ON THE ASYMPTOTIC BEHAVIOUR OF THE EIGENVALUES OF A ROBIN PROBLEM

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Abstract. We prove that every eigenvalue of a Robin problem with boundary parameter α on a sufficiently smooth domain behaves asymptotically like $-\alpha^2$ as $\alpha \to \infty$. This generalises an existing result for the first eigenvalue.

1. Introduction and Main Results

We are interested in the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \alpha u && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where we assume $\Omega \subset \mathbb{R}^N$ is a bounded domain, that is, a bounded open set, without loss of generality connected, and $\alpha > 0$. The problem (1.1) is usually referred to as a Robin problem (in comparison with the case $\alpha < 0$) or sometimes as a generalised Neumann problem. This problem has received considerable attention in the last few years; see for example [1, 4, 5, 6, 8, 9, 10] and the references therein. It is well-known that if Ω is Lipschitz then there is a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \ldots \to \infty$, which we repeat according to their multiplicities, where $\lambda_1 < 0$ is simple and is the unique eigenvalue with a positive eigenfunction ψ_1. Our main result is as follows.

Theorem 1.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^1. Then for every $n \geq 1$ we have*

$$\lim_{\alpha\to\infty} \frac{\lambda_n(\alpha)}{-\alpha^2} = 1. \tag{1.2}$$

It was shown in [8] that for Ω piecewise-C^1 the first eigenvalue λ_1 has the asymptotic behaviour $\liminf_{\alpha\to\infty} -\lambda_1(\alpha)/\alpha^2 \geq 1$, with equality if

Mathematics Subject Classification (2000). 35P15 (35B40, 35J05).

Key words and phrases. Laplacian, Robin boundary conditions, eigenvalue asymptotics.

$\partial\Omega$ is equivalent in some sense to a sphere. It was also observed in [8] that when Ω is a ball of radius 1, there are $\lfloor \alpha \rfloor + 1$ negative eigenvalues of (1.1), and they satisfy $\sqrt{-\lambda_n(\alpha)} \sim \alpha + O(1)$ as $\alpha \to \infty$. It was subsequently shown in [10] that in fact

$$\lim_{\alpha\to\infty} \frac{\lambda_1(\alpha)}{-\alpha^2} = 1 \tag{1.3}$$

for every bounded and C^1 domain Ω. Related results have been obtained in [5, 6]. The C^1 assumption in (1.3) is optimal: the authors in [8] constructed examples of domains with "corners" for which the limit in (1.3) is a constant larger than one. Such results were generalised and further studied in [9].

Remark 1.2. One can also consider the same problem with the boundary condition $\frac{\partial u}{\partial \nu} = \alpha b u$, where $b \in C(\partial\Omega)$ is a weight function which is positive somewhere. In this case, if Ω is bounded and C^1, then

$$\lim_{\alpha\to\infty} \frac{\lambda_1(\alpha)}{-\alpha^2 (\max_{\partial\Omega} b)^2} = 1$$

(see [10, Remark 1.1]). It seems the same should be true for λ_n, $n \geq 1$. However all we can say at present is that Theorem 1.1 together with the monotonic behaviour of λ_n with respect to changes in b imply that

$$\limsup_{\alpha\to\infty} \frac{\lambda_n(\alpha)}{-\alpha^2 (\max_{\partial\Omega} b)^2} \leq 1.$$

We will also prove the following result on the eigenfunctions of (1.1).

Proposition 1.3. *Suppose $\Omega \subset \mathbb{R}^N$ is bounded and C^1. Fix $2 \leq p < \infty$ and let ψ_n be any eigenfunction associated with λ_n, normalised so that $\|\psi_n\|_{L^p(\Omega)} = 1$. Then*
- (i) *$\psi_n \to 0$ in $L^p_{loc}(\Omega)$ as $\alpha \to \infty$;*
- (ii) *$\|\psi_n\|_{L^q(\Omega)} \to 0$ as $\alpha \to \infty$ for $1 \leq q < p$;*
- (iii) *$\|\psi_n\|_{L^r(\Omega)} \to \infty$ as $\alpha \to \infty$ for $r > p$.*

We will prove Theorem 1.1 in the next section and defer the proof of Proposition 1.3 until Section 3. We will use the result of Theorem 1.1 to obtain Proposition 1.3; however, the former is only needed to show that $\lambda_n(\alpha) \to -\infty$ as $\alpha \to \infty$. Proposition 1.3 is valid for Lipschitz domains whenever we have this more general asymptotic behaviour.

2. Proof of Theorem 1.1

We first discuss the theory related to (1.1) that will be needed to prove Theorem 1.1. The form associated with (1.1) is given by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \alpha u v \, dx,$$

where $u, v \in H^1(\Omega)$. We understand eigenvalues λ and associated eigenfunctions ψ of (1.1) in the weak sense, as satisfying $a(\psi, v) =$

$\lambda\langle\psi, v\rangle$ for all $v \in H^1(\Omega)$. Here and throughout $\langle\, . \, , \, . \,\rangle$ denotes the usual inner product on $L^2(\Omega)$. The eigenfunctions $\psi_1, \psi_2, \dots$ can be chosen orthogonal in $L^2(\Omega)$. To see this, note first that if $\lambda_i \neq \lambda_j$ for some $i, j \geq 1$, then $a(\psi_i, \psi_j) = \lambda_i\langle\psi_i, \psi_j\rangle = \lambda_j\langle\psi_i, \psi_j\rangle$ implies $\langle\psi_i, \psi_j\rangle = 0$. If instead λ_n is a repeated eigenvalue, we may apply the Gram-Schmidt process to its eigenfunctions. We also impose the scaling $\|\psi_n\|_{L^2(\Omega)} = 1$ in this section. With the eigenvalues ordered by increasing size and repeated according to their multiplicities, the nth eigenvalue may be characterised variationally as

$$\lambda_n(\alpha) = \inf_{0 \neq v \in M_n} \frac{a(v,v)}{\|v\|^2_{L^2(\Omega)}}, \tag{2.1}$$

where M_n is the subspace of $H^1(\Omega)$ of codimension $n-1$ obtained by taking the orthogonal complement of the L^2-span of the first $n-1$ eigenfunctions $\psi_1, \dots, \psi_{n-1}$ (see [3, Section VI.1]). If we set $v_n := v - \sum_{i=1}^{n-1}\langle v, \psi_i\rangle\psi_i$, then $v_n \in M_n$ and so provided $v_n \neq 0$, that is, provided v is not in the L^2-span of $\psi_1, \dots, \psi_{n-1}$, we may use v_n as a test function in (2.1) to estimate λ_n from above.

We will use this representation, together with an appropriate choice of v and an induction argument on n, to prove Theorem 1.1. Our choice of test function is due to an argument in [5, Theorem 2.3], though also cf. [9, Example 2.4]. We will assume throughout that $\Omega \subset \mathbb{R}^N$ is bounded and C^1, although some of the results, including the next lemma, are valid for Lipschitz domains with the same proof.

Lemma 2.1. *Let $d \in \mathbb{R}^N$, $\|d\| = 1$ be any unit vector. Set $u_d(x, \alpha) := ce^{\alpha x \cdot d} \in C^\infty(\mathbb{R}^N) \cap H^1(\Omega)$, where $c = c(d, \alpha)$ is a constant chosen so that $\|u_d\|_{L^2(\Omega)} = 1$. Then $a(u_d, u_d) \leq -\alpha^2$ for all $\alpha > 0$.*

Proof. For $x \in \mathbb{R}^N$ writing $x = (x_1, \dots, x_N)$, we may without loss of generality rotate our coordinate system if necessary so that $d = (0, \dots, 0, 1)$. In this case $u_d = ce^{\alpha x_N}$ and $\nabla u_d = (0, \dots, 0, c\alpha e^{\alpha x_N})$. Hence

$$a(u_d, u_d) = c^2\alpha^2 \int_\Omega e^{2\alpha x_N}\, dx - c^2\alpha \int_{\partial\Omega} e^{2\alpha x_N}\, d\sigma.$$

We will now use the divergence theorem on $V := (0, \dots, 0, e^{2\alpha x_N}) \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and the domain Ω (see for example [11, Théorème 3.1.1]). Denoting the outer unit normal to Ω by $\nu_\Omega(x) = (\nu_1(x), \dots, \nu_N(x))$, $x \in \partial\Omega$, we have

$$\begin{aligned}\int_{\partial\Omega} e^{2\alpha x_N}\, d\sigma &\geq \int_{\partial\Omega} e^{2\alpha x_N}\nu_N\, d\sigma = \int_{\partial\Omega} V \cdot \nu_\Omega\, d\sigma \\ &= \int_\Omega \operatorname{div} V\, dx = 2\alpha \int_\Omega e^{2\alpha x_N}\, dx.\end{aligned}$$

Multiplying through by $\alpha > 0$ and combining this with the expression for $a(u_d, u_d)$ yields

$$a(u_d, u_d) \leq -\alpha^2 c^2 \int_\Omega e^{2\alpha x_N}\, dx = -\alpha^2,$$

where the last equality follows from the definition of c. $\square$

Remark 2.2. An easy calculation shows that the function $u(x) := e^{\alpha x_N}$ is a positive eigenfunction, with eigenvalue $-\alpha^2$, of (1.1) on the half-space $T = \{x \in \mathbb{R}^N : x_N < 0\}$.

For $d \in \mathbb{R}^N$ a fixed unit vector and $n \geq 1$ also fixed, set $u_{n+1} := u_d - \sum_{i=1}^n \langle u_d, \psi_i \rangle \psi_i \in M_{n+1}$. We will use u_{n+1} as a test function in the variational characterisation in order to establish (1.2). To that end, we estimate λ_{n+1} in terms of the previous n eigenvalues and functions.

Lemma 2.3. *Suppose* $u_d \notin \mathit{span}\{\psi_1, \dots, \psi_n\}$. *Then*

$$\lambda_{n+1}(\alpha) \leq \frac{-\alpha^2 - \sum_{i=1}^n \lambda_i \langle u_d, \psi_i \rangle^2}{1 - \sum_{i=1}^n \langle u_d, \psi_i \rangle^2}. \tag{2.2}$$

Proof. Since u_d is not a linear combination of the first n eigenfunctions, we can use $u_{n+1} = u_d - \sum_{i=1}^n \langle u_d, \psi_i \rangle \psi_i \not\equiv 0$ as a test function in (2.1). A simple calculation using the orthonormality of the eigenfunctions shows that

$$0 < \langle u_{n+1}, u_{n+1} \rangle = 1 - \sum_{i=1}^n \langle u_d, \psi_i \rangle^2.$$

We now estimate $a(u_{n+1}, u_{n+1})$. Using the definition of u_{n+1} and the bilinearity of the form a, we see that $a(u_{n+1}, u_{n+1})$ is given by

$$a(u_d, u_d) - 2 \sum_{i=1}^n \langle u_d, \psi_i \rangle\, a(u_d, \psi_i) + \sum_{i=1}^n \sum_{j=1}^n \langle u_d, \psi_i \rangle^2 a(\psi_i, \psi_j).$$

Since $a(u_d, \psi_i) = \lambda_i \langle u_d, \psi_i \rangle$, and since $a(\psi_i, \psi_j) = \lambda_i$ if $i = j$ and 0 otherwise, we obtain

$$a(u_{n+1}, u_{n+1}) = a(u_d, u_d) - \sum_{i=1}^n \lambda_i \langle u_d, \psi_i \rangle^2.$$

(Cf. the abstract theory in [7, Section I.6.10].) Using the estimate of $a(u_d, u_d)$ from Lemma 2.1 and putting everything together yields

$$\lambda_{n+1}(\alpha) \leq \frac{a(u_{n+1}, u_{n+1})}{\|u_{n+1}\|^2_{L^2(\Omega)}} \leq \frac{-\alpha^2 - \sum_{i=1}^n \lambda_i \langle u_d, \psi_i \rangle^2}{1 - \sum_{i=1}^n \langle u_d, \psi_i \rangle^2},$$

establishing (2.2). $\square$

Roughly speaking, to prove Theorem 1.1 using the estimate of λ_{n+1} in Lemma 2.3 we have to prove that we can find a direction d such that $\langle u_d, \psi_i \rangle$ stays small as $\alpha \to \infty$ for all $1 \leq i \leq n$. To that end we will study the functions u_d more carefully. We start by observing that, for any given $\alpha > 0$, the upper level sets of u_d are restrictions to Ω of half-planes of the form $\{x \in \mathbb{R}^N : x \cdot d > \kappa\}$, where $\kappa \in \mathbb{R}$. The key place where we will use the assumption that Ω has C^1 boundary is in parts (iii) and (iv) of the next lemma.

Lemma 2.4. *Let $d \in \mathbb{R}^N$, $\|d\| = 1$. For $\kappa \in \mathbb{R}$ set*

$$\begin{aligned} U_d(\kappa) &:= \{x \in \Omega : x \cdot d > \kappa\}, \\ \kappa_d &:= \sup\{\kappa \in \mathbb{R} : U_d(\kappa) \neq \emptyset\}, \\ K_d &:= \{x \in \overline{\Omega} : x \cdot d = \kappa_d\}. \end{aligned} \tag{2.3}$$

Then

(i) *the $U_d(\kappa)$ are open, nested (i.e. $U_d(\kappa) \subset U_d(\kappa')$ if $\kappa > \kappa'$), nonempty if and only if $\kappa < \kappa_d$, and $0 \neq |U_d(\kappa)| \to 0$ as $\kappa \to \kappa_d$ from below;*

(ii) *$\emptyset \neq K_d \subset \partial\Omega$;*

(iii) *if $z \in K_d$, then $d = \nu_\Omega(z)$, the outer unit normal to Ω at z;*

(iv) *if $d \neq e \in \mathbb{R}^N$, $\|e\| = 1$ is another unit vector with $U_e(\kappa)$ and κ_e defined as in* (2.3)*, then there exists $\varepsilon > 0$ such that $U_d(\kappa) \cap U_e(\tilde{\kappa}) = \emptyset$ for all $\kappa \in (\kappa_d - \varepsilon, \kappa_d)$ and all $\tilde{\kappa} \in (\kappa_e - \varepsilon, \kappa_e)$.*

Proof. (i) is obvious. For (ii), to show $K_d \neq \emptyset$ we note that $K_d = \cap_{\kappa < \kappa_d} \overline{U_d(\kappa)}$, that is, K_d is the intersection of nested, compact and nonempty sets. That $K_d \subset \partial\Omega$ is immediate from the definitions and the fact that the U_d are open. For (iii), we assume as in the proof of Lemma 2.1 that $d = (0, \ldots, 0, 1)$, so that $U_d(\kappa) = \{x \in \Omega : x_N > \kappa\}$. Then $z = (z_1, \ldots, z_N) \in K_d$ means $z_N = \kappa_d$, that is, $z_N = \max\{x_N : x \in \overline{\Omega}\}$. Since Ω is C^1, this means the tangent plane to Ω at $z \in K_d$ must be horizontal. Thus $\nu_\Omega(z)$ points in the direction x_N, that is, $\nu_\Omega(z) = (0, \ldots, 0, 1)$. For (iv), suppose for a contradiction that there exist $\kappa_j \nearrow \kappa_d$ and $\tilde{\kappa}_j \nearrow \kappa_e$ such that, for each $j \geq 1$, there exists $x_j \in U_d(\kappa_j) \cap U_e(\tilde{\kappa}_j)$. Since $\overline{\Omega}$ is compact, a subsequence of the x_j converges to some $z \in \overline{\Omega}$. Since $x_j \in U_d(\kappa_j)$ and $\cap_{j \geq 1} \overline{U_d(\kappa_j)} = K_d$, we see $z \in K_d$. By a similar argument, $z \in K_e$. This contradicts (iii) since $d \neq e$. ☐

We now show that for d fixed, all the mass of u_d becomes concentrated in an arbitrarily small region of Ω as $\alpha \to \infty$.

Lemma 2.5. *Let $d \in \mathbb{R}^N$ and $u_d(x) = ce^{\alpha x \cdot d}$ be as in Lemma 2.1 and let $U_d(\kappa)$ and κ_d be as in Lemma 2.4. For every $\varepsilon > 0$ and $\kappa' < \kappa_d$ there exists $\alpha_\varepsilon := \alpha(\varepsilon, \kappa') > 0$ such that*

$$\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} < \varepsilon \tag{2.4}$$

for all $\alpha > \alpha_\varepsilon$.

Proof. Since $u_d(x) \leq ce^{\alpha\kappa'}$ for all $x \in \Omega \setminus U_d(\kappa')$, we have

$$\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} \leq ce^{2\alpha\kappa'} |\Omega|.$$

Choose $\kappa'' \in (\kappa', \kappa_d)$. Then $U_d(\kappa'') \subset U_d(\kappa')$ with $|U_d(\kappa'')| \neq 0$ and

$$1 = \|u_d\|^2_{L^2(\Omega)} \geq \|u_d\|^2_{L^2(U_d(\kappa''))} \geq ce^{2\alpha\kappa''}|U_d(\kappa'')|.$$

For $\varepsilon > 0$ fixed, choose $\alpha_\varepsilon > 0$ such that

$$e^{2\alpha_\varepsilon\kappa'}|\Omega| < \varepsilon e^{2\alpha_\varepsilon\kappa''}|U_d(\kappa'')|, \tag{2.5}$$

which we can do since $\kappa' < \kappa''$. Then (2.5) will hold uniformly in $\alpha > \alpha_\varepsilon$ and so

$$\|u_d\|^2_{L^2(\Omega\setminus U_d(\kappa'))} < ce^{2\alpha\kappa'}|\Omega| < \varepsilon ce^{2\alpha\kappa''}|U_d(\kappa'')| < \varepsilon$$

for all $\alpha > \alpha_\varepsilon$. □

Lemma 2.5 implies that for fixed d, $u_d \rightharpoonup 0$ weakly in $L^2(\Omega)$ as $\alpha \to \infty$; it turns out that the same is true of the ψ_i (see Proposition 1.3). But this is not enough to show directly that $\langle u_d, \psi_i\rangle$ is uniformly small, since both u_d and ψ_i vary with α. Instead, we will use the following rather technical result concerning the u_d. Since this does not use any specific properties of the ψ_i, we set it up so it works for arbitrary L^2-functions.

Lemma 2.6. *Fix $n \geq 1$ and $\delta > 0$. Suppose we have a sequence $\alpha_k \to \infty$ and for each $k \in \mathbb{N}$ a family of n functions $\varphi_i(k) \in L^2(\Omega)$, $1 \leq i \leq n$, such that $\|\varphi_i(k)\|_{L^2(\Omega)} = 1$ for all $1 \leq i \leq n$ and $k \in \mathbb{N}$. Then there exists a unit vector $d \in \mathbb{R}^N$ and a subsequence $\alpha_{k_l} \to \infty$ of the (α_k) such that*

$$\sum_{i=1}^{n} \langle u_d(k_l), \varphi_i(k_l)\rangle^2 \leq \delta, \tag{2.6}$$

for all $l \in \mathbb{N}$, where $u_d(k_l) = u_d(x, \alpha_{k_l})$ is as in Lemma 2.1.

Proof. Fix $n \geq 1$, $\delta > 0$ and a sequence $\alpha_k \to \infty$. Choose $m \geq 1$ and $\varepsilon > 0$, to be specified precisely later on. Now choose any m distinct unit vectors $d_j \in \mathbb{R}^N$, $1 \leq j \leq m$, and for each j let $u_j := u_{d_j}(x, \alpha_k)$ be as in Lemma 2.1. For each j choose a nonempty open set $U_j := U_{d_j}(\kappa_j)$ as in Lemma 2.4. By making an appropriate choice of κ_j we may assume the U_j are pairwise disjoint. Using Lemma 2.5, we find an $\alpha_\varepsilon > 0$ such that

$$\|u_j\|^2_{L^2(\Omega\setminus U_j)} < \varepsilon$$

for all $\alpha > \alpha_\varepsilon$ and all $1 \leq j \leq m$. By discarding at most finitely many k, we may assume $\alpha_k > \alpha_\varepsilon$ for all $k \in \mathbb{N}$. Now for each $k \in \mathbb{N}$, we have

$$\int_\Omega \sum_{i=1}^{n} |\varphi_i(k)|^2\,dx = \sum_{i=1}^{n} \|\varphi_i(k)\|^2_{L^2(\Omega)} = n.$$

Since the U_j are disjoint, it follows that for each $k \in \mathbb{N}$, there exists at least one $j = j_k$ such that

$$\int_{U_{j_k}} \sum_{i=1}^{n} |\varphi_i(k)|^2\,dx \leq \frac{n}{m}.$$

For this j_k, using Hölder's inequality, for each $1 \le i \le n$ we have

$$\begin{aligned}|\langle u_{j_k}, \varphi_i(k)\rangle| &\le \int_{U_{j_k}} |u_j\varphi_i|\,dx + \int_{\Omega\setminus U_{j_k}} |u_j\varphi_i|\,dx \\ &\le \|u_j\|_{L^2(\Omega)}\Big(\frac{n}{m}\Big)^{\frac12} + \varepsilon^{\frac12}\|u_j\|_{L^2(\Omega)}\|\varphi_i\|_{L^2(\Omega)} \\ &= \Big(\frac{n}{m}\Big)^{\frac12} + \varepsilon^{\frac12},\end{aligned}$$

where we have used the bound $\int_{U_j} |\varphi_i|^2\,dx \le n/m$. We now specify $m \ge 1$ and $\varepsilon > 0$ to be such that

$$n\Big(\Big(\frac{n}{m}\Big)^{\frac12} + \varepsilon^{\frac12}\Big)^2 \le \delta,$$

noting that this depends only on n and δ. Squaring the above estimate for $|\langle u_{j_k}, \varphi_i(k)\rangle|$ and summing over i, this implies that for all but finitely many $k \in \mathbb{N}$, (2.6) holds for at least one of the m fixed u_j.

By a simple counting argument, there must exist at least one j^* between 1 and m such that (2.6) holds for this fixed u_{j^*} and infinitely many α_k. This gives us our u_d and (α_{k_l}). □

Proof of Theorem 1.1. The proof is by induction on n. The step when $n = 1$ is given by [10, Theorem 1.1]. Now fix $n \ge 1$ and suppose we know that for all $1 \le i \le n$, $-\lambda_i(\alpha_k)/\alpha_k^2 \to 1$ as $k \to \infty$ for every sequence $\alpha_k \to \infty$. It suffices to prove that for every such sequence $\alpha_k \to \infty$, there exists a subsequence $\alpha_{k_l} \to \infty$ such that $-\lambda_{n+1}(\alpha_{k_l})/\alpha_{k_l}^2 \to 1$ as $l \to \infty$.

So fix a particular sequence $\alpha_k \to \infty$ and also fix $0 < \delta < 1$. Let u_d satisfy the conclusion of Lemma 2.6 for a subsequence which we will still denote by (α_k), this $\delta > 0$ and the family of n functions $\psi_i(\alpha_k) =: \varphi_i(k)$, $1 \le i \le n$. Then by Lemma 2.6 we know that

$$\sum_{i=1}^{n} \langle u_d(\alpha_k), \psi_i(\alpha_k)\rangle^2 \le \delta \tag{2.7}$$

for all $k \in \mathbb{N}$ and the fixed direction d. In particular, (2.7) implies $u_d \notin \operatorname{span}\{\psi_1(\alpha_k), \dots, \psi_n(\alpha_k)\}$ for any $k \in \mathbb{N}$, since $\delta < 1$. Applying Lemma 2.3 to u_d for each $k \in \mathbb{N}$, we obtain

$$\lambda_{n+1}(\alpha_k) \le \frac{-\alpha_k^2 - \sum_{i=1}^n \lambda_i \langle u_d, \psi_i\rangle^2}{1 - \sum_{i=1}^n \langle u_d, \psi_i\rangle^2}$$

for all $k \in \mathbb{N}$. This implies

$$\frac{\lambda_1(\alpha_k)}{-\alpha_k^2} \ge \frac{\lambda_{n+1}(\alpha_k)}{-\alpha_k^2} \ge \frac{1 - \sum_{i=1}^n \frac{\lambda_i(\alpha_k)}{-\alpha_k^2}\langle u_d, \psi_i\rangle^2}{1 - \sum_{i=1}^n \langle u_d, \psi_i\rangle^2}. \tag{2.8}$$

Using the bound (2.7), which holds independently of $k \in \mathbb{N}$, together with the induction assumption $-\lambda_i(\alpha_k^2)/\alpha_k^2 \to 1$ as $k \to \infty$ for all $i \le n$ it follows that the term on the right in (2.8) converges to 1 as $k \to \infty$.

This establishes the desired limit for $-\lambda_{n+1}(\alpha_k)/\alpha_k^2$, which completes the proof. $\square$

3. Proof of Proposition 1.3

Fix $n \geq 1$ and $p \geq 2$. We first obtain the following interior estimate for ψ_n, from which the proof of the proposition will follow easily.

Lemma 3.1. *Under the assumptions of Proposition 1.3, if $\varphi \in C_c^\infty(\Omega)$, then*

$$\lambda_n \geq -(p-1)^{-1} \frac{\int_\Omega |\psi_n|^p \, |\nabla\varphi|^2 \, dx}{\int_\Omega |\psi_n|^p \, \varphi^2 \, dx}$$

for all $\alpha > 0$ and all $n \geq 1$.

Proof. Given $\varphi \in C_c^\infty(\Omega)$, we will use $\phi := \varphi^2 |\psi_n|^{p-2}\psi_n$ as a test function in the weak form of (1.1) given by

$$\lambda_n \int_\Omega \psi_n v \, dx = a(\psi_n, v) = \int_\Omega \nabla\psi_n \cdot \nabla v \, dx - \int_{\partial\Omega} \alpha\psi_n v \, d\sigma \qquad (3.1)$$

for all $v \in H^1(\Omega)$. We first note that if $p \geq 2$, then since $\psi_n \in C(\overline{\Omega})$ (see [4, Corollary 4.2]) we have $\phi \in H^1(\Omega)$ with $\nabla\phi = 2\varphi|\psi_n|^{p-2}\psi_n\nabla\varphi + (p-1)\varphi^2|\psi_n|^{p-2}\nabla\psi_n$. Moreover $\langle \phi, \psi_n\rangle = \int_\Omega \varphi^2|\psi_n|^p \, dx \neq 0$, since ψ_n cannot vanish identically on an open set (see [2]). Hence, by completing the square,

$$\begin{aligned}
&\int_\Omega \nabla\psi_n \cdot \nabla\phi \, dx \\
&\qquad = \int_\Omega 2\varphi|\psi_n|^{p-2}\psi_n\nabla\varphi \cdot \nabla\psi_n + (p-1)\varphi^2|\psi_n|^{p-2}|\nabla\psi_n|^2 \, dx \\
&\qquad = \int_\Omega \Big|(p-1)^{\frac{1}{2}}|\psi_n|^{\frac{p}{2}-1}\varphi\nabla\psi_n + (p-1)^{-\frac{1}{2}}|\psi_n|^{\frac{p}{2}-1}\psi_n\nabla\varphi\Big|^2 \, dx \\
&\qquad\qquad - \int_\Omega (p-1)^{-1}|\psi_n|^p|\nabla\varphi|^2 \, dx.
\end{aligned}$$

Substituting this into (3.1), and using that $\varphi \equiv 0$ on $\partial\Omega$,

$$\lambda_n \int_\Omega \varphi^2|\psi_n|^p \, dx = \int_\Omega \nabla\psi_n \cdot \nabla\phi \, dx \geq -\int_\Omega (p-1)^{-1}|\psi_n|^p|\nabla\varphi|^2 \, dx.$$

Rearranging gives the conclusion of the lemma. $\square$

To prove the proposition, part (i) uses the result of Theorem 1.1, that $\lambda_n \to -\infty$ as $\alpha \to \infty$; parts (ii) and (iii) follow directly from (i).

Proof of Proposition 1.3. (i) Fix $p \geq 2$, $n \geq 1$ and $\Omega_0 \subset\subset \Omega$ and assume $\|\psi_n\|_{L^p(\Omega)} = 1$. Let $\varphi \in C_c^\infty(\Omega)$ be such that $0 \leq \varphi \leq 1$ in Ω and $\varphi \equiv 1$ in Ω_0. Setting $K := (p-1)^{-1}\|\nabla\varphi\|^2_{L^\infty(\Omega)} > 0$, which depends only on p and Ω_0, by Lemma 3.1,

$$\lambda_n \geq \frac{-K}{\int_{\Omega_0} |\psi_n|^p \, dx}$$

for all $\alpha > 0$. Since $\lambda_n \to -\infty$ as $\alpha \to \infty$ by Theorem 1.1, this forces $\int_{\Omega_0} |\psi_n|^p \, dx \to 0$ as $\alpha \to \infty$.

(ii) Fix $1 \le q < p$ and $\varepsilon > 0$. Choose $\Omega_\varepsilon \subset\subset \Omega$ such that $|\Omega \backslash \Omega_\varepsilon|^{\frac{p-q}{p}} < \varepsilon/2$, which we may do since $p > q$. Also choose $\alpha_\varepsilon > 0$ such that

$$\|\psi_n\|^q_{L^p(\Omega_\varepsilon)} < \frac{\varepsilon}{2} |\Omega_\varepsilon|^{\frac{q-p}{p}}$$

for all $\alpha > \alpha_\varepsilon$, which we may do by (i). Noting that p/q and $p/(p-q)$ are dual exponents, Hölder's inequality implies

$$\begin{aligned}
\|\psi_n\|^q_{L^q(\Omega)} &= \int_{\Omega_\varepsilon} |\psi_n|^q \, dx + \int_{\Omega\backslash\Omega_\varepsilon} |\psi_n|^q \, dx \\
&\le \Big(\int_{\Omega_\varepsilon} |\psi_n|^p \, dx \Big)^{\frac{q}{p}} |\Omega_\varepsilon|^{\frac{p-q}{p}} + \Big(\int_{\Omega\backslash\Omega_\varepsilon} |\psi_n|^p \, dx \Big)^{\frac{q}{p}} |\Omega \setminus \Omega_\varepsilon|^{\frac{p-q}{p}} \\
&= \|\psi_n\|^q_{L^p(\Omega_\varepsilon)} |\Omega_\varepsilon|^{\frac{p-q}{p}} + \|\psi_n\|^q_{L^p(\Omega\backslash\Omega_\varepsilon)} |\Omega \setminus \Omega_\varepsilon|^{\frac{p-q}{p}} < \varepsilon
\end{aligned}$$

for all $\alpha > \alpha_\varepsilon$, by choice of Ω_ε and α_ε, and since $\|\psi_n\|^q_{L^p(\Omega\backslash\Omega_\varepsilon)} \le 1$.

(iii) Fix $r > p$. If we normalise ψ_n so that $\|\psi_n\|_{L^r(\Omega)} = 1$, then (ii) implies $\|\psi_n\|_{L^p(\Omega)} \to 0$, so that

$$\frac{\|\psi_n\|_{L^r(\Omega)}}{\|\psi_n\|_{L^p(\Omega)}} \longrightarrow \infty \tag{3.2}$$

as $\alpha \to \infty$. Now re-normalise so that $\|\psi_n\|_{L^p(\Omega)} = 1$. Since this does not affect (3.2), in this case $\|\psi_n\|_{L^r(\Omega)} \to \infty$. □

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