]$. We introduce the *Littlewood–Richardson polynomials* $c_{\lambda\mu}^{\nu}(a)$ as the structure coefficients of the ring Λ in the basis of double Schur functions,

$$s_\lambda(x\,\|\,a)\, s_\mu(x\,\|\,a) = \sum_\nu c_{\lambda\mu}^{\nu}(a)\, s_\nu(x\,\|\,a). \tag{1.3}$$

In the specialization $a = (0)$ the polynomials $c_{\lambda\mu}^{\nu}(a)$ become the classical Littlewood–Richardson coefficients $c_{\lambda\mu}^{\nu}$; see [11]. These are remarkable nonnegative integers which occupy a prominent place in combinatorics, representation theory and geometry; see e.g. Fulton [4], Macdonald [13] and Sagan [19].

The main result of this paper is a combinatorial rule for the calculation of the Littlewood–Richardson polynomials which provides a manifestly positive formula in the sense that $c_{\lambda\mu}^{\nu}(a)$ is written as a polynomial in the differences $a_i - a_j$, $i < j$, with positive integer coefficients; cf. Graham [8].

The double Schur function $s_\lambda(x\,\|\,a)$ can be regarded as the sequence of the *double Schur polynomials*

$$s_\lambda(x_1, \dots, x_n\,\|\,a), \qquad n = 1, 2, \dots, \tag{1.4}$$

which are compatible with respect to the homomorphisms (1.1),

$$\varphi_n : s_\lambda(x_1, \dots, x_n\,\|\,a) \mapsto s_\lambda(x_1, \dots, x_{n-1}\,\|\,a). \tag{1.5}$$

The polynomials (1.4) are closely related to the "factorial" or "double" Schur polynomials $s_\lambda(x|u)$ with $x = (x_1, \dots, x_n)$. The latter were introduced by Goulden and Greene [7] and Macdonald [12] as a generalization of the factorial Schur polynomials introduced by Biedenharn and Louck [1, 2], and they are also a special case of the double Schubert polynomials of Lascoux and Schützenberger; see Lascoux [10]. We follow Chen, Li and Louck [3] and Fulton [5] and use the name "double Schur polynomials" for the related polynomials $s_\lambda(x\,\|\,a)$ as well.

In a more detail, consider a *diagram* λ which is a sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers λ_i such that $\lambda_1 \geqslant \dots \geqslant \lambda_n \geqslant 0$. We will view λ as the array of left justified rows of unit boxes with λ_1 boxes in the top row, λ_2 boxes in the second row, etc. The total number of boxes in λ will be denoted by $|\lambda|$ so that λ may also be regarded as a partition of $|\lambda|$. The transposed diagram $\lambda' = (\lambda'_1, \dots, \lambda'_p)$ is obtained from λ by applying the symmetry with respect to the main diagonal, so that λ'_j is the number of boxes in column j of λ.

Let $u = (u_1, u_2, \dots)$ be a sequence of variables. The polynomials $s_\lambda(x|u)$ can be defined by

$$s_\lambda(x|u) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - u_{T(\alpha)+c(\alpha)}), \tag{1.6}$$

where T runs over all semistandard (column-strict) tableaux of shape λ with entries in $\{1, \dots, n\}$, $T(\alpha)$ is the entry of T in the box $\alpha \in \lambda$ and $c(\alpha) = j - i$ is the content of the box $\alpha = (i, j)$ in row i and column j.

By a *reverse λ-tableau* T we will mean the tableau obtained by filling in the boxes of λ with the numbers $1, 2, \dots, n$ in such a way that the entries weakly decrease along the rows and strictly decrease down the columns. If $\alpha = (i, j)$ is a box of λ we let $T(\alpha) = T(i, j)$ denote the entry of T in the box α. We define the double Schur polynomials $s_\lambda(x\,\|\,a)$ by

$$s_\lambda(x\,\|\,a) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}), \tag{1.7}$$

summed over the reverse λ-tableaux T. Then we have

$$s_\lambda(x\,\|\,a) = s_\lambda(x|u) \tag{1.8}$$

for the sequences a and u related by $a_{n-i+1} = u_i$ with $i = 1, 2, \dots$. In particular, the polynomial $s_\lambda(x\,\|\,a)$ only depends on the variables a_i with $i \leqslant n$, $i \in \mathbb{Z}$. The relation (1.8) is verified easily by replacing x_i with x_{n-i+1} in (1.6) for all $i = 1, \dots, n$ and using the fact that $s_\lambda(x|u)$ is a symmetric polynomial in x.

In the particular case where a is the sequence of zeros, the polynomials $s_\lambda(x\,\|\,a)$ coincide with the Schur polynomials $s_\lambda(x)$. For another specialization of the sequence

a with $a_i = n - i$ formula (1.7) defines the shifted Schur polynomials of Okounkov and Olshanski [16, 17]. The use of the *reverse* tableaux was significant in their study of the vanishing and stability properties of these polynomials and associated central elements of the universal enveloping algebra for the Lie algebra $\mathfrak{gl}_n$; see also Section 3.2 below.

The double Schur polynomials $s_\lambda(x\|a)$ parameterized by the diagrams λ with at most n rows form a basis of the ring Λ_n. Due to the stability property (1.5), the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ can be defined by the expansion (1.3), where x is understood as the set of variables $x = (x_1, \dots, x_n)$ for any positive integer n such that the diagrams λ, μ and ν have at most n rows. This allows us to work with a finite set of variables for the determination of the polynomials $c_{\lambda\mu}^{\nu}(a)$.

It was observed by Goulden and Greene [7] and Macdonald [12] that $s_\lambda(x|u)$, regarded as a formal power series in the infinite sets of variables x and u, admits a "supertableaux" representation. We will show that this representation has its natural "finite" counterpart where x is a finite set of variables. The corresponding formula is implied by the results of [12], although it was not stated there in an explicit form. This representation leads to a "supertableau" expression for the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$, although that expression is neither manifestly positive, nor stable.

It was realized by Knutson and Tao [9] that under an appropriate specialization, the polynomials $c_{\lambda\mu}^{\nu}(a)$ describe the multiplication rule for the equivariant Schubert classes on Grassmannians; see also Fulton [5] for a more direct argument. The aforementioned positivity property was established in an earlier paper of Graham [8] in the general context of the equivariant Schubert calculus. The first manifestly positive formula for the coefficients in the expansion (1.3) was obtained by Knutson and Tao [9] by using combinatorics of *puzzles*. An earlier rule of Molev and Sagan [15] lacks the positivity property (with the exception of the classical case $a = (0)$). The structure coefficients provided by either of these rules depend on the number of variables and so do not exhibit the stability property. Still, this property can be derived directly from the puzzle rule as observed by Fulton [6].

For the proof of the main theorem (Theorem 2.1) we follow the general approach of [15], using the techniques of "barred" tableaux and modify the corresponding arguments in order to obtain manifestly positive polynomials. This is achieved by imposing an *admissibility condition* on the barred tableaux. This gives a manifestly positive and stable formula for the expansion of the product of two equivariant Schubert classes on the Grassmannian. As pointed out by Fulton [6], the stability property also extends to the equivariant Schubert calculus on the flag manifold. In particular, (1.5) is implied by a similar property of the double Schubert polynomials.

As another application, we obtain a rule for the positive integer expansion of the product of two (virtual) quantum immanants (or the corresponding higher Capelli operators) of Okounkov and Olshanski [16, 17]; cf. [15].

This work was inspired by Bill Fulton's lectures [5]. I am grateful to Bill for stimulating discussions.

2 Multiplication rule

Let R denote a sequence of diagrams

$$\mu = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \nu, \tag{2.1}$$

where $\rho \to \sigma$ means that σ is obtained from ρ by adding one box. Let r_i denote the row number of the box added to the diagram $\rho^{(i-1)}$. The sequence $r_1 r_2 \dots r_l$ is called the *Yamanouchi symbol* of R. Introduce the ordering on the set of boxes of a diagram λ by reading them by columns from left to right and from bottom to top in each column. We call this the *column order*. We shall write $\alpha \prec \beta$ if α (strictly) precedes β with respect to the column order. Given a sequence R, construct the set $\mathcal{T}(\lambda, R)$ of *barred* reverse λ-tableaux T with entries from $\{1, 2, \dots\}$ such that T contains boxes $\alpha_1, \dots, \alpha_l$ with

$$\alpha_1 \prec \cdots \prec \alpha_l \qquad \text{and} \qquad T(\alpha_i) = r_i, \quad 1 \leqslant i \leqslant l.$$

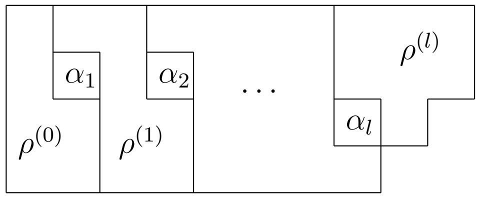
We will distinguish the entries in $\alpha_1, \dots, \alpha_l$ by barring each of them. So, an element of $\mathcal{T}(\lambda, R)$ is a pair consisting of a reverse λ-tableau and a chosen sequence of barred entries compatible with R. We shall keep the notation T for such a pair. For example, let R be the sequence

$$(3, 1) \to (3, 2) \to (3, 2, 1) \to (3, 3, 1) \to (4, 3, 1)$$

so that the Yamanouchi symbol is $2\,3\,2\,1$. Then for $\lambda = (5, 5, 3)$ the following barred λ-tableau belongs to $\mathcal{T}(\lambda, R)$:

5	5	4	$\overline{2}$	2
4	$\overline{3}$	2	1	$\overline{1}$
$\overline{2}$	1	1		

For each box α with $\alpha_i \prec \alpha \prec \alpha_{i+1}$, $0 \leqslant i \leqslant l$, set $\rho(\alpha) = \rho^{(i)}$. The boxes $\alpha_1, \dots, \alpha_l$ of λ divide the diagram into regions marked by the elements of the sequence R, as illustrated:



Finally, a reverse λ-tableau T will be called *ν-admissible* if

$$T(1,j) \leqslant \nu'_j \qquad \text{for all} \quad j = 1, \dots, \lambda_1.$$

Note that ν-admissible tableaux exist only if $\lambda \subseteq \nu$.

We are now in a position to state a rule for the calculation of the Littlewood-Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ defined by (1.3).

Theorem 2.1. *The polynomial $c_{\lambda\mu}^{\nu}(a)$ is zero unless $\mu \subseteq \nu$. If $\mu \subseteq \nu$ then*

$$c_{\lambda\mu}^{\nu}(a) = \sum_{R}\sum_{T} \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{unbarred}}} \Big(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)} \Big), \tag{2.2}$$

summed over all sequences R of the form (2.1) *and all ν-admissible reverse λ-tableaux $T \in \mathcal{T}(\lambda, R)$. Moreover, for all differences of the form $a_i - a_j$ occurring in the formula we have $i < j$.*

Before proving the theorem, let us point out some properties of the Littlewood-Richardson polynomials which are immediate from the rule and consider some examples. The polynomial $c_{\lambda\mu}^{\nu}(a)$ is zero unless both diagrams λ and μ are contained in ν and $|\lambda| + |\mu| \geqslant |\nu|$. In this case $c_{\lambda\mu}^{\nu}(a)$ is a homogeneous polynomial in the a_i of degree $|\lambda| + |\mu| - |\nu|$. If $|\lambda| + |\mu| - |\nu| = 0$ then the theorem reproduces a version of the classical Littlewood-Richardson rule; see e.g. [15]. Note also that by the definition, the polynomials have the symmetry $c_{\lambda\mu}^{\nu}(a) = c_{\mu\lambda}^{\nu}(a)$ which is not apparent from the rule.

Example 2.2. For the product of the double Schur functions $s_{(2)}(x\,\|\,a)$ and $s_{(2,1)}(x\,\|\,a)$ we have

$$\begin{aligned} s_{(2)}(x\,\|\,a)\, s_{(2,1)}(x\,\|\,a) = {} & s_{(4,1)}(x\,\|\,a) + s_{(3,2)}(x\,\|\,a) + s_{(3,1,1)}(x\,\|\,a) + s_{(2,2,1)}(x\,\|\,a) \\ & + \big(a_{-1} - a_2 + a_{-2} - a_0\big)\, s_{(3,1)}(x\,\|\,a) + \big(a_{-1} - a_2\big)\, s_{(2,2)}(x\,\|\,a) \\ & + \big(a_{-1} - a_0\big)\, s_{(2,1,1)}(x\,\|\,a) + \big(a_{-1} - a_2\big)\, \big(a_{-1} - a_0\big)\, s_{(2,1)}(x\,\|\,a). \end{aligned}$$

For instance, the coefficient of $s_{(3,1)}(x\,\|\,a)$ is calculated by the following barred (2)-tableaux

$$\begin{array}{|c|c|}\hline 1 & \overline{1} \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline \overline{1} & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & \overline{1} \\ \hline\end{array}$$

compatible with the sequence $(2,1) \to (3,1)$. They contribute respectively $a_{-1} - a_1$, $a_{-2} - a_0$, $a_1 - a_2$ which sums up to the coefficient $a_{-1} - a_2 + a_{-2} - a_0$. Alternatively, using the symmetry $c_{\lambda\mu}^{\nu}(a) = c_{\mu\lambda}^{\nu}(a)$ we can calculate the coefficient of $s_{(3,1)}(x\,\|\,a)$ by considering the barred $(2,1)$-tableaux

$$\begin{array}{|c|c|}\hline \overline{2} & 1 \\ \hline \overline{1} \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline \overline{2} & \overline{1} \\ \hline 1 \\ \cline{1-1}\end{array}$$

compatible with the sequences $(2) \to (3) \to (3,1)$ and $(2) \to (2,1) \to (3,1)$, respectively. Their contributions to the coefficient are $a_{-2} - a_0$ and $a_{-1} - a_2$.

Example 2.3. For the calculation of $c_{(4,2,1)\,(2,2)}^{(5,2,2)}(a)$ take $\lambda = (4,2,1)$, $\mu = (2,2)$ and $\nu = (5,2,2)$. We have ten sequences R of the form (2.1) but the set $\mathcal{T}(\lambda, R)$ contains ν-admissible tableaux only for three of them. For the sequence R_1 with the Yamanouchi symbol $1\,3\,3\,1\,1$, the set $\mathcal{T}(\lambda, R_1)$ contains two admissible barred tableaux

$$\begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{1} & \overline{1} \\ \hline 2 & 2 \\ \cline{1-2} \overline{1} \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{1} & \overline{1} \\ \hline 2 & 1 \\ \cline{1-2} \overline{1} \\ \cline{1-1}\end{array}$$

whose contributions to the Littlewood–Richardson polynomial are $(a_0 - a_3)(a_0 - a_2)$ and $(a_0 - a_3)(a_{-2} - a_1)$, respectively. For the sequence R_2 with the Yamanouchi symbol $1\,3\,1\,3\,1$, the set $\mathcal{T}(\lambda, R_2)$ contains the admissible tableaux

$$\begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{1} & 1 \\ \hline 2 & \overline{1} \\ \cline{1-2} \overline{1} \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & 1 & \overline{1} \\ \hline 2 & \overline{1} \\ \cline{1-2} \overline{1} \\ \cline{1-1}\end{array}$$

with the respective contributions $(a_0 - a_3)(a_{-4} - a_{-2})$ and $(a_0 - a_3)(a_{-3} - a_{-1})$. For the sequence R_3 with the Yamanouchi symbol $3\,1\,3\,1\,1$, the set $\mathcal{T}(\lambda, R_3)$ contains the only admissible tableau

$$\begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{1} & \overline{1} \\ \hline 2 & \overline{1} \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

with the contribution $(a_{-1}-a_3)(a_0-a_3)$. Hence,

$$c_{(4,2,1)\,(2,2)}^{(5,2,2)}(a) = (a_0-a_3)\,(a_{-4}+a_{-3}+a_0-a_1-a_2-a_3).$$

Taking $\lambda=(2,2)$, $\mu=(4,2,1)$ and $\nu=(5,2,2)$ we get two sequences with the Yamanouchi symbols $1\,3$ and $3\,1$. The corresponding sets $\mathcal{T}(\lambda,R)$ consist of five and four admissible barred tableaux, respectively, thus leading to a slightly longer calculation. $\square$

Proof of Theorem 2.1. We present the proof as a sequence of lemmas. Due to the stability property (1.5), we may (and will) work with a finite set of variables $x=(x_1,\dots,x_n)$. Accordingly, possible entries of the tableaux are now elements of the set $\{1,\dots,n\}$. Introduce another sequence of variables $b=(b_i)$, $i\in\mathbb{Z}$, and define the Littlewood–Richardson type coefficients $c_{\lambda\mu}^{\nu}(a,b)$ by the expansion

$$s_\lambda(x\,\|\,b)\,s_\mu(x\,\|\,a)=\sum_\nu c_{\lambda\mu}^{\nu}(a,b)\,s_\nu(x\,\|\,a). \tag{2.3}$$

Lemma 2.4. *The coefficient $c_{\lambda\mu}^{\nu}(a,b)$ is zero unless $\mu\subseteq\nu$. If $\mu\subseteq\nu$ then*

$$c_{\lambda\mu}^{\nu}(a,b)=\sum_R\sum_T\prod_{\substack{\alpha\in\lambda\\ T(\alpha)\ \text{unbarred}}}\Big(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}}-b_{T(\alpha)-c(\alpha)}\Big), \tag{2.4}$$

summed over all sequences R of the form (2.1) *and all reverse λ-tableaux $T\in\mathcal{T}(\lambda,R)$.*

Proof. This is essentially a reformulation of the main result of [15] (Theorem 3.1). Note that the summation in (2.4) is taken over *all* barred tableaux $T\in\mathcal{T}(\lambda,R)$ (not just over the ν-admissible ones as in (2.2)). Rather than repeating the whole argument of [15], we only sketch the main steps of the proof and indicate the necessary changes to be made. We refer the reader to [15] for the details.

We assume that all diagrams here have at most n rows. If $\rho=(\rho_1,\dots,\rho_n)$ is a such diagram, we set

$$a_\rho=(a_{1-\rho_1},\dots,a_{n-\rho_n})\qquad\text{and}\qquad |a_\rho|=a_{1-\rho_1}+\dots+a_{n-\rho_n}.$$

Under the correspondence (1.8) we have $a_\rho=u_\rho=(u_{\rho_1+n},\dots,u_{\rho_n+1})$, the latter notation was used in [15].

The starting point is the Vanishing Theorem of [16] whose proof was also reproduced in [15]. By that theorem,

$$s_\lambda(a_\rho\,\|\,a)=0\qquad\text{unless}\quad\lambda\subseteq\rho,$$

and

$$s_\lambda(a_\lambda \,\|\, a) = \prod_{(i,j)\in\lambda} \big(a_{i-\lambda_i} - a_{\lambda'_j - j+1}\big).$$

The first claim of the lemma follows from the Vanishing Theorem which also implies

$$c^{\mu}_{\lambda\mu}(a,b) = s_\lambda(a_\mu \,\|\, b).$$

This proves (2.4) for the case $\nu = \mu$. Now we suppose that $|\nu| - |\mu| \geqslant 1$ and proceed by induction on $|\nu| - |\mu|$. The induction step is based on the recurrence relation

$$c^{\nu}_{\lambda\mu}(a,b) = \frac{1}{|a_\nu| - |a_\mu|}\left(\sum_{\mu\to\mu^+} c^{\nu}_{\lambda\mu^+}(a,b) - \sum_{\nu^-\to\nu} c^{\nu^-}_{\lambda\mu}(a,b)\right) \tag{2.5}$$

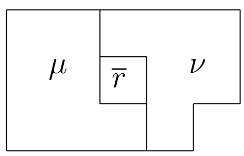
which was proved in [15, Proposition 3.4]; see also [9]. Suppose that the diagram ν is obtained from μ by adding one box in row r. Then

$$c^{\nu}_{\lambda\mu}(a,b) = \frac{s_\lambda(a_\nu \,\|\, b) - s_\lambda(a_\mu \,\|\, b)}{(a_\nu)_r - (a_\mu)_r}. \tag{2.6}$$

Now use the definition (1.7) of the double Schur polynomials. Since the n-tuples a_ν and a_μ only differ at the r-th component, the ratio on the right hand side of (2.6) can be expanded by taking into account the entries r of the reverse λ-tableaux T. We need the following formula, where we are thinking of $y = (a_\nu)_r$, $z = (a_\mu)_r$ and $m_i = b_{T(\alpha)-c(\alpha)}$ as α runs over the boxes of T with $T(\alpha) = r$ in column order:

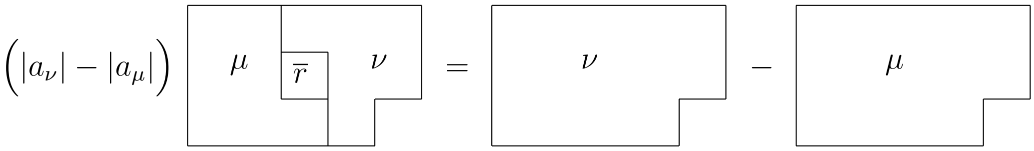
$$\frac{\prod_{i=1}^k (y - m_i) - \prod_{i=1}^k (z - m_i)}{y - z} = \sum_{j=1}^{k} (z - m_1)\dots(z - m_{j-1})(y - m_{j+1})\dots(y - m_k).$$

The right hand side of (2.6) can now be interpreted as the right hand side of (2.4), where R is the only sequence $\mu \to \nu$ and the sum is taken over the reverse λ-tableaux T with one barred entry r, as illustrated:



Here $\rho(\alpha) = \mu$ for all boxes α preceding the box occupied by the barred r, and $\rho(\alpha) = \nu$ for all boxes α which follow that box in column order. Note that the variables y and z are now swapped on the right hand side of the above expansion, as compared to [15] (this does not change the polynomial due to the symmetry in y and z). Consequently, the column order used in [15] is the opposite to the order on the boxes of λ we use here.

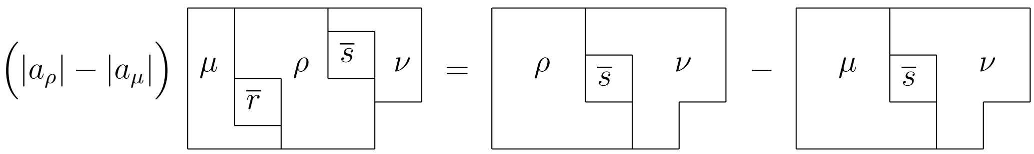
We can represent the above calculation of $c^{\nu}_{\lambda\mu}(a)$ as the "diagrammatic" relation



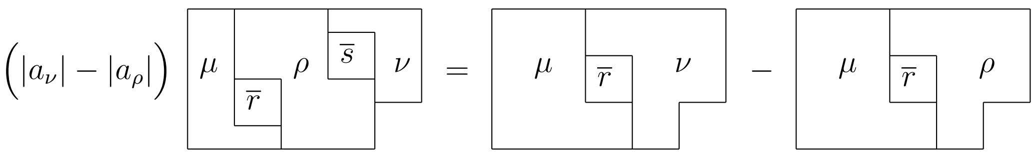
Consider now the next case where $|\nu| - |\mu| = 2$ and apply the recurrence relation (2.5). We have three subcases: the diagram ν is obtained from μ by adding two boxes in different rows and columns; by adding two boxes in the same row; or by adding two boxes in the same column. The first two subcases are dealt with in a way similar to the case $|\nu| - |\mu| = 1$. An additional care is needed for the third subcase where we suppose that ν is obtained from μ by adding the boxes in rows r and $r + 1$. Denote by ρ the diagram obtained from μ by adding the box in row r. Then (2.5) gives

$$c^{\nu}_{\lambda\mu}(a, b) = \frac{c^{\nu}_{\lambda\rho}(a, b) - c^{\rho}_{\lambda\mu}(a, b)}{|a_\nu| - |a_\mu|}.$$

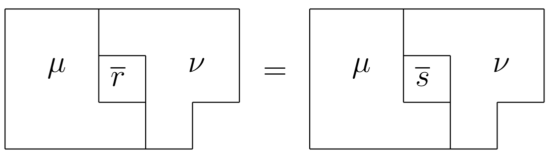
Set $s = r + 1$. Exactly as in the case $|\nu| - |\mu| = 1$, we have the following diagrammatic relations:



and



Hence, the desired formula for $c^{\nu}_{\lambda\mu}(a, b)$ will follow if we prove the relation



We construct a "weight-preserving" bijection between the barred reverse λ-tableaux which are represented by the left and right hand sides of this diagrammatic relation. Here the weight is the product on the right hand side of (2.4) corresponding to a barred tableau. Let such a tableau with a barred entry r in the box (i, j) be given. Suppose first that the box $(i - 1, j)$ belongs to the diagram and it is occupied by

$s = r + 1$. Then the image of the tableau under the map is the same tableau but the entry $T(i, j) = r$ is now unbarred while $T(i - 1, j) = r + 1$ is barred. Since

$$(a_\nu)_{r+1} = (a_\mu)_r \qquad \text{and} \qquad T(i-1,j) - c(i-1,j) = T(i,j) - c(i,j),$$

the weights of the tableaux are preserved under the map.

Suppose now that the entry in the box $(i - 1, j)$ is greater than $r + 1$, or this box is outside the diagram. Consider all entries r in the row i to the left of the box (i, j) and suppose that they occupy the boxes $(i, j - m)$, $(i, j - m + 1), \dots, (i, j - 1)$. Then the image of the tableau under the map is the tableau obtained by replacing the entries in each of the boxes $(i, j - m), \dots, (i, j)$ with $s = r + 1$ and barring the entry in the box $(i, j - m)$. The weights of the tableaux are again preserved.

The inverse map is described in a similar way. This gives the desired weight-preserving bijection. The general argument uses similar calculations with the barred diagrams and a similar bijection described in [15]. □

Remark 2.5. (i) A cohomological interpretation of the coefficients $c^\nu_{\lambda\mu}(a, b)$ and their puzzle computation can be found in [9].

(ii) The definition (2.3) of the coefficients $c^\nu_{\lambda\mu}(a, b)$ can be extended to the case where λ is a skew diagram. Lemma 2.4 and its proof remain valid; see [15].

(iii) In contrast with the Littlewood–Richardson polynomials $c^\nu_{\lambda\mu}(a)$, the coefficients $c^\nu_{\lambda\mu}(a, b)$ do not have the stability property as they depend on n. □

Lemma 2.4 implies that the Littlewood–Richardson polynomials can be calculated by (2.4) with $b = a$, that is, $c^\nu_{\lambda\mu}(a) = c^\nu_{\lambda\mu}(a, a)$. Our strategy now is to show that (unlike the formula of Theorem 3.1 in [15]), the formula (2.4) (with $b = a$) is "nonnegative" in the sense that all nonzero products which occur in the formula are polynomials in the $a_i - a_j$ with $i < j$. Then we demonstrate that the ν-admissibility condition serves to eliminate the unwanted zero terms.

Lemma 2.6. *Let R be a sequence of the form* (2.1) *and let $T \in \mathcal{T}(\lambda, R)$. Suppose that*

$$\prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{unbarred}}} \Big(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)} \Big) \neq 0. \tag{2.7}$$

Then $\rho(\alpha)_{T(\alpha)} > c(\alpha)$ for all $\alpha \in \lambda$ with unbarred $T(\alpha)$.

Proof. Suppose on the contrary that there exists a box $\alpha = (i, j)$ with an unbarred $T(i, j)$ and the condition $\rho(i, j)_{T(i,j)} < j - i$; the equality $\rho(i, j)_{T(i,j)} = j - i$ is excluded since this would violate (2.7). Choose such a box with the minimum possible value of j. If all the entries $T(i, 1), \dots, T(i, j - 1)$ of T are barred then $\rho(i, j)$ is obtained from μ by adding boxes in rows $T(i,1) \geqslant \cdots \geqslant T(i,j-1)$ and, possibly, by adding other boxes. Since $T(i,j-1) \geqslant T(i,j)$, we have $\rho(i,j)_{T(i,j)} \geqslant j-1$, a contradiction. So, at least one of the entries $T(i,1),\dots,T(i,j-1)$ must be unbarred. Take such an unbarred entry $T(i,k)$ which is the closest to $T(i,j)$, that is, all entries $T(i,k+1),\dots,T(i,j-1)$ are barred. Then $\rho(i,j)$ is obtained from $\rho(i,k)$ by adding boxes in rows $T(i,k+1) \geqslant \cdots \geqslant T(i,j-1)$ and, possibly, by adding other boxes. Hence,

$$\rho(i,j)_{T(i,j)} \geqslant \rho(i,k)_{T(i,k)} + j - k - 1$$

which implies $\rho(i,k)_{T(i,k)} < k - i + 1$. However, if $\rho(i,k)_{T(i,k)} = k - i$ then the factor in (2.7) corresponding to $\alpha = (i,k)$ is zero, which is impossible. Therefore $\rho(i,k)_{T(i,k)} < k - i$ which contradicts the choice of j. □

Lemma 2.7. *Suppose that R is a sequence of the form* (2.1) *and $T \in \mathcal{T}(\lambda, R)$. If* (2.7) *holds then T is ν-admissible.*

Proof. By Lemma 2.6, for all unbarred entries $T(1,k)$ of the first row of the tableau T we have $\rho(1,k)_{T(1,k)} \geqslant k$. This implies $\nu_{T(1,k)} \geqslant k$. If the entry $T(1,j)$ is barred then $\rho(1,k)_{T(1,k)} \geqslant k$ for the nearest unbarred entry $T(1,k)$ on its left (if it exists). Then ν is obtained from $\rho(1,k)$ by adding boxes in rows $T(1,k+1) \geqslant \cdots \geqslant T(1,j)$ and, possibly, by adding other boxes. This implies $\nu_{T(1,j)} \geqslant j$. Thus, this inequality holds for all $j = 1,\dots,\lambda_1$. This is equivalent to the ν-admissibility of T. □

Lemma 2.8. *Suppose that R is a sequence of the form* (2.1) *and $T \in \mathcal{T}(\lambda, R)$ is ν-admissible. Then $\rho(\alpha)_{T(\alpha)} > c(\alpha)$ for all $\alpha \in \lambda$ with unbarred $T(\alpha)$.*

Proof. We argue by contradiction. Taking into account Lemma 2.6, we find that for some $\alpha = (i,j)$ with unbarred $T(\alpha)$ we have $\rho(i,j)_{T(i,j)} = j - i$. Set $t = T(i,j)$ and consider all barred entries of T (assuming for now they exist) which are equal to t and occur to the right of the column j. Since T is a reverse tableau, these entries $\bar{t}$ can only occur in rows $1,2,\dots,i$. Let (r,k) be the box with the maximum column number k containing $\bar{t}$. Then the total number of such entries $\bar{t}$ does not exceed $k-j$. This implies that the number of boxes ν_t in row t of ν does not exceed $\rho(i,j)_t + k - j = k - i$. Hence, $\nu'_k \leqslant t-1$. On the other hand, by the ν-admissibility of T we have $t = T(r,k) \leqslant T(1,k) \leqslant \nu'_k$, a contradiction.

If none of the boxes to the right of the column j contains $\bar{t}$ then $\nu_t = \rho(i,j)_t = j-i$. However, by the assumption, $\nu_t \geqslant \nu_{T(1,j)} \geqslant j$, a contradiction. □

This completes the proof of the theorem. □

By the column word of a tableau T we will mean the sequence of all entries of T written in the column order.

Corollary 2.9. *Suppose that $|\nu| = |\lambda| + |\mu|$. The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ equals the number of ν-admissible reverse λ-tableaux T whose column word coincides with the Yamanouchi symbol of a certain sequence R of the form* (2.1). □

This is a well-known version of the Littlewood–Richardson rule which also holds with the ν-admissibility condition dropped; see Lemma 2.7.

Remark 2.10. Due to (1.8), the multiplication rule for the polynomials $s_\lambda(x|u)$ is obtained from Theorem 2.1 by replacing a_i with u_{n-i+1} for each i. The corresponding coefficients are polynomials in the $u_i - u_j$, $i > j$, with positive integer coefficients. □

Corollary 2.11. *Suppose that the polynomials $c^{\nu}_{\lambda\mu}(a)$ are defined by the expansion* (1.3) *with $x = (x_1, \dots, x_n)$. Then $c^{\nu}_{\lambda\mu}(a)$ is independent of n as soon as $n \geqslant \nu'_1$. Moreover, if $n < \nu'_1$ then $c^{\nu}_{\lambda\mu}(a) = 0$.*

Proof. This follows from the admissibility condition on the reverse tableaux. □

We conclude this section with one more rule for the calculation of the Littlewood–Richardson polynomials $c^{\nu}_{\lambda\mu}(a)$. It relies on a supertableau representation of the double Schur polynomials $s_\lambda(x\,\|\,a)$ which is implied by the results of [12] although is not explicitly stated there. This representation provides a "finite" version of the supertableau formulas of [7] and [12]; cf. [3].

Fix a positive integer n. For $r \geqslant 1$ set $u^{(r)} = (u_1, \dots, u_r)$ and use the 9th Variation in [12] with the indeterminates h_{rs} specialized by

$$h_{rs} = h_r(u^{(n-r-s+1)}) \qquad \text{if} \quad r + s \leqslant n,$$

and 0 otherwise, where h_r denotes the r-th complete symmetric polynomial. Let us write $\widehat{s}_{\lambda/\mu}(u)$ for the corresponding Schur functions. Then (8.2) and (9.1) in [12] give

$$\widehat{s}_{\lambda/\mu}(u) = \sum_T \prod_{\alpha\in\lambda} u_{T(\alpha)},$$

summed over semistandard tableaux T of shape λ/μ, such that the entries of the i-th row do not exceed $n - \lambda_i + i$. Furthermore, using (6.18)[1] and (9.6′) in [12] we get

$$s_\lambda(x|u) = \sum_{\mu\subseteq\lambda} s_\mu(x)\, \widehat{s}_{\lambda'/\mu'}(-u). \tag{2.8}$$

Equivalently, this can be interpreted as a combinatorial expression for the polynomials $s_\lambda(x|u)$ in terms of "supertableaux". Identify the indices of u with the symbols $1', 2', \dots$. A *supertableau* T is obtained by filling in the diagram of λ with the indices

[1]This formula in [12] should be corrected by replacing $a^{(\lambda_j+n-j)}$ with $a^{(\lambda_i+n-i)}$.

$1, \dots, n, 1', 2', \dots$ in such a way that in each row (resp. column) each primed index is to the right (resp. below) of each unprimed index; unprimed indices weakly increase along the rows and strictly increase down the columns; primed indices strictly increase along the rows and weakly increase down the columns; primed indices in column j do not exceed $n - \lambda'_j + j$. Relation (2.8) implies the following.

Proposition 2.12. *We have*

$$s_\lambda(x|u) = \sum_T \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{unprimed}}} x_{T(\alpha)} \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{primed}}} (-u_{T(\alpha)}), \tag{2.9}$$

summed over all λ-supertableaux T. □

Using (1.8), we get an analogous representation for the double Schur polynomials $s_\lambda(x\,\|\,a)$. A *reverse supertableau* T is obtained by filling in the diagram of λ with the indices $1, \dots, n, n', (n-1)', \dots$ (including non-positive primed indices) in such a way that in each row (resp. column) each primed index is to the right (resp. below) of each unprimed index; unprimed indices weakly decrease along the rows and strictly decrease down the columns; primed indices strictly decrease along the rows and weakly decrease down the columns; primed indices in column j are not less than $\lambda'_j - j + 1$. The following supertableau representation of the polynomials $s_\lambda(x\,\|\,a)$ follows from Proposition 2.12.

Corollary 2.13. *We have*

$$s_\lambda(x\,\|\,a) = \sum_T \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{unprimed}}} x_{T(\alpha)} \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{primed}}} (-a_{T(\alpha)}), \tag{2.10}$$

summed over all reverse λ-supertableaux T. □

Example 2.14. Let $n = 2$ and $\lambda = (2,1)$. By the definition (1.7),

$$s_{(2,1)}(x\,\|\,a) = (x_2 - a_2)(x_1 - a_0)(x_1 - a_2) + (x_2 - a_2)(x_2 - a_1)(x_1 - a_2).$$

On the other hand, the reverse $(2,1)$-supertableaux are

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 0' \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 1' \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2' \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 0' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 1' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & 2' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 0' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1 & 2' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2' & 0' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2' & 1' \\ \hline 2' \\ \cline{1-1}\end{array}$$

which yield

$$
\begin{aligned}
s_{(2,1)}(x\,\|\,a) = {} & x_1^2x_2 + x_1x_2^2 - x_1x_2a_2 - x_2^2a_2 - x_1^2a_2 - x_1x_2a_0 - x_1x_2a_1 - x_1x_2a_2 \\
& + x_2a_0a_2 + x_2a_1a_2 + x_2a_2^2 + x_1a_0a_2 + x_1a_1a_2 + x_1a_2^2 - a_0a_2^2 - a_1a_2^2.
\end{aligned}
$$

Formula (2.8) implies a supertableau representation of the coefficients $c_{\lambda\mu}^{\nu}(a,b)$ and hence, of the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$. The representation for the latter is neither manifestly positive, nor stable; it provides an expression for $c_{\lambda\mu}^{\nu}(a)$ as an alternating sum of monomials in the a_i. Given a sequence R of the form (2.1), construct the set $\mathcal{S}(\lambda,R)$ of barred reverse λ-supertableaux by analogy with $\mathcal{T}(\lambda,R)$. A tableau $T\in\mathcal{S}(\lambda,R)$ must contain boxes $\alpha_1,\dots,\alpha_l$ occupied by unprimed indices $r_1,r_2,\dots,r_l$ listed in the column order which is restricted to the subtableau of T formed by the unprimed indices. As before, we distinguish the entries in $\alpha_1,\dots,\alpha_l$ by barring each of them. For each box α with $\alpha_i\prec\alpha\prec\alpha_{i+1}$, $0\leqslant i\leqslant l$, which is occupied by an unprimed index, set $\rho(\alpha)=\rho^{(i)}$.

Corollary 2.15. *The coefficients $c_{\lambda\mu}^{\nu}(a,b)$ defined in* (2.3) *can be given by*

$$
c_{\lambda\mu}^{\nu}(a,b) = \sum_R \sum_T \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{unprimed, unbarred}}} a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{primed}}} (-b_{T(\alpha)}), \tag{2.11}
$$

summed over sequences R of the form (2.1) *and reverse supertableaux $T\in\mathcal{S}(\lambda,R)$.*

Proof. Applying formula (2.8) we can reduce the calculation of $c_{\lambda\mu}^{\nu}(a,b)$ to the particular case of the sequence $b=(0)$. Now (2.11) follows from Lemma 2.4. □

Example 2.16. In order to calculate the Littlewood–Richardson polynomial $c_{(2,1)\,(2)}^{(2,1)}(a)$ we may take $n=2$; see Corollary 2.11. The barred reverse supertableaux compatible with the sequence $(2)\to(2,1)$ are

$$
\begin{array}{|c|c|}\hline \bar{2} & 1 \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 1 \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline 2' \\ \cline{1-1}\end{array}
$$

$$
\begin{array}{|c|c|}\hline \bar{2} & 0' \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 1' \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 2' \\ \hline 1 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 0' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 1' \\ \hline 2' \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline \bar{2} & 2' \\ \hline 2' \\ \cline{1-1}\end{array}
$$

so that

$$
\begin{aligned}
c_{(2,1)\,(2)}^{(2,1)}(a) &= a_{-1}^2 + a_{-1}a_1 + a_{-1}a_2 - a_{-1}a_2 - a_1a_2 - a_2^2 \\
&\quad - a_{-1}a_0 - a_{-1}a_1 - a_{-1}a_2 + a_0a_2 + a_1a_2 + a_2^2 \\
&= a_{-1}^2 - a_{-1}a_0 - a_{-1}a_2 + a_0a_2,
\end{aligned}
$$

which agrees with Example 2.2. □

3 Applications

3.1 Equivariant Schubert calculus on the Grassmannian

Let n and N be nonnegative integers with $n \leqslant N$ and let $\mathrm{Gr}(n,N)$ denote the Grassmannian of the n-dimensional vector subspaces of $\mathbb{C}^N$. The torus $T = (\mathbb{C}^*)^N$ acts naturally on $\mathrm{Gr}(n,N)$. The equivariant cohomology ring $H_T^*(\mathrm{Gr}(n,N))$ has a natural basis of the equivariant Schubert classes σ_λ parameterized by all diagrams λ contained in the $n \times m$ rectangle, $m = N - n$; see e.g. [4, 5]. Due to [5, Lecture 8, Proposition 1.1] (see also Mihalcea [14]) the classes σ_λ can be expressed in terms of the Chern roots as follows. Consider the tautological sequence on $\mathrm{Gr}(n,N)$,

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

where E denotes the trivial bundle $\mathbb{C}^N_{\mathrm{Gr}(n,N)}$. Let $t_1, \dots, t_N$ be the Chern roots of E and let $y_1, \dots, y_n$ be the Chern roots of the dual $S^\vee$ of the subbundle S so that for the total Chern classes we have $c(E) = \prod_{i=1}^N (1 + t_i)$ and $c^T(S) = \prod_{i=1}^n (1 - y_i)$. Then

$$\sigma_\lambda = s_\lambda(y|u), \qquad u = (-t_N, \dots, -t_1, 0, \dots).$$

Hence, Theorem 2.1 yields a multiplication rule for the equivariant Schubert classes. The corresponding stability property is implied by Corollary 2.11.

Corollary 3.1. *We have*

$$\sigma_\lambda \, \sigma_\mu = \sum_\nu d_{\lambda\mu}^\nu \, \sigma_\nu,$$

where $d_{\lambda\mu}^\nu = c_{\lambda\mu}^\nu(a)$ with the sequence a specialized by

$$a_n = -t_N, \quad a_{n-1} = -t_{N-1}, \quad \dots, \quad a_{n-N+1} = -t_1$$

while the remaining a_i are zero. The $d_{\lambda\mu}^\nu$ are polynomials in the $t_i - t_j$, $i > j$ with positive integer coefficients. Moreover, if λ and μ are fixed then for all ν the coefficients $d_{\lambda\mu}^\nu$, regarded as polynomials in the a_i, are independent of n and m, as soon as the inequalities $n \geqslant \lambda_1' + \mu_1'$ and $m \geqslant \lambda_1 + \mu_1$ hold. □

Example 3.2. For any $n \geqslant 3$ and $m \geqslant 4$ we have

$$\begin{aligned}\sigma_{(2)}\, \sigma_{(2,1)} = {} & \sigma_{(4,1)} + \sigma_{(3,2)} + \sigma_{(3,1,1)} + \sigma_{(2,2,1)} \\ & + (t_{m+2} - t_{m-1} + t_m - t_{m-2})\, \sigma_{(3,1)} + (t_{m+2} - t_{m-1})\, \sigma_{(2,2)} \\ & + (t_m - t_{m-1})\, \sigma_{(2,1,1)} + (t_{m+2} - t_{m-1})\,(t_m - t_{m-1})\, \sigma_{(2,1)}.\end{aligned}$$

This follows from Example 2.2. □

The first manifestly positive rule for the expansion of $\sigma_\lambda \, \sigma_\mu$ was given by Knutson and Tao [9] by using combinatorics of puzzles. As pointed out by Fulton [6], the stability property can be deduced from the puzzle rule as well. Corollary 3.1 provides an alternative manifestly positive and stable rule for this expansion. It would be interesting to construct a weight-preserving bijection between the puzzles and the barred tableaux.

3.2 Quantum immanants and higher Capelli operators

Let $\mathfrak{gl}_n$ denote the general linear Lie algebra over $\mathbb{C}$. Consider the center $\mathrm{Z}(\mathfrak{gl}_n)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. The algebra $\mathrm{U}(\mathfrak{gl}_n)$ is equipped with the natural filtration. For all n we identify $\mathfrak{gl}_{n-1}$ as a subalgebra of $\mathfrak{gl}_n$ in a usual way and denote by $\mathfrak{gl}_\infty$ the corresponding inductive limit

$$\mathfrak{gl}_\infty = \bigcup_n \mathfrak{gl}_n.$$

Due to Olshanski [18], there exist filtration-preserving homomorphisms

$$o_n : \mathrm{Z}(\mathfrak{gl}_n) \to \mathrm{Z}(\mathfrak{gl}_{n-1}), \qquad n \geqslant 1, \tag{3.1}$$

which allow one to define the algebra Z of the *virtual Casimir elements* for the Lie algebra $\mathfrak{gl}_\infty$ as the inverse limit

$$\mathrm{Z} = \lim_{\longleftarrow} \mathrm{Z}(\mathfrak{gl}_n), \qquad n \to \infty,$$

in the category of filtered algebras.

The *quantum immanants* $\mathbb{S}_{\lambda|n}$ are elements of the center $\mathrm{Z}(\mathfrak{gl}_n)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ parameterized by the diagrams λ with at most n rows; see [16]. The elements $\mathbb{S}_{\lambda|n}$ form a basis of $\mathrm{Z}(\mathfrak{gl}_n)$ and they are consistent with the Olshanski homomorphisms (3.1) so that

$$o_n : \mathbb{S}_{\lambda|n} \mapsto \mathbb{S}_{\lambda|n-1}, \tag{3.2}$$

where we assume $\mathbb{S}_{\lambda|n} = 0$ if the number of rows of λ exceeds n. For any diagram λ, the corresponding *virtual quantum immanant* $\mathbb{S}_\lambda$ is then defined as the sequence

$$\mathbb{S}_\lambda = (\, \mathbb{S}_{\lambda|n} \,|\, n \geqslant 0).$$

The elements $\mathbb{S}_\lambda$ parameterized by all diagrams λ form a basis of the algebra Z so that we can define the coefficients $f_{\lambda\mu}^{\nu}$ by the expansion

$$\mathbb{S}_\lambda \, \mathbb{S}_\mu = \sum_\nu f_{\lambda\mu}^{\nu} \, \mathbb{S}_\nu.$$

Note that the same coefficients $f_{\lambda\mu}^{\nu}$ determine the multiplication rule for the *higher Capelli operators* Δ_λ, which are defined as the sequences of the images of the quantum immanants $\mathbb{S}_{\lambda|n}$, where each image is taken under a natural representation of $\mathfrak{gl}_n$ by differential operators; see [16, 17].

Corollary 3.3. *The coefficient $f_{\lambda\mu}^{\nu}$ is zero unless $\mu \subseteq \nu$. If $\mu \subseteq \nu$ then*

$$f_{\lambda\mu}^{\nu} = \sum_{R} \sum_{T} \prod_{\substack{\alpha\in\lambda \\ T(\alpha) \text{ unbarred}}} \Big(\rho(\alpha)_{T(\alpha)} - c(\alpha)\Big), \tag{3.3}$$

summed over all sequences R of the form (2.1) *and all ν-admissible reverse λ-tableaux $T \in \mathcal{T}(\lambda, R)$. In particular, the $f_{\lambda\mu}^{\nu}$ are nonnegative integers.*

Proof. Due to the stability property (3.2) of the quantum immanants, it suffices to calculate the corresponding coefficients for the expansion of the products $\mathbb{S}_{\lambda|n}\, \mathbb{S}_{\mu|n}$. The images of the quantum immanants $\mathbb{S}_{\lambda|n}$ under the Harish-Chandra isomorphism can be identified with the double Schur polynomials $s_\lambda(x \,\|\, a)$ where the sequence a is specialized to $a_i = n - i$; see [16]. Therefore, the coefficients in question coincide with the corresponding specializations of the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$. It remains to apply Corollary 2.11 and observe that the polynomials $c_{\lambda\mu}^{\nu}(a)$ are invariant with respect to the simultaneous shifts $a_i \mapsto a_i + h$ for any fixed h. □

Example 3.4. Using Example 2.2 we get

$$\mathbb{S}_{(2)}\, \mathbb{S}_{(2,1)} = \mathbb{S}_{(4,1)} + \mathbb{S}_{(3,2)} + \mathbb{S}_{(3,1,1)} + \mathbb{S}_{(2,2,1)} + 5\ \mathbb{S}_{(3,1)} + 3\ \mathbb{S}_{(2,2)} + \mathbb{S}_{(2,1,1)} + 3\ \mathbb{S}_{(2,1)}.$$

In the course of the proof of Corollary 3.3 we also calculated the coefficients for the expansion of the products $\mathbb{S}_{\lambda|n}\, \mathbb{S}_{\mu|n}$ for any n. Some other formulas for these coefficients were obtained in [15]. In particular, it was shown that the $f_{\lambda\mu}^{\nu}$ are integers, although their positivity property was not established there.

Note also that the algebra of virtual Casimir elements Z is isomorphic to the algebra of *shifted symmetric functions* Λ^*; see [17]. Corollary 3.3 shows that the multiplication in this algebra is described by the specialization of the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ at $a_i = -i$ for all $i \in \mathbb{Z}$. However, the corresponding Schur functions $s_\lambda(x \,\|\, a)$ are not defined under this specialization. Informally speaking, the shifted Schur functions $s^*_\lambda(y)$ of [17] can still be identified with the $s_\lambda(x \,\|\, a)$ under the "change of variables" $y_i = x_i + i$ for all $i = 1, 2, \dots$. Consequently, the algebra Λ^* may be viewed as a "virtual" specialization of the algebra Λ at $a_i = -i$ for all $i \in \mathbb{Z}$.

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