GRADED REPRESENTATION THEORY OF THE CYCLOTOMIC QUIVER HECKE ALGEBRAS OF TYPE A

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Contents

Introduction

The *quiver Hecke algebras*, or *KLR algebras*, $\mathscr{R}_n(\Gamma)$ are a remarkable family of graded algebras which were introduced independently by Khovanov and Lauda [67, 68] and Rouquier [114], where $n \geq 0$ and Γ is an oriented quiver of Kac-Moody type. The algebras $\mathscr{R}_n(\Gamma)$ are $\mathbb{Z}$-graded and they categorify the negative part of the associated quantum group $U_q(\mathfrak{g}_\Gamma)$ [115, 125]. That is, there are natural isomorphisms

$$U_q(\mathfrak{g}_\Gamma)^- \cong \bigoplus_{n\geq 0}[\mathrm{Rep}(\mathscr{R}_n(\Gamma))],$$

where $[\mathrm{Rep}(\mathscr{R}_n(\Gamma))]$ is the Grothendieck group of the finitely generated graded $\mathscr{R}_n(\Gamma)$-modules. For each dominant weight Λ the quiver algebra $\mathscr{R}_n(\Gamma)$ has a cyclotomic quotient $\mathscr{R}_n^\Lambda(\Gamma)$ which categorifes the highest

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weight module $L(\Lambda)$ [62, 115, 127]. These results can be thought of as far reaching generalizations of Ariki's Categorification Theorem in type A [2].

The quiver Hecke algebras attached to the quivers of type A are distinguished because these are the only quiver Hecke algebras which already existed in the literature — all of the other quiver Hecke algebras are "new" algebras. In type A, when we are working over a field, the quiver Hecke algebras are isomorphic to affine Hecke algebras of type A [115] and the cyclotomic quiver Hecke algebras are isomorphic to the cyclotomic Hecke algebras of type A [19]. The cyclotomic Hecke algebras of type A have a uniform description but, historically, they been studied either as *Ariki-Koike* algebras ($v \neq 1$), or as degenerate Ariki-Koike algebras ($v = 1$). These algebras include as special cases the group algebras of the symmetric groups and the Iwahori–Hecke algebras of types A and B. The existence of gradings on Hecke algebras, as least in the "abelian defect case", was predicted Rouquier [113, Remark 3.11] and Turner [123].

The cyclotomic quiver Hecke algebras of type A are better understood than other types because we already know a lot about the isomorphic, but ungraded, cyclotomic Hecke algebras [100]. For example, by piggybacking on the existing theory, homogeneous bases have been constructed for the cyclotomic quiver Hecke algebras of type A [49] but such bases are not yet known in other types. Many of the major results for general quiver Hecke algebras were first proved in type A and then generalized to other types. In fact, the type A algebras, through Ariki's theorem and Chuang and Roouquier's seminal work on $\mathfrak{sl}_2$-categorifications [26], has motivated many of these developments.

This chapter brings together the "classical" ungraded representation theory and the emerging graded representation theory of the cyclotomic Hecke algebras of type A so that people can see how the two theories interact. With the advent of the KLR algebras these algebras can now be studied from the following different perspectives:

a) As ungraded cyclotomic Hecke algebras.
b) As graded cyclotomic quiver Hecke algebras or KLR algebras.
c) Geometrically as the ext-algebras of Lusztig sheaves [91, 116, 125].
d) Through the lens of 2-representation theory using Rouquier's theory of 2-Kac Moody algebras [62, 114, 127].

Here we focus on (a) and (b) taking an unashamedly combinatorial approach, although we will see shadows of geometry and 2-representation theory. Kleshchev [73] has written a nice survey of the applications of quiver Hecke algebras to symmetric groups which takes a takes a slightly different path to that given here.

The first section starts by giving a uniform description of the degenerate and non-degenerate cyclotomic Hecke algebras. We quickly recall some important structural results from the representation theory of these algebras. Everything that we mention in this section is applied later in the graded setting.

The second section introduces the cyclotomic KLR algebras as abstract algebras given by generators and relations. We use the relations in a series of extended examples to try and give the reader a feel for these algebras. In particular, using just the relations we show that the semisimple cyclotomic quiver Hecke algebras of type A are always direct sums of matrix rings. From this we deduce Brundan and Kleshchev's Graded Isomorphism Theorem in the semisimple case.

The third section starts with Brundan and Kleshchev's Graded Isomorphism Theorem [19]. We then develop the representation theory of the cyclotomic quiver Hecke algebras as graded cellular algebras, focusing on the graded Specht modules. The highlight of this section is a self-contained proof of Brundan and Kleshchev's Graded Categorification Theorem [20], starting from the graded branching rules for the graded Specht modules and then using Ariki's Categorification Theorem [2] to make the link with canonical bases. We also give a new treatment of graded adjustment matrices using a cellular algebra approach.

In the final section we sketch one way of proving Brundan and Kleshchev's Graded Isomorphism Theorem using the classical theory of seminormal forms. As an application we describe how to construct a new graded cellular basis for the cyclotomic quiver Hecke algebras which appears to have remarkable properties. We end with a conjecture for the q-characters of the graded simple modules.

Although the experts will find some new results here most of the novelty is in our approach and our arguments. We include many examples and a comprehensive survey of the literature. We apologize for any sins and omissions that remain.

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1. Cyclotomic Hecke algebras of type A

This sections surveys the representation theory of the cyclotomic Hecke algebras of type A and, at the same time, introduces the results and the combinatorics that we need later.

1.1. Cyclotomic Hecke algebras and Ariki-Koike algebras.

Hecke algebras of the complex reflections groups $G_{\ell,n} = \mathbb{Z}/\ell\mathbb{Z} \wr \mathfrak{S}_n$ of type $G(\ell,1,n)$ were introduced by Ariki-Koike [8], motivated by the Iwahori-Hecke algebras of Coxeter groups [53]. Soon afterwards, Broué and Malle [14] defined Hecke algebras for arbitrary complex reflection groups. The following refinement of the definition of these algebras unifies the treatment of the degenerate and non-degenerate algebras.

Let $\mathcal{Z}$ be a commutative domain with one.

1.1.1. **Definition** (Hu-Mathas [52, Definition 2.2])**.** *Fix integers $n \ge 0$ and $\ell \ge 1$. The* ***cyclotomic Hecke algebra of type*** *A, with* ***Hecke parameter*** *$v \in \mathcal{Z}^\times$ and* ***cyclotomic parameters*** *$Q_1, \dots, Q_\ell \in \mathcal{Z}$, is the unital associative $\mathcal{Z}$-algebra $\mathscr{H}_n = \mathscr{H}_n(\mathcal{Z}, v, Q_1, \dots, Q_\ell)$ with generators $L_1, \dots, L_n$, $T_1, \dots, T_{n-1}$ and relations*

$$\begin{aligned}
&\textstyle\prod_{l=1}^{\ell}(L_1 - Q_l) = 0, && (T_r + v^{-1})(T_r - v) = 0, && \\
&L_r L_t = L_t L_r, && T_r T_s = T_s T_r && \text{if } |r-s| > 1,\\
&T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, && T_r L_t = L_t T_r, && \text{if } t \ne r, r+1,\\
&&& L_{r+1} = T_r L_r T_r + T_r, &&
\end{aligned}$$

where $1 \le r < n$, $1 \le s < n-1$ and $1 \le t \le n$.

By definition, $\mathscr{H}_n$ is generated by $L_1, T_1, \dots, T_{n-1}$ but we prefer including $L_2, \dots, L_n$ in the generating set.

Let $\mathfrak{S}_n$ be the **symmetric group** on n letters. For $1 \le r < n$ let $s_r = (r, r+1)$ be the corresponding simple transposition. Then $\{s_1, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. A **reduced expression** for $w \in \mathfrak{S}_n$ is a word $w = s_{r_1}, \dots s_{r_k}$ with k minimal and $1 \le r_j < n$ for $1 \le j \le k$. If $w = s_{r_1} \dots s_{r_k}$ is reduced then set $T_w = T_{r_1} \dots T_{r_k}$. Then T_w is independent of the choice of reduced expression by Matsumoto's Monoid Lemma [103] since the braid relations hold in $\mathscr{H}_n$; see, for example, [97, Theorem 1.8]. Arguing as in [8, Theorem 3.3], it follows that $\mathscr{H}_n$ is free as a $\mathcal{Z}$-module with basis

$$\{\, L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_1, \dots, a_n < \ell \text{ and } w \in \mathfrak{S}_n \,\}. \tag{1.1.2}$$

Consequently, $\mathscr{H}_n$ is free as a $\mathcal{Z}$-module of rank $\ell^n n!$, which is the order of the complex reflection group $G_{\ell,n} = \mathbb{Z}/\ell\mathbb{Z} \wr \mathfrak{S}_n$ of type $G(\ell,1,n)$.

Definition 1.1.1 is different to Ariki and Koike's [8] definition of the cyclotomic Hecke algebras of type $G(\ell,1,n)$ because we have changed the commutation relation for T_r and L_r. Ariki and Koike [8] defined their algebra to be the unital associative algebra generated by $T_0, T_1, \dots, T_{n-1}$ subject to the relations

$$\begin{aligned}
&\textstyle\prod_{l=1}^{\ell}(T_0 - Q_l') = 0, && (T_r + v^{-1})(T_r - v) = 0,\\
&T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0 && T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1},\\
&& T_r T_s = T_s T_r \text{ if } |r-s| > 1 &
\end{aligned}$$

We have renormalised the quadratic relation for the T_r, for $1 \le r < n$, so that $q = v^2$ in the notation of [8]. Ariki and Koike then defined $L_1' = T_0$ and set $L_{r+1}' = T_r L_r' T_r$ for $1 \le r < n$. In fact, if $v - v^{-1}$ is invertible in $\mathcal{Z}$ then $\mathscr{H}_n$ is (isomorphic to) the Ariki-Koike algebra with parameters $Q_l' = 1 + (v - v^{-1})Q_l$ for $1 \le l \le \ell$. To see this set $L_r' = 1 + (v - v^{-1})L_r$ in $\mathscr{H}_n$, for $1 \le r \le n$. Then $T_r L_r' T_r = (v - v^{-1}) T_r L_r T_r + T_r^2 = L_{r+1}'$, which implies our claim. Therefore, over a field, $\mathscr{H}_n$ is an Ariki-Koike algebra whenever $v^2 \ne 1$. On the other hand, if $v^2 = 1$ then $\mathscr{H}_n$ is a *degenerate* cyclotomic Hecke algebra [11, 72]. In general, $\mathscr{H}_n$ is *not* isomorphic to an Ariki-Koike algebra when $v^2 = 1$.

We note that the Ariki-Koike algebras with $v^2 = 1$ include as a special the group algebras $\mathcal{Z}G_{\ell,n}$ of the complex reflection groups $G_{\ell,n}$, for $n \ge 0$. One consequence of the last paragraph is that $\mathcal{Z}G_{\ell,n}$ is *not* a specialization of $\mathscr{H}_n$. This said, if F is a field such that $\mathscr{H}_n$ and $FG_{\ell,n}$ are both split semisimple then $\mathscr{H}_n \cong FG_{\ell,n}$. On the other hand, the algebras $\mathscr{H}_n$ always fit into the *spetses* framework of Broué, Malle and Michel's [15].

The algebras $\mathscr{H}_n$ with $v^2 = 1$ are the *degenerate* cyclotomic Hecke algebras of type $G(\ell,1,n)$ whereas if $v^2 \ne 1$ then $\mathscr{H}_n$ is an Ariki-Koike algebra in the sense of [8]. The definition of $\mathscr{H}_n$ that we have given is more natural because many features of the Hecke algebras $\mathscr{H}_n$ have a uniform description in the degenerate and non-degenerate cases:

- The centre of $\mathscr{H}_n$ is the set of symmetric polynomials in $L_1, \dots, L_n$ (Brundan [17] in the degenerate case when $v^2 = 1$ and announced when $v^2 \ne 1$ by Graham and Francis building on [39]).
- The blocks of $\mathscr{H}_n$ are indexed by the same combinatorial data (Lyle and Mathas [89] when $v^2 \ne 1$ and Brundan [17] when $v^2 = 1$).

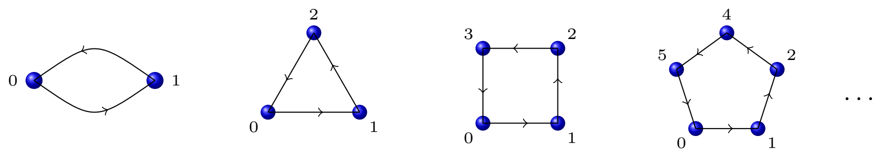
- The irreducible $\mathscr{H}_n$-modules are indexed by the crystal graph of the integral highest weight module $L(\Lambda)$ for $U_q(\widehat{\mathfrak{sl}}_e)$ (Ariki [2] when $v^2 \neq 1$ and Brundan and Kleshchev [21] when $v^2 = 1$).
- The algebras $\mathscr{H}_n$ categorify $L(\Lambda)$. Moreover, in characteristic zero the projective indecomposable $\mathscr{H}_n$-modules correspond to the canonical basis of $L(\Lambda)$. (Ariki [2] when $v^2 \neq 1$ and Brundan and Kleshchev [21] when $v^2 = 1$).
- The algebra $\mathscr{H}_n$ is isomorphic to a cyclotomic quiver Hecke algebras of type A (Brundan and Kleshchev [19]).

In contrast, the Ariki-Koike algebras with $v^2 = 1$ do not share any of these properties: their center can be larger than the set of symmetric polynomials in $L_1, \dots, L_n$ (Ariki [2]); they have only one block (Lyle and Mathas [89]); their irreducible modules are indexed by a different set (Mathas [96]); they do not categorify $L(\Lambda)$ and no non-trivial grading on these algebras is known. In this sense, the definition of the Ariki-Koike algebras from [8] gives the wrong algebras when $v^2 = 1$. Definition 1.1.1 corrects for this.

We remark that many results for the cyclotomic Hecke algebras $\mathscr{H}_n^\Lambda$ were proved separately in the degenerate ($v^2 = 1$) and non-degenerate cases ($v^2 \neq 1$). Using Definition 1.1.1 it should now be possible to give uniform proofs in all cases. In fact, all of arguments that we have checked can be extended to include the $v^2 = 1$ case. One of the aims of this article is to give a uniform proof of the Ariki-Brundan-Kleshchev Graded Categorification Theorem [2, 20, 21] for the *integral* cyclotomic Hecke algebras $\mathscr{H}_n^\Lambda$.

1.2. **Quivers of type A and integral parameters.** Rather than work with arbitrary cyclotomic parameters $Q_1, \dots, Q_\ell$, as in Definition 1.1.1, we now specialize to the *integral case* using the Morita equivalence results of Dipper and the author [30] (when $v^2 \neq 1$) and Brundan and Kleshchev [18] (when $v^2 = 1$). First, however, we need to introduce quivers and quantum integers.

Fix an integer $e \geq 2$ and let Γ_e be the oriented quiver with vertex set $I_i = \mathbb{Z}/(e\mathbb{Z} \cap \mathbb{Z})$ and edges $i \longrightarrow i+1$, for $i \in I_e$. If $i, j \in I_e$ and i and j are not connected by an edge in Γ_e then we write $i \not\relbar j$. When e is fixed we write $\Gamma = \Gamma_e$ and $I = I_e$. Hence, we are considering either the linear quiver $\mathbb{Z}$ ($e = \infty$) or a cyclic quiver ($e < \infty$):



$e = 2$ $e = 3$ $e = 4$ $e = 5$

In the literature the case $e = \infty$ is often written as $e = 0$, however, we prefer $e = \infty$ because then $e = |I_e|$. There are also several results which hold when $e > n$ — using the "$e = 0$ convention" this condition must be written as $e > n$ or $e = 0$. Below, if $e = \infty$ then the coset $i + e\mathbb{Z} \in I$ should be read as $i + (e\mathbb{Z} \cap \mathbb{Z}) = \{i\}$ and identified with $i \in \mathbb{Z}$. We write $e \geq 2$ to mean $e \in \{2, 3, 4, 5, \dots\} \cup \{\infty\}$.

To the quiver Γ_e we attach the symmetric Cartan matrix $(c_{ij})_{i,j \in I}$, where

$$c_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \to j \text{ or } i \leftarrow j, \\ -2, & \text{if } i \leftrightarrows j, \\ 0, & \text{otherwise,} \end{cases}$$

Let $\widehat{\mathfrak{sl}}_e$ be the corresponding Kac-Moody algebra [61] with fundamental weights $\{\, \Lambda_i \mid i \in I \,\}$, positive weight lattice $P^+ = P_e^+ = \sum_{i \in I} \mathbb{N}\Lambda_i$ and positive root lattice $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$. Let $(\cdot, \cdot)$ be the bilinear form determined by

$$(\alpha_i, \alpha_j) = c_{ij} \qquad \text{and} \qquad (\Lambda_i, \alpha_j) = \delta_{ij}, \qquad \text{for } i, j \in I.$$

More details can be found, for example, in [61, Chapter 1].

Fix a sequence $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$, the **multicharge**, and define $\Lambda = \Lambda(\boldsymbol{\kappa}) = \Lambda_{\overline{\kappa}_1} + \dots + \Lambda_{\overline{\kappa}_\ell}$, where $\overline{a} = a + e\mathbb{Z} \in I$ for $a \in \mathbb{Z}$. Then $\Lambda \in P^+$ is dominant weight of **level** ℓ. The integral cyclotomic Hecke algebras defined below depend only on Λ, however, the bases and much of the combinatorics that we introduce will depend upon the choice of multicharge $\boldsymbol{\kappa}$.

Recall that $\mathcal{Z}$ is an integral domain. For $t \in \mathcal{Z}^\times$ and $k \in \mathbb{Z}$ define the ***t*-quantum integer** $[k]_t$ by

$$[k]_t = \begin{cases} t + t^3 + \dots + t^{2k-1}, & \text{if } k \geq 0, \\ -(t^{-1} + t^{-3} + \dots + t^{2k-1}), & \text{if } k < 0. \end{cases}$$

When t is understood we simply write $[k] = [k]_t$. Unpacking the definition, if $t^2 \neq 1$ then $[k] = (t^{2k} - 1)/(t - t^{-1})$ whereas $[k] = \pm k$ if $t = \pm 1$.

The **quantum characteristic** of v is the smallest non-negative integer $e \in \{2, 3, 4, 5, \dots\} \cup \{\infty\}$ such that $[e]_v = 0$, where we set $e = \infty$ if $[k]_v \neq 0$ for all $k \geq 0$.

1.2.1. **Definition.** *Suppose that $\Lambda=\Lambda(\boldsymbol{\kappa})\in P^+$, for $\boldsymbol{\kappa}\in\mathbb{Z}^\ell$, and that $v\in\mathcal{Z}$ has quantum characteristic e. The **integral** cyclotomic Hecke algebra of type A of weight Λ is the cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda=\mathscr{H}_n(\mathcal{Z},v,Q_1,\dots,Q_r)$ with Hecke parameter v and cyclotomic parameters $Q_r=[\kappa_r]_v$, for $1\le r\le\ell$.*

If the reader finds the choice of cyclotomic parameters in Definition 1.2.1 surprising then, as discussed in §1.1, observe that if $v^2\ne1$ then these parameters correspond to the Ariki-Koike parameters $Q'_r=v^{2\kappa_r}$, for $1\le r\le\ell$.

As observed in [52, §2.2], translating the Morita equivalence theorems of [30, Theorem 1.1] and [18, Theorem 5.19] into the current setting explains the significance of the integral cyclotomic Hecke algebras.

1.2.2. **Theorem** (Dipper-Mathas [30], Brundan-Kleshchev [18]). *Every cyclotomic quiver Hecke algebra $\mathscr{H}_n$ is Morita equivalent to a direct sum of tensor products of integral cyclotomic Hecke algebras.*

Brundan and Kleshchev treated the degenerate case when $v^2=1$ using very different arguments than those in [30]. With the benefit of Definition 1.1.1 the argument of [30] now applies uniformly to both the degenerate and non-degenerate cases. The Morita equivalences in [18, 30] are described explicitly, with the equivalence being determined by orbits of the cyclotomic parameters. See [18, 30] for more details.

In view of Theorem 1.2.2, it is enough to consider the integral cyclotomic Hecke algebras $\mathscr{H}_n^\Lambda$ where $v\in\mathcal{Z}^\times$ has quantum characteristic e and $\Lambda\in P^+$. This said, in this section we continue to consider the general case of a not necessarily integral cyclotomic Hecke algebra because we will need this generality in §4.2.

1.3. **Cellular algebras.** For convenience we recall Graham and Lehrer's cellular algebra framework [45]. This will allow us to define Specht modules for $\mathscr{H}_n$ as cell modules. Significantly, the cellular algebra machinery endows the Specht modules with an associative bilinear form. Here is the definition.

1.3.1. **Definition** (Graham and Lehrer [45]). *Suppose that A is a $\mathcal{Z}$-algebra that is $\mathcal{Z}$-free and of finite rank as a $\mathcal{Z}$-module. A **cell datum** for A is an ordered triple $(\mathcal{P},T,C)$, where $(\mathcal{P},\rhd)$ is the **weight poset**, $T(\lambda)$ is a finite set for $\lambda\in\mathcal{P}$, and*

$$C\colon\coprod_{\lambda\in\mathcal{P}}T(\lambda)\times T(\lambda)\longrightarrow A;(\mathsf{s},\mathsf{t})\mapsto c_{\mathsf{st}},$$

is an injective map of sets such that:

(GC$_1$) *$\{\,c_{\mathsf{st}}\mid \mathsf{s},\mathsf{t}\in T(\lambda)$ for $\lambda\in\mathcal{P}\,\}$ is a $\mathcal{Z}$-basis of A.*

(GC$_2$) *If $\mathsf{s},\mathsf{t}\in T(\lambda)$, for some $\lambda\in\mathcal{P}$, and $a\in A$ then there exist scalars $r_{\mathsf{tv}}(a)$, which do not depend on s, such that*

$$c_{\mathsf{st}}a=\sum_{\mathsf{v}\in T(\lambda)}r_{\mathsf{tv}}(a)c_{\mathsf{sv}}\ (mod\ A^{\rhd\lambda}),$$

where $A^{\rhd\lambda}$ is the $\mathcal{Z}$-submodule of A spanned by $\{\,c_{\mathsf{ab}}\mid\mu\rhd\lambda$ and $\mathsf{a},\mathsf{b}\in T(\mu)\,\}$.

(GC$_3$) *The $\mathcal{Z}$-linear map $*\colon A\longrightarrow A$ determined by $(c_{\mathsf{st}})^*=c_{\mathsf{ts}}$, for all $\lambda\in\mathcal{P}$ and all $\mathsf{s},\mathsf{t}\in T(\lambda)$, is an anti-isomorphism of A.*

*A **cellular algebra** is an algebra which has a cell datum. If A is a cellular algebra with cell datum $(\mathcal{P},T,C)$ then the basis $\{\,c_{\mathsf{st}}\mid\lambda\in\mathcal{P}$ and $\mathsf{s},\mathsf{t}\in T(\lambda\,\}$ is a **cellular basis** of A with cellular algebra anti-isomorphism* 8.

König and Xi [80] have given a basis free definition of cellular algebras. Goodman and Graber [42] have shown that (GC$_3$) can be relaxed to the requirement that there exists an anti-isomorphism $*$ of A such that $(c_{\mathsf{st}})^*\equiv c_{\mathsf{ts}}\pmod{A}^{\rhd\lambda}$.

The prototypical example of a cellular algebra is a matrix algebra with its basis of matrix units, which we call a *Wedderburn basis*. As any split semisimple algebra is isomorphic to a direct sum of matrix algebras it follows that every split semisimple algebra is cellular. The cellular algebra framework is, however, most useful in studying non-semisimple algebras which are not isomorphic to a direct sum of matrix rings. In general, a cellular basis can be thought of as approximation, or weakening of, a basis of matrix units. (This is idea is made more explicit in [101].)

The cellular basis axioms, like the basis of matrix units of a split semisimple algebra, determines a filtration of the cellular algebra, via the ideals $A^{\rhd\lambda}$. As we will see, this leads to a quick construction of its irreducible representations.

For a cellular algebra A we let $A^{\unrhd\lambda}$ be the two-sided ideal of A spanned by $\{\,c_{\mathsf{ab}}\mid\mu\unrhd\lambda$ and $\mathsf{a},\mathsf{b}\in T(\mu)\,\}$.

Fix $\lambda\in\mathcal{P}$. The **cell module** $\underline{C}^\lambda$ is the (right) A-module with basis $\{\,c_{\mathsf{t}}\mid\mathsf{t}\in T(\lambda)\,\}$ and where $a\in A$ acts on $\underline{C}^\lambda$ by:

$$c_{\mathsf{t}}a=\sum_{\mathsf{v}\in T(\lambda)}r_{\mathsf{tv}}(a)c_{\mathsf{v}},\qquad\text{for }\mathsf{t}\in T(\lambda),$$

where the scalars $r_{\mathsf{tv}}(a)\in\mathcal{Z}$ are those appearing in (GC$_2$). It follows immediately from Definition 1.3.1 that $\underline{C}^\lambda$ is an A-module. Indeed, if $\mathsf{s}\in T(\lambda)$ then $\underline{C}^\lambda$ is isomorphic to the submodule $(c_{\mathsf{st}}+A^{\rhd\lambda})A$ of A/A^λ via the map $c_{\mathsf{t}} \mapsto c_{\mathsf{st}} + A^{\lambda}$, for $\mathsf{t} \in T(\lambda)$. The cell module $\underline{C}^{\lambda}$ comes with a symmetric bilinear form $\langle\ ,\ \rangle_{\lambda}$ that is uniquely determined by

$$\langle c_{\mathsf{t}}, c_{\mathsf{v}}\rangle_{\lambda} c_{\mathsf{ab}} \equiv c_{\mathsf{at}} c_{\mathsf{vb}} \pmod{A^{\rhd\lambda}}, \tag{1.3.2}$$

for $\mathsf{a}, \mathsf{b}, \mathsf{t}, \mathsf{v} \in T(\lambda)$. By (GC$_2$) of Definition 1.3.1, the inner product $\langle c_{\mathsf{t}}, c_{\mathsf{v}}\rangle_{\lambda}$ depends only on t and v, and not on the choices of a and b. In addition, $\langle xa, y\rangle = \langle x, ya^{*}\rangle_{\lambda}$, for all $x, y \in \underline{C}^{\lambda}$ and $a \in A$. Therefore,

$$\operatorname{rad} \underline{C}^{\lambda} = \{\, x \in \underline{C}^{\lambda} \mid \langle x, y\rangle_{\lambda} = 0 \text{ for all } y \in \underline{C}^{\lambda} \,\} \tag{1.3.3}$$

is an A-submodule of $\underline{C}^{\lambda}$. Set $\underline{D}^{\lambda} = \underline{D}^{\lambda}/\operatorname{rad} \underline{C}^{\lambda}$. Then $\underline{D}^{\lambda}$ is an A-module.

The following theorem summarizes some of the main properties of a cellular algebra. The proof is surprisingly easy given the strength of the result. In applications the main difficulty is in showing that a given algebra is cellular.

1.3.4. **Theorem** (Graham and Lehrer [45]). *Suppose that $\mathcal{Z} = F$ is a field. Then:*

a) *Suppose that $\mu \in \mathcal{P}$. Then $\underline{D}^{\lambda}$ is either zero or absolutely irreducible.*
b) *Let $\mathcal{P}_0 = \{\, \mu \in \mathcal{P} \mid \underline{D}^{\mu} \neq 0 \,\}$. Then $\{\, \underline{D}^{\mu} \mid \mu \in \mathcal{P}_0 \,\}$ is a complete set of pairwise non-isomorphic irreducible A-modules.*
c) *If $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{P}_0$ then $[\underline{C}^{\lambda}{:}\underline{D}^{\mu}] \neq 0$ only if $\lambda \unrhd \mu$. Moreover, $[\underline{C}^{\mu}{:}\underline{D}^{\mu}] = 1$.*

In part (c), $[\underline{C}^{\lambda}{:}\underline{D}^{\mu}]$ is the decomposition multiplicity of the simple module $\underline{D}^{\mu}$ in $\underline{C}^{\lambda}$. If $\mu \in \mathcal{P}_0$ let $\underline{P}^{\mu}$ be the projective cover of $\underline{D}^{\mu}$. It follows from Definition 1.3.1 that $\underline{P}^{\mu}$ has a filtration in which the quotients are cell modules such that $\underline{C}^{\lambda}$ appears with multiplicity $[\underline{C}^{\lambda}{:}\underline{D}^{\mu}]$. Consequently, an analogue of Brauer-Humphrey's reciprocity holds for A. In particular, the Cartan matrix of A is symmetric.

1.4. **Multipartitions and tableaux.** A **partition** of m is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $|\lambda| = \lambda_1 + \lambda_2 + \cdots = m$. An ($\ell$-)**multipartition** of n is an ℓ-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. We identify the multipartition $\boldsymbol{\lambda}$ with its **diagram** which is the set of **nodes** $[\![\boldsymbol{\lambda}]\!] = \{\, (l, r, c) \mid 1 \le c \le \lambda^{(l)}_r \text{ for } 1 \le l \le \ell \,\}$. In this way, we think of $\boldsymbol{\lambda}$ as an ordered ℓ-tuple of arrays of boxes in the plane and we talk of the **components**, **rows** and **columns** of $\boldsymbol{\lambda}$. For example, if $\boldsymbol{\lambda} = (3, 1^2|2, 1|3, 2)$ then

$$\boldsymbol{\lambda} = [\![\boldsymbol{\lambda}]\!] = \left(\begin{array}{|c|c|c|}\hline &&\\ \hline &\multicolumn{2}{c}{}\\ \cline{1-1} &\multicolumn{2}{c}{}\\ \cline{1-1}\end{array} \;\Bigg|\; \begin{array}{|c|c|}\hline &\\ \hline &\multicolumn{1}{c}{}\\ \cline{1-1}\end{array} \;\Bigg|\; \begin{array}{|c|c|c|}\hline &&\\ \hline &&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \right).$$

A node A is an **addable node** of $\boldsymbol{\lambda}$ if $A \notin \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cup \{A\}$ is the (diagram of) a multipartition of $n+1$. Similarly, a node B is a **removable node** of $\boldsymbol{\lambda}$ if $B \in \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \setminus \{B\}$ is a multipartition of $n-1$. If A is an addable node of $\boldsymbol{\lambda}$ let $\boldsymbol{\lambda} + A$ be the multipartition $\boldsymbol{\lambda} \cup \{A\}$ and, similarly, if B is a removable node let $\boldsymbol{\lambda} - A = \boldsymbol{\lambda} \setminus \{B\}$. Order the nodes lexicographically by $\le$.

The set of multipartitions of n becomes a poset under dominance where $\boldsymbol{\lambda}$ **dominates** $\boldsymbol{\mu}$, written as $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$, if

$$\sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda^{(l)}_j \ge \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu^{(l)}_j,$$

for $1 \le l \le \ell$ and $i \ge 1$. If $\boldsymbol{\lambda} \unrhd \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$ then write $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$. Let $\mathcal{P}_n = \mathcal{P}_{\ell,n}$ be the set of multipartitions of n. We consider $\mathcal{P}_n$ as a poset ordered by dominance.

Fix a multipartition $\boldsymbol{\lambda}$. A **$\boldsymbol{\lambda}$-tableau** is a bijective map $\mathsf{t}\colon [\![\boldsymbol{\lambda}]\!] \longrightarrow \{1, 2, \dots, n\}$, which we identify with a labelling of (the diagram of) $\boldsymbol{\lambda}$ by $\{1, 2, \dots, n\}$. For example,

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&\multicolumn{2}{c}{}\\ \cline{1-1} 5&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array} \;\Bigg|\; \begin{array}{|c|c|}\hline 6&7\\ \hline 8&\multicolumn{1}{c}{}\\ \cline{1-1}\end{array} \;\Bigg|\; \begin{array}{|c|c|c|}\hline 9&10&11\\ \hline 12&13&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \right) \quad\text{and}\quad \left(\begin{array}{|c|c|c|}\hline 9&12&13\\ \hline 10&\multicolumn{2}{c}{}\\ \cline{1-1} 11&\multicolumn{2}{c}{}\\ \cline{1-1}\end{array} \;\Bigg|\; \begin{array}{|c|c|}\hline 6&8\\ \hline 7&\multicolumn{1}{c}{}\\ \cline{1-1}\end{array} \;\Bigg|\; \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4&\multicolumn{1}{c}{}\\ \cline{1-2}\end{array} \right)$$

are both $\boldsymbol{\lambda}$-tableaux when $\boldsymbol{\lambda} = (3, 1^2|2, 1|3, 2)$.

A $\boldsymbol{\lambda}$-tableau is **standard** if its entries increase along rows and down columns in each component. For example, the two tableaux above are standard. Let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux. If $\mathcal{P}$ is any set of multipartitions let $\operatorname{Std}(\mathcal{P}) = \bigcup_{\boldsymbol{\lambda} \in \mathcal{P}} \operatorname{Std}(\boldsymbol{\lambda})$. Similarly set $\operatorname{Std}^2(\mathcal{P}) = \{\, (\mathsf{s}, \mathsf{t}) \mid \mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P} \,\}$.

If t is a $\boldsymbol{\lambda}$-tableau set $\operatorname{Shape}(\mathsf{t}) = \boldsymbol{\lambda}$ and let $\mathsf{t}_{\downarrow m}$ be the subtableau of t which contains the numbers $\{1, 2, \dots, m\}$. If t is a standard $\boldsymbol{\lambda}$-tableau then $\operatorname{Shape}(\mathsf{t}_{\downarrow m})$ is a multipartition for all $m \ge 0$. We extend the dominance ordering to $\operatorname{Std}(\mathcal{P}_n)$, the set of all standard tableaux, by defining $\mathsf{s} \unrhd \mathsf{t}$ if $\operatorname{Shape}(\mathsf{s}_{\downarrow m}) \unrhd \operatorname{Shape}(\mathsf{t}_{\downarrow m})$, for $1 \le m \le n$. As before, write $\mathsf{s} \rhd \mathsf{t}$ if $\mathsf{s} \unrhd \mathsf{t}$ and $\mathsf{s} \neq \mathsf{t}$. Finally, define the **strong dominance ordering** on $\operatorname{Std}^2(\mathcal{P}_n)$ by $(\mathsf{s}, \mathsf{t}) \mathrel{\underline{\blacktriangleright}} (\mathsf{u}, \mathsf{v})$ if $\mathsf{s} \unrhd \mathsf{u}$ and $\mathsf{t} \unrhd \mathsf{v}$. Similarly, $(\mathsf{s}, \mathsf{t}) \blacktriangleright (\mathsf{u}, \mathsf{v})$ if $(\mathsf{s}, \mathsf{t}) \mathrel{\underline{\blacktriangleright}} (\mathsf{u}, \mathsf{v})$ and $(\mathsf{s}, \mathsf{t}) \neq (\mathsf{u}, \mathsf{v})$

It is easy to see that there are unique standard $\boldsymbol{\lambda}$-tableaux $\mathsf{t}^{\boldsymbol{\lambda}}$ and $\mathsf{t}_{\boldsymbol{\lambda}}$ such that $\mathsf{t}^{\boldsymbol{\lambda}} \trianglerighteq \mathsf{t} \trianglerighteq \mathsf{t}_{\boldsymbol{\lambda}}$, for all $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The tableau $\mathsf{t}^{\boldsymbol{\lambda}}$ has the numbers $1, 2, \dots, n$ entered in order from left to right along the rows of $\mathsf{t}^{\lambda^{(1)}}$, and then $\mathsf{t}^{\lambda^{(2)}}, \dots, \mathsf{t}^{\lambda^{(\ell)}}$. Similarly, $\mathsf{t}_{\boldsymbol{\lambda}}$ is the tableau with the numbers $1, \dots, n$ entered in order down the columns of $\mathsf{t}^{\lambda^{(\ell)}}, \dots, \mathsf{t}^{\lambda^{(2)}}, \mathsf{t}^{\lambda^{(1)}}$. If $\boldsymbol{\lambda} = (3, 1^2|2, 1|3, 2)$ then the two $\boldsymbol{\lambda}$-tableaux displayed above are $\mathsf{t}^{\boldsymbol{\lambda}}$ and $\mathsf{t}_{\boldsymbol{\lambda}}$, respectively.

Given a standard $\boldsymbol{\lambda}$-tableau t define permutations $d(\mathsf{t}), d'(\mathsf{t}) \in \mathfrak{S}_n$ by $\mathsf{t}^{\boldsymbol{\lambda}} d(\mathsf{t}) = \mathsf{t} = \mathsf{t}_{\boldsymbol{\lambda}} d'(\mathsf{t})$. Then $d(\mathsf{t})d'(\mathsf{t})^{-1} = d(\mathsf{t}_{\boldsymbol{\lambda}})$ with $\ell(d(\mathsf{t})) + \ell(d'(\mathsf{t})) = \ell(d(\mathsf{t}_{\boldsymbol{\lambda}}))$, for all $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Let $\le$ be the Bruhat order on $\mathfrak{S}_n$ with the convention that $1 \le w$ for all $w \in \mathfrak{S}_n$. Independently, Ehresmann and James [54] showed that if $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\mathsf{s} \trianglerighteq \mathsf{t}$ if and only if $d(\mathsf{s}) \le d(\mathsf{t})$ and if and only if $d'(\mathsf{t}) \le d'(\mathsf{s})$. A proof can be found, for example, in [97, Theorem 3.8].

Finally, we will need to know how to conjugate multipartitions and tableaux. The conjugate of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ where $\lambda'_r = \#\{\, s \ge 1 \mid \lambda_s \ge r \,\}$. That is, we swap the rows and columns of λ. The **conjugate** of a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}|\dots|\lambda^{(\ell)})$ is the multipartition $\boldsymbol{\lambda}' = (\lambda^{(\ell)\prime}|\dots|\lambda^{(1)\prime})$. Similarly, the conjugate of a $\boldsymbol{\lambda}$-tableau $\mathsf{t} = (\mathsf{t}^{(1)}|\dots|\mathsf{t}^{(\ell)})$ is the $\boldsymbol{\lambda}'$-tableau $\mathsf{t}' = (\mathsf{t}^{(\ell)\prime}|\dots|\mathsf{t}^{(1)\prime})$ where $\mathsf{t}^{(k)\prime}$ is the tableau obtained by swapping the rows and columns of $\mathsf{t}^{(k)}$, for $1 \le k \le \ell$. Then $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}' \trianglerighteq \boldsymbol{\lambda}'$ and that $\mathsf{s} \trianglerighteq \mathsf{t}$ if and only if $\mathsf{t}' \trianglerighteq \mathsf{s}'$.

1.5. **The Murphy basis of $\mathscr{H}_n^\Lambda$.** Graham and Lehrer [45] showed that the cyclotomic Hecke algebras (when $v^2 \ne 1$) are cellular algebras. In this section we recall another cellular basis for these algebras which was constructed in [29] when $v^2 \ne 1$ and in [11] when $v^2 = 1$. When $\ell = 1$ these result are due to Murphy [106].

First observe that Definition 1.1.1 implies that there is a unique anti-isomorphism $*$ on $\mathscr{H}_n$ which fixes each of the generators $T_1, \dots, T_{n-1}, L_1, \dots, L_n$ of $\mathscr{H}_n$. It is easy to see that $T_w^* = T_{w^{-1}}$, for $w \in \mathfrak{S}_n$

Fix a multipartition $\boldsymbol{\lambda} \in \mathcal{P}_n$. Following [29, Definition 3.14] and [11, §6], if $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ define $m_{\mathsf{st}} = T_{d(\mathsf{s})^{-1}} m_{\boldsymbol{\lambda}} T_{d(\mathsf{t})}$, where $m_{\boldsymbol{\lambda}} = u_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}}$,

$$u_{\boldsymbol{\lambda}} = \prod_{1 \le l < \ell} \prod_{r=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(l)}|} \frac{1}{Q'_{l+1}}(L_r - [\kappa_{l+1}]) \quad \text{and} \quad x_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} v^{\ell(w)} T_w,$$

where $Q'_l = 1 + (v - v^{-1})Q_l$ as in §1.1. The renormalization of $u_{\boldsymbol{\lambda}}$ by $1/Q'_{l+1}$ is not strictly necessary. When $Q'_{l+1} = 0$ this factor can be omitted from the definition of $u_{\boldsymbol{\lambda}}$, at the expense of some aesthetics in some of the formulas which follow. In the integral case, which i what we care most about, this problem does not arise because $Q'_l = q^{\kappa_l} \ne 0$ since $Q_l = [\kappa_l]$, for $1 \le l \le \ell$.

Using the relations in $\mathscr{H}_n^\Lambda$ it is not hard to show that $u_{\boldsymbol{\lambda}}$ and $x_{\boldsymbol{\lambda}}$ commute. Consequently, $m_{\mathsf{st}}^* = m_{\mathsf{ts}}$, for all $(\mathsf{s}, \mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$.

1.5.1. **Theorem** ([29, Theorem 3.26] and [11, Theorem 6.3])**.** *The cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda$ is free as a $\mathcal{Z}$-module with cellular basis $\{\, m_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n \,\}$ with respect to the poset $(\mathcal{P}_n, \trianglerighteq)$.*

Consequently, $\mathscr{H}_n^\Lambda$ is a cellular algebra so all of theory in §1.3 applies. In particular, for each $\boldsymbol{\lambda} \in \mathcal{P}_n$ there exists a Specht module $\underline{S}^{\boldsymbol{\lambda}}$ with basis $\{\, m_{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$. Concretely, we could take $m_{\mathsf{t}} = m_{\mathsf{t}^{\boldsymbol{\lambda}}\mathsf{t}} + \mathscr{H}_n^{\rhd\boldsymbol{\lambda}}$, for $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$.

Let $\underline{D}^{\boldsymbol{\lambda}} = \underline{S}^{\boldsymbol{\lambda}} / \operatorname{rad} \underline{S}^{\boldsymbol{\lambda}}$ be the quotient of $\underline{S}^{\boldsymbol{\lambda}}$ by the radical of its bilinear form. Set $\mathcal{K}_n^\Lambda = \{\, \boldsymbol{\mu} \in \mathcal{P}_n \mid \underline{D}^{\boldsymbol{\mu}} \ne 0 \,\}$. Then by Theorem 1.3.4 we obtain:

1.5.2. **Corollary** ([29, 45])**.** *Suppose that $\mathcal{Z} = F$ is a field. Then $\{\, \underline{D}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_n^\Lambda$-modules.*

The set of multipartitions $\mathcal{K}_n^\Lambda$ has been determined by Ariki [3]. We describe and recover his classification of the irreducible $\mathscr{H}_n^\Lambda$-modules in Corollary 3.5.12 below. When $\ell \ge 3$ the only known descriptions of $\mathcal{K}_n^\Lambda$ are recursive. See [9, 27] for the cases when $\ell \le 2$.

1.6. **Semisimple cyclotomic Hecke algebras of type A.** We now explicitly describe the semisimple representation theory of $\mathscr{H}_n^\Lambda$ using the *seminormal coefficient systems* introduced in [52]. As we are ultimately interested in the cyclotomic quiver Hecke algebras, which are intrinsically non-semisimple algebras, it is a little surprising that we are interested in these results. we will see, however, that the semisimple representation theory of $\mathscr{H}_n^\Lambda$ and the KLR grading are closely intertwined.

The **Gelfand-Zetlin subalgebra** of $\mathscr{H}_n$ is the subalgebra $\mathscr{L}_n = \mathscr{L}_n(\mathcal{Z}) = \langle L_1, L_2, \dots, L_n \rangle$. We believe that understanding this subalgebra is crucial to understanding the representation theory of $\mathscr{H}_n$. To explain how $\mathscr{L}_n$ acts on $\mathscr{H}_n^\Lambda$ define two **content** functions for $\mathsf{t} \in \operatorname{Std}(\mathcal{P}_n)$ and $1 \le r \le n$ by

$$c_r^{\mathcal{Z}}(\mathsf{t}) = v^{2(c-b)} Q_l + [c - b]_v \in \mathcal{Z} \qquad \text{and} \qquad c_r^{\mathbb{Z}}(\mathsf{t}) = \kappa_l + c - b \in \mathbb{Z}, \tag{1.6.1}$$

where $\mathsf{t}(l, b, c) = r$. In the special case of the integral parameters, where $Q_l = [\kappa_l]_v$ for $1 \le l \le \ell$, the reader can check that $c_r^{\mathcal{Z}}(\mathsf{t}) = [c_r^{\mathbb{Z}}(\mathsf{t})]_v$, for $1 \le r \le n$.

The next result is well-known and extremely useful.

1.6.2. **Lemma** (James-Mathas [57, Proposition 3.7]). *Suppose that $1 \le r \le n$ and that $\mathsf{s},\mathsf{t} \in \operatorname{Std}(\boldsymbol\lambda)$, for $\boldsymbol\lambda \in \mathcal{P}_n$. Then*

$$m_{\mathsf{st}} L_r \equiv c_r^{\mathcal{Z}}(\mathsf{t}) m_{\mathsf{st}} + \sum_{\substack{\mathsf{v} \rhd \mathsf{t}\\ \mathsf{v} \in \operatorname{Std}(\boldsymbol\lambda)}} a_{\mathsf{v}} m_{\mathsf{sv}} \pmod{\mathscr{H}_n^{\rhd\boldsymbol\lambda}},$$

for some $a_{\mathsf{v}} \in \mathcal{Z}$.

Proof. Let $(l,b,c) = \mathsf{t}^{-1}(r)$. Using our notation, [57, Proposition 3.7] says that $m_{\mathsf{st}} L'_r = Q'_l v^{2(c-b)} m_{\mathsf{st}}$ plus linear combination of more dominant terms, where $Q'_l = 1 + (v - v^{-1})Q_l$. As $L_r = 1 + (v - v^{-1})L'_r$ this easily implies the result when $v^2 \ne 1$. The case when $v^2 = 1$ now follows by specialization – or, see [11, Lemma 6.6]. □

In the integral case this implies that $m_{\mathsf{st}} L_r \equiv [c_r^{\mathbb{Z}}(\mathsf{t})] m_{\mathsf{st}} + \sum_{\mathsf{v} \rhd \mathsf{t}} a_{\mathsf{v}} m_{\mathsf{st}} \pmod{\mathscr{H}_n^{\rhd\boldsymbol\lambda}}$. This agrees with [52, Lemma 2.9].

The Hecke algebra $\mathscr{H}_n$ is **content separated** if whenever $\mathsf{s},\mathsf{t} \in \operatorname{Std}(\mathcal{P}_n)$ are standard tableaux, not necessarily of the same shape, then $\mathsf{s} = \mathsf{t}$ if and only if $c_r^{\mathcal{Z}}(\mathsf{s}) = c_r^{\mathcal{Z}}(\mathsf{t})$, for $1 \le r \le n$. The following is an immediate corollary of Lemma 1.6.2 using the theory of JM-elements developed in [101, Theorem 3.7].

1.6.3. **Corollary** ([52, Proposition 3.4]). *Suppose that $\mathcal{Z} = F$ is a field and that $\mathscr{H}_n$ is content separated. Then, as an $(\mathscr{L}_n, \mathscr{L}_n)$-bimodule,*

$$\mathscr{H}_n = \bigoplus_{(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)} H_{\mathsf{st}},$$

where $H_{\mathsf{st}} = \{\, h \in \mathscr{H}_n \mid L_r h = c_r^{\mathcal{Z}}(\mathsf{s}) h \text{ and } h L_r = c_r^{\mathcal{Z}}(\mathsf{t}) h, \text{ for } 1 \le r \le n \,\}$.

For the rest of §1.6 we assume that $\mathscr{H}_n$ is content separated. Corollary 1.6.3 motivates the following definition.

1.6.4. **Definition** (Hu-Mathas [52, Definition 3.7]). *Suppose that $\mathcal{Z} = K$ is a field. Then a $*$-seminormal basis of $\mathscr{H}_n$ is any basis of the form $\{\, f_{\mathsf{st}} \mid 0 \ne f_{\mathsf{st}} \in H_{\mathsf{st}} \text{ and } f_{\mathsf{st}}^* = f_{\mathsf{ts}}, \text{ for } (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n) \,\}$.*

There is a vast literature on seminormal bases. This story started with Young's seminormal forms for the symmetric groups [130] and has now been extended to Hecke algebras and many other diagram algebras including the Brauer, BMW and partition algebras; see, for example, [98, 108, 111].

Suppose that $\{f_{\mathsf{st}}\}$ is a $*$-seminormal basis and that $(\mathsf{s},\mathsf{t}), (\mathsf{u},\mathsf{v}) \in \operatorname{Std}^2(\mathcal{P}_n)$. Let $\mathscr{C}_n = \{\, c_r^{\mathcal{Z}}(\mathsf{s}) \mid \mathsf{s} \in \operatorname{Std}(\mathcal{P}_n) \text{ for } 1 \le r \le n \,\}$ be the set of all possible contents for tableaux in $\operatorname{Std}(\mathcal{P}_n)$. Following Murphy [101, 105], for a standard tableau $\mathsf{s} \in \operatorname{Std}(\mathcal{P}_n)$ define

$$F_{\mathsf{s}} = \prod_{r=1}^{n} \prod_{\substack{c \in \mathscr{C}_n\\ c \ne c_r^{\mathcal{Z}}(\mathsf{s})}} \frac{L_r - c}{c_r^{\mathcal{Z}}(\mathsf{s}) - c}.$$

By Definition 1.6.4, if $(\mathsf{s},\mathsf{t}), (\mathsf{u},\mathsf{v}) \in \operatorname{Std}^2(\mathcal{P}_n)$ then $f_{\mathsf{st}} = \delta_{\mathsf{su}} \delta_{\mathsf{tv}} F_{\mathsf{u}} f_{\mathsf{st}} F_{\mathsf{v}}$. In particular, F_{s} is a non-zero element of $\mathscr{H}_n$. It follows that F_{s} is a scalar multiple of f_{ss} which in turn implies that $\{\, F_{\mathsf{s}} \mid \mathsf{s} \in \operatorname{Std}(\mathcal{P}_n) \,\}$ is a set of pairwise orthogonal idempotents in $\mathscr{H}_n$. (In fact, in [101] these properties are used to establish Corollary 1.6.3.) Consequently, there exists a non-zero scalar $\gamma_{\mathsf{s}} \in F$ such that $F_{\mathsf{s}} = \frac{1}{\gamma_{\mathsf{s}}} f_{\mathsf{ss}}$. Therefore, if $(\mathsf{s},\mathsf{t}), (\mathsf{u},\mathsf{v}) \in \operatorname{Std}^2(\mathcal{P}_n)$ then

$$f_{\mathsf{st}} f_{\mathsf{uv}} = f_{\mathsf{st}} F_{\mathsf{t}} F_{\mathsf{v}} f_{\mathsf{uv}} = \delta_{\mathsf{tv}} \gamma_{\mathsf{t}} f_{\mathsf{sv}}, \tag{1.6.5}$$

The next definition will allow us to classify all seminormal bases and to describe how $\mathscr{H}_n^\Lambda$ acts on them.

1.6.6. **Definition** (Hu-Mathas [52, §3]). *A **$*$-seminormal coefficient system** is a collection of scalars*

$$\boldsymbol\alpha = \{\, \alpha_r(\mathsf{t}) \mid \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n) \text{ and } 1 \le r \le n \,\}$$

such that $\alpha_r(\mathsf{t}) = 0$ if $\mathsf{v} = \mathsf{t}(r, r+1)$ is not standard, if $\mathsf{v} \in \operatorname{Std}(\mathcal{P}_n)$ then

$$\alpha_r(\mathsf{v})\alpha_r(\mathsf{t}) = \frac{\big(1 - v^{-1}c_r^{\mathcal{Z}}(\mathsf{t}) + v c_r^{\mathcal{Z}}(\mathsf{v})\big)\big(1 + v c_r^{\mathcal{Z}}(\mathsf{t}) - v^{-1}c_r^{\mathcal{Z}}(\mathsf{v})\big)}{\big(c_r^{\mathcal{Z}}(\mathsf{t}) - c_r^{\mathcal{Z}}(\mathsf{v})\big)\big(c_r^{\mathcal{Z}}(\mathsf{v}) - c_r^{\mathcal{Z}}(\mathsf{t})\big)},$$

and if $1 \le r < n$ then $\alpha_r(\mathsf{t})\alpha_{r+1}(\mathsf{t}s_r)\alpha_r(\mathsf{t}s_r s_{r+1}) = \alpha_{r+1}(\mathsf{t})\alpha_r(\mathsf{t}s_{r+1})\alpha_{r+1}(\mathsf{t}s_{r+1}s_r)$.

As the reader might guess, the two conditions on the scalars $\alpha_r(\mathsf{t})$ in Definition 1.6.6 correspond to the quadratic relations $(T_r - v)(T_r + v^{-1}) = 0$ and the braid relations $T_r T_{r+1} T_r = T_{r+1} T_r T_{r+1}$ in $\mathscr{H}_n$, respectively. The simplest example of a seminormal coefficient system is

$$\alpha_r(\mathsf{t}) = \frac{\big(1 - v^{-1}c_{r+1}^{\mathcal{Z}}(\mathsf{t}) + v c_r^{\mathcal{Z}}(\mathsf{t})\big)}{\big(c_{r+1}^{\mathcal{Z}}(\mathsf{t}) - c_r^{\mathcal{Z}}(\mathsf{t})\big)},$$

whenever $1 \le r < n$ and $\mathsf{t}, \mathsf{t}(r, r+1) \in \mathrm{Std}(\mathcal{P}_n)$. Another seminormal coefficient system is given in (1.7.1) below.

Seminormal coefficient systems arise because they describe the action of $\mathscr{H}_n$ on a seminormal basis. More precisely, we have the following:

1.6.7. **Theorem** (Hu-Mathas [52]). *Suppose that $\mathcal{Z} = K$ is a field and that $\mathscr{H}_n$ is content separated and that $\{\, f_{\mathsf{st}} \mid (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n) \,\}$ is a seminormal basis of $\mathscr{H}_n$. Then $\{f_{\mathsf{st}}\}$ is a cellular basis of $\mathscr{H}_n$ and there exists a unique seminormal coefficient system $\boldsymbol{\alpha}$ such that*

$$f_{\mathsf{st}} T_r = \alpha_r(\mathsf{t}) f_{\mathsf{sv}} + \frac{1 + (v - v^{-1}) c^{\mathcal{Z}}_{r+1}(\mathsf{t})}{c^{\mathcal{Z}}_{r+1}(\mathsf{t}) - c^{\mathcal{Z}}_r(\mathsf{t})} f_{\mathsf{st}},$$

where $\mathsf{v} = \mathsf{t}(r, r+1)$. Moreover, if $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $F_{\mathsf{s}} = \frac{1}{\gamma_{\mathsf{s}}} f_{\mathsf{ss}}$ is a primitive idempotent and $\underline{S}^{\boldsymbol{\lambda}} \cong F_{\mathsf{s}} \mathscr{H}_n$ is irreducible for all $\boldsymbol{\lambda} \in \mathcal{P}_n$.

Sketch of proof. By definition, $\{f_{\mathsf{st}}\}$ is a basis of $\mathscr{H}_n$ such that $f^*_{\mathsf{st}} = f_{\mathsf{ts}}$ for all $(\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n)$. Therefore, it follows from (1.6.5) that $\{f_{\mathsf{st}}\}$ is a cellular basis of $\mathscr{H}_n$ with cellular automorphism $*$.

It is an amusing application of the relations in Definition 1.1.1 to show that there exists a seminormal coefficient system which describes the action of T_r on the seminormal basis; [52, Lemma 3.13] for details. The uniqueness of $\boldsymbol{\alpha}$ is clear.

We have already observed that $F_{\mathsf{s}} = \frac{1}{\gamma_{\mathsf{s}}}$, for $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$, so it remains to show that F_{s} is primitive and that $\underline{S}^{\boldsymbol{\lambda}} \cong F_{\mathsf{s}} \mathscr{H}_n$. By what we have already shown, $F_{\mathsf{s}} \mathscr{H}_n$ is contained in the span of $\{\, f_{\mathsf{st}} \mid \mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$. On the other hand, if $f = \sum_{\mathsf{t}} r_{\mathsf{t}} f_{\mathsf{st}} \in F_{\mathsf{s}} \mathscr{H}_n$ and $r_{\mathsf{v}} \neq 0$ then $r_{\mathsf{v}} f_{\mathsf{sv}} = f F_{\mathsf{v}} \in F_{\mathsf{s}} \mathscr{H}_n$. It follows that $F_{\mathsf{s}} \mathscr{H}_n = \sum_{\mathsf{t}} K f_{\mathsf{st}}$, as a vector space. Consequently, $F_{\mathsf{s}} \mathscr{H}_n$ is irreducible and F_{s} is a primitive idempotent in $\mathscr{H}_n$. Finally, $\underline{S}^{\boldsymbol{\lambda}} \cong F_{\mathsf{s}} \mathscr{H}_n$ by Lemma 1.6.2 since $\mathscr{H}_n$ is content separated. □

1.6.8. **Corollary** ([52, Corollary 3.7]). *Suppose that $\boldsymbol{\alpha}$ is a seminormal coefficient system and that $\mathsf{s} \rhd \mathsf{t} = \mathsf{s}(r, r+1)$, for tableaux $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mathcal{P}_n)$ and $1 \le r < n$. Then $\alpha_r(\mathsf{s}) \gamma_{\mathsf{t}} = \alpha_r(\mathsf{t}) \gamma_{\mathsf{s}}$.*

Consequently, if the seminormal coefficient system $\boldsymbol{\alpha}$ is known then fixing γ_{t}, for some $\mathsf{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, determines γ_{s} for all $\mathsf{s} \in \mathrm{Std}(\boldsymbol{\lambda})$. Conversely, these scalars, together with $\boldsymbol{\alpha}$, determines the seminormal basis.

1.6.9. **Corollary** (Classification of seminormal bases [52, Theorem 3.14]). *There is a one-to-one correspondence between the $*$-seminormal bases of $\mathscr{H}_n$ and the pairs $(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ where $\boldsymbol{\alpha} = \{\, \alpha_r(\mathsf{s}) \mid 1 \le r < n$ and $\mathsf{s} \in \mathrm{Std}(\mathcal{P}_n) \,\}$ is a seminormal coefficient system and $\boldsymbol{\gamma} = \{\, \gamma_{\mathsf{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n \,\}$.*

Finally, the seminormal basis machinery in this section can be used to classify the semisimple cyclotomic Hecke algebras $\mathscr{H}_n$, thus reproving Ariki's semisimplicity criterion [1], when $v^2 \neq 1$ and [11, Theorem 6.11], when $v^2 = 1$.

1.6.10. **Theorem** (Ariki [1] and [11, Theorem 6.11]). *Suppose that F is a field. The following are equivalent:*

a) *$\mathscr{H}_n = \mathscr{H}_n(F, v, Q_1, \dots, Q_\ell)$ is semisimple.*
b) *$\mathscr{H}_n$ is content separated.*
c) $\displaystyle [1]_v [2]_v \dots [n]_v \prod_{1 \le r < s \le \ell} \prod_{-n < d < n} (v^{2d} Q_r + [d]_v - Q_s) \neq 0.$

We want to rephrase the semisimplicity criterion of Theorem 1.6.10 for the integral cyclotomic Hecke algebras $\mathscr{H}^\Lambda_n$, for $\Lambda \in P^+$. For each $i \in I$ define the ***i*-string of length** $n+1$ to be $\alpha_{i,n} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+n}$. Then $\alpha_{i,n} \in Q^+$.

1.6.11. **Corollary.** *Suppose that $\Lambda \in P^+$ and that $\mathcal{Z} = F$ is a field. Then $\mathscr{H}^\Lambda_n$ is semisimple if and only if $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$.*

Proof. As $Q_r = [\kappa_r]$, for $1 \le r \le \ell$, the statement of Theorem 1.6.10(c) simplifies because $v^{2d} Q_r + [d]_v - Q_s = v^{-2\kappa_s} [d + \kappa_r - \kappa_s]_v$. Therefore, $\mathscr{H}^\Lambda_n$ is semisimple if and only if

$$[1]_v [2]_v \dots [n]_v \prod_{1 \le r < s \le \ell} \prod_{-n < d < n} [d + \kappa_r - \kappa_s]_v \neq 0.$$

On the other hand, $(\Lambda, \alpha_{i,n}) \le 1$ for all $i \in I$ if and only if $(\Lambda, \alpha_i) \le 1$, for all $i \in I$, and whenever $(\Lambda, \alpha_i) \neq 0$ then $(\Lambda, \alpha_{i+k}) \neq 0$, for $1 \le k \le n$. The result follows. □

In particular, note that $e\ell > n$ if $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$.

1.7. **Gram determinants and the cyclotomic Jantzen sum formula.** For future use, we now recall the closed formula for the Gram determinants of the Specht modules $\underline{S}^{\boldsymbol\lambda}$ and the connection between these formulas and Jantzen filtrations. Throughout this section we assume that $\mathscr{H}_n$ is content separated over the field $K = \mathcal{Z}$.

For $\boldsymbol\lambda \in \mathcal{P}_n$ let $\underline{G}^{\boldsymbol\lambda} = \big(\langle m_{\mathsf s}, m_{\mathsf s}\rangle\big)_{\mathsf{s},\mathsf{t}\in\operatorname{Std}(\boldsymbol\lambda)}$ be the **Gram matrix** of the Specht module $\underline{S}^{\boldsymbol\lambda}$, where we fix an arbitrary ordering of the rows and columns of $\underline{G}^{\boldsymbol\lambda}$.

For $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$ set $f_{\mathsf{st}} = F_{\mathsf s} m_{\mathsf{st}} F_{\mathsf t}$. Then by Lemma 1.6.2 and (1.6.5),

$$f_{\mathsf{st}} = m_{\mathsf{st}} + \sum_{(\mathsf{u},\mathsf{v})\triangleright(\mathsf{s},\mathsf{t})} r_{\mathsf{uv}} m_{\mathsf{uv}},$$

for some $r_{\mathsf{uv}} \in K$. By construction, $\{f_{\mathsf{st}}\}$ is a seminormal basis of $\mathscr{H}_n$. By [52, Proposition 3.18] this basis corresponds to the seminormal coefficient system given by

$$\alpha_r(\mathsf t) = \begin{cases} 1, & \text{if } \mathsf t \triangleright \mathsf t(r,r+1),\\ \dfrac{\big(1-v^{-1}c_r(\mathsf t)+vc_r(\mathsf v)\big)\big(1+vc_r(\mathsf t)-v^{-1}c_r(\mathsf v)\big)}{\big(c_r(\mathsf t)-c_r(\mathsf v)\big)\big(c_r(\mathsf v)-c_r(\mathsf t)\big)}, & \text{otherwise,}\end{cases} \tag{1.7.1}$$

for $\mathsf t \in \operatorname{Std}(\mathcal{P}_n)$ and $1 \le r < n$ such that $\mathsf t s_r$ is standard. The γ-coefficients $\{\gamma_{\mathsf t}\}$ for this basis are explicitly known by [57, Corollary 3.29]. Moreover,

$$\det \underline{G}^{\boldsymbol\lambda} = \prod_{\mathsf t\in\operatorname{Std}(\boldsymbol\lambda)} \gamma_{\mathsf t} \tag{1.7.2}$$

By explicitly computing the scalars $\gamma_{\mathsf t}$, and using an intricate inductive argument based on the semisimple branching rules for the Specht modules, James and the author proved the following:

1.7.3. **Theorem** (James-Mathas [57, Corollary 3.38]). *Suppose that $\mathscr{H}_n$ is content separated. Then there exist explicitly known scalars $g_{\boldsymbol{\lambda\mu}}$ and signs $\varepsilon_{\boldsymbol{\lambda\mu}} = \pm1$ such that*

$$\det \underline{G}^{\boldsymbol\lambda} = \prod_{\substack{\boldsymbol\mu\in\mathcal{P}_n\\ \boldsymbol\mu\triangleright\boldsymbol\lambda}} g_{\boldsymbol{\lambda\mu}}^{\varepsilon_{\boldsymbol{\lambda\mu}}\dim \underline{S}^{\boldsymbol\lambda}}.$$

The scalars $g_{\boldsymbol{\lambda\mu}}$ are described combinatorially as the quotient of at most two *hook lengths* which are determined by $\boldsymbol\lambda$ and $\boldsymbol\mu$. The sign $\varepsilon_{\boldsymbol{\lambda\mu}}$ is the parity of the sum of the leg lengths of these hooks.

Theorem 1.7.3 is a very pretty closed formula for the Gram determinant $\underline{G}^{\boldsymbol\lambda}$ which generalizes a classical result of James and Murphy [59]. One problem with this formula is that $\det \underline{G}^{\boldsymbol\lambda}$ is a *polynomial* in $v, v^{-1}, Q_1, \dots, Q_\ell$ whereas Theorem 1.7.3 computes this determinant as a *rational function* in $v, Q_1, \dots, Q_\ell$. On the other hand, as we now recall, Theorem 1.7.3 has an impressive module theoretic application in terms of the *Jantzen sum formula*.

Fix a modular system $(K, \mathcal{Z}, F)$, where $\mathcal{Z}$ discrete valuation ring with maximal ideal $\mathfrak p$ and such that $\mathcal{Z}$ contains $v, v^{-1}, Q_1, \dots, Q_\ell$, Let K be the field of fractions of $\mathcal{Z}$ and let $F = \mathcal{Z}/\mathfrak p$ be the residue field of $\mathcal{Z}$. Let $\mathscr{H}_n^{\mathcal Z}$, $\mathscr{H}_n^K \cong \mathscr{H}_n^{\mathcal Z} \otimes_{\mathcal Z} K$ and $\mathscr{H}_n^F = \mathscr{H}_n^{\mathcal Z} \otimes_{\mathcal Z} F$ be the corresponding Hecke algebras. Therefore, $\mathscr{H}_n^F$ has Hecke parameter $v + \mathfrak p$ and cyclotomic parameters $Q_l + \mathfrak p$, for $1 \le l \le \ell$.

Let $\boldsymbol\lambda \in \mathcal{P}_n$ and let $\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}$ and $\underline{S}^{\boldsymbol\lambda}_F \cong \underline{S}^{\boldsymbol\lambda}_{\mathcal Z} \otimes_{\mathcal Z} F$ be the corresponding Specht modules for $\mathscr{H}_n^{\mathcal Z}$ and $\mathscr{H}_n^F$, respectively. Define a filtration $\{\, J_k(\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}) \mid k \ge 0\,\}$ of $\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}$ by $J_k(\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}) = \{\, x \in \underline{S}^{\boldsymbol\lambda}_{\mathcal Z} \mid \langle x, y\rangle_{\boldsymbol\lambda} \in \mathfrak p^k \,\}$, for $k \ge 0$. The **Jantzen filtration** of $\underline{S}^{\boldsymbol\lambda}$ is the filtration $\underline{S}^{\boldsymbol\lambda} = J_0(\underline{S}^{\boldsymbol\lambda}_F) \supseteq J_1(\underline{S}^{\boldsymbol\lambda}_F) \supseteq \dots \supseteq J_z(\underline{S}^{\boldsymbol\lambda}_F) = 0$, where $J_k(\underline{S}^{\boldsymbol\lambda}_F) = \big(J_k(\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}) + \mathfrak p\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}\big)/\mathfrak p\underline{S}^{\boldsymbol\lambda}_{\mathcal Z}$ for $k \ge 0$.

Let $\operatorname{Rep}(\mathscr{H}_n)$ be the category of finitely generated $\mathscr{H}_n$-modules and let $[\operatorname{Rep}(\mathscr{H}_n)]$ be its Grothendieck group. Let $[M]$ be the image of the $\mathscr{H}_n$-module M in $[\operatorname{Rep}(\mathscr{H}_n)]$. Let $\nu_{\mathfrak p}$ be the $\mathfrak p$-adic valuation map on $\mathcal{Z}^\times$.

1.7.4. **Theorem** (James-Mathas [57, Theorem 4.6]). *Suppose that $(K, \mathcal{Z}, F)$ is a modular system and that $\boldsymbol\lambda \in \mathcal{P}_n$. Then, in $[\operatorname{Rep}(\mathscr{H}_n^F)]$,*

$$\sum_{k>0} [J_k(\underline{S}^{\boldsymbol\lambda}_F)] = \sum_{\boldsymbol\mu\triangleright\boldsymbol\lambda} \varepsilon_{\boldsymbol{\lambda\mu}} \nu_{\mathfrak p}(g_{\boldsymbol{\lambda\mu}})[\underline{S}^{\boldsymbol\mu}_F].$$

Intuitively, the proof of Theorem 1.7.4 amounts to taking the $\mathfrak p$-adic valuation of the formula in Theorem 1.7.3. In fact, this is exactly how Theorem 1.7.4 is proved except that you need the corresponding formulas for the Gram determinants of the weight spaces of the Weyl modules of the cyclotomic Schur algebras of [29]. This is enough because the dimensions of the weight spaces of a module uniquely determine the image of the module in the Grothendieck group. The proof given in [57] is stated only for the non-degenerate case $v^2 \ne 1$, however, the arguments apply equally well for the degenerate case when $v^2 = 1$.

The main point that we want to emphasize in this section is that the *rational* formula for $\det \underline{G}^{\boldsymbol\lambda}_F$ in Theorem 1.7.3 corresponds to writing the lefthand side of the Jantzen sum formula sum as a $\mathbb{Z}$-linear combination of Specht modules. Therefore, when the righthand side of the sum formula is written as a linear combination of simple modules some of the terms must cancel. We give a cancellation free sum formula in §4.1.

Theorem 1.7.4 is a useful inductive tool because it gives an upper bound on the decomposition numbers of $S^{\boldsymbol\lambda}$. Let $j_{\boldsymbol{\lambda\mu}} = \varepsilon_{\boldsymbol{\lambda\mu}}\nu_{\mathfrak p}(g_{\boldsymbol{\lambda\mu}})$, for $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal{P}_n$ and set $d^F_{\boldsymbol{\lambda\mu}} = [\underline{S}^{\boldsymbol\lambda}_F : \underline{D}^{\boldsymbol\mu}_F]$. Using Theorem 1.7.4 to compute the multiplicity of $\underline{D}^{\boldsymbol\mu}_F$ in $\bigoplus_{k>0} J_k(\underline{S}^{\boldsymbol\lambda})$ yields the following.

1.7.5. **Corollary.** *Suppose that $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal{P}_n$. Then* $0 \le d^F_{\boldsymbol{\lambda\mu}} \le \sum_{\substack{\boldsymbol\nu \in \mathcal{P}_n \\ \boldsymbol\lambda \rhd \boldsymbol\nu \unrhd \boldsymbol\mu}} j_{\boldsymbol{\lambda\nu}} d^F_{\boldsymbol{\nu\mu}}$.

A second application, Theorem 1.7.4 classifies the irreducible Specht modules $\underline{S}^{\boldsymbol\lambda}$, for $\boldsymbol\lambda \in \mathcal{K}^\Lambda_n$.

1.7.6. **Corollary** (James-Mathas [57, Theorem 4.7])**.** *Suppose that $\boldsymbol\lambda \in \mathcal{K}^\Lambda_n$. Then the Specht module $\underline{S}^{\boldsymbol\lambda}$ is irreducible if and only if $j_{\boldsymbol{\lambda\mu}} = 0$ for all $\boldsymbol\mu \rhd \boldsymbol\lambda$.*

1.8. **The blocks of $\mathscr{H}_n$.** The most important application of the Jantzen Sum Formula (Theorem 1.7.4) is to the classification of the blocks of $\mathscr{H}^F_n$. The algebra $\mathscr{H}_n$, and in fact any algebra over a field, can be written as a direct sum of indecomposable two-sided ideals: $\mathscr{H}^F_n = B_1 \oplus \dots \oplus B_d$. The subalgebras $B_1, \dots, B_z$, which are the **blocks** of $\mathscr{H}_n$, are uniquely determined up to permutation. Any $\mathscr{H}^F_n$-module M splits into a direct sum of block components $M = MB_1 \oplus \dots \oplus MB_d$, where we allow some of the summands to be zero. The module M **belongs** to the block B_r if $M = MB_r$. It is a standard fact that two simple modules $\underline{D}^{\boldsymbol\lambda}$ and $\underline{D}^{\boldsymbol\mu}$ belong to the same block if and only if they are in the same **linkage class**. That is, there exists a sequence of multipartitions $\boldsymbol\nu_0 = \boldsymbol\lambda$, $\boldsymbol\nu_1, \dots, \boldsymbol\nu_z = \boldsymbol\mu$ such that $[\underline{S}^{\boldsymbol\nu_r} : \underline{D}^{\boldsymbol\nu_{r+1}}] \ne 0$ or $[\underline{S}^{\boldsymbol\nu_{r+1}} : \underline{D}^{\boldsymbol\nu_r}] \ne 0$, for $0 \le r < z$.

We want an explicit combinatorial description of the blocks of $\mathscr{H}^F_n$. Define two equivalence relations $\sim_C$ and $\sim_J$ on $\mathcal{P}_n$ as follows. First, $\boldsymbol\lambda \sim_C \boldsymbol\mu$ if there is an equality of *multisets* $\{\, c^{\mathcal{Z}}_{\mathfrak{t}^{\boldsymbol\lambda}}(r) \mid 1 \le r \le n \,\} = \{\, c^{\mathcal{Z}}_{\mathfrak{t}^{\boldsymbol\mu}}(r) \mid 1 \le r \le n \,\}$. The second relation, **Jantzen equivalence**, is more involved: $\boldsymbol\lambda \sim_J \boldsymbol\mu$ if there exists a sequence $\boldsymbol\nu_0 = \boldsymbol\lambda, \boldsymbol\nu_1, \dots, \boldsymbol\nu_z = \boldsymbol\mu$ of multipartitions in $\mathcal{P}_n$ such that $j_{\boldsymbol\nu_r\boldsymbol\nu_{r+1}} \ne 0$ or $j_{\boldsymbol\nu_{r+1}\boldsymbol\nu_r} \ne 0$, for $0 \le r < z$.

1.8.1. **Theorem** (Lyle-Mathas [89], Brundan [17])**.** *Suppose that $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal{P}_n$. Then the following are equivalent:*

a) *$\underline{D}^{\boldsymbol\lambda}$ and $\underline{D}^{\boldsymbol\mu}$ are in the same $\mathscr{H}^F_n$-block.*
b) *$\underline{S}^{\boldsymbol\lambda}$ and $\underline{S}^{\boldsymbol\mu}$ are in the same $\mathscr{H}^F_n$-block.*
c) *$\boldsymbol\lambda \sim_J \boldsymbol\mu$.*
d) *$\boldsymbol\lambda \sim_C \boldsymbol\mu$.*

Parts (a) and (b) are equivalent by the general theory of cellular algebras [45] whereas the equivalence of parts (b) and (c) is a general property of Jantzen filtrations from [89]. (In fact, part (c) is general property of the standard modules of a quasi-hereditary algebra.) In practice, part (d) is the most useful because it easy to compute.

The hard part in proving Theorem 1.8.1 is in showing that parts (c) and (d) are equivalent. The argument is purely combinatorial with work of Fayers [34, 35] playing an important role.

In the integral case, when $\mathscr{H}^F_n = \mathscr{H}^\Lambda_n$ for some $\Lambda \in P^+$, there is a nice reformulation of Theorem 1.8.1. The **residue sequence** of a standard tableau $\mathfrak t$ is $\mathbf{i}^{\mathfrak t} = (i^{\mathfrak t}_1, \dots, i^{\mathfrak t}_n) \in I^n$ where $i^{\mathfrak t}_r = c^{\mathbb{Z}}_r(\mathfrak t) + e\mathbb{Z}$. If $\mathfrak t \in \operatorname{Std}(\boldsymbol\lambda)$, for $\boldsymbol\lambda \in \mathcal{P}_n$, define

$$\beta^{\boldsymbol\lambda} = \sum_{r=1}^n \alpha_{i^{\mathfrak t}_r} = \sum_{r=1}^n \alpha_{i^{\boldsymbol\lambda}_r} \in Q^+.$$

It is easy to see that $\beta^{\boldsymbol\lambda}$ depends only on $\boldsymbol\lambda$, and not on the choice of $\mathfrak t$. By definition, $\beta^{\boldsymbol\lambda} \in Q^+$. Moreover, $\boldsymbol\lambda \sim_C \boldsymbol\mu$ if and only if $\beta^{\boldsymbol\lambda} = \beta^{\boldsymbol\mu}$. Hence, we have the following:

1.8.2. **Corollary.** *Suppose that $\Lambda \in P^+$ and $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal{P}_n$. Then $\underline{S}^{\boldsymbol\lambda}$ and $\underline{S}^{\boldsymbol\mu}$ are in the same $\mathscr{H}^\Lambda_n$-block if and only if $\beta^{\boldsymbol\lambda} = \beta^{\boldsymbol\mu}$.*

2. Cyclotomic quiver Hecke algebras of type A

This section introduces the quiver Hecke algebras, and their cyclotomic quotients. We use the relations to reveal some of the properties of these algebras. The main aim of this section is to give the reader an appreciation of, and some familiarity with, the KLR relations without appealing to any general theory.

2.1. **Graded algebras.** In this section we quickly review the theory of graded (cellular) algebras. For more details the reader is referred to [13, 49, 107]. Throughout, $\mathcal{Z}$ is a commutative integral domain. Unless otherwise stated, all modules and algebras will be free and of finite rank as $\mathcal{Z}$-modules.

In this chapter a **graded module** will always mean a $\mathbb{Z}$-graded module. That is, a $\mathcal{Z}$-module M which has a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ as a $\mathcal{Z}$-module. A **positively graded** module is a graded module $M = \bigoplus_d M_d$ such that $M_d = 0$ if $d < 0$.

A **graded algebra** is a unital associative $\mathcal{Z}$-algebra $A=\bigoplus_{d\in\mathbb{Z}}A_d$ that is a graded $\mathcal{Z}$-module such that $A_dA_e\subseteq A_{d+e}$, for all $d,e\in\mathbb{Z}$. It follows that $1\in A_0$ and that A_0 is a graded subalgebra of A. A graded (right) A-module is a graded $\mathcal{Z}$-module M such that $\underline{M}$ is an $\underline{A}$-module and $M_dA_e\subseteq M_{d+e}$, for all $d,e\in\mathbb{Z}$, where $\underline{M}$ and $\underline{A}$ are the ungraded modules obtained by forgetting the $\mathbb{Z}$-grading on M and A respectively. Graded submodules, graded left A-modules and so on are all defined in the obvious way.

Suppose that M is a graded A-module. If $m\in M_d$, for $d\in\mathbb{Z}$, then m is **homogeneous** of **degree** d and we set $\deg m=d$. Every element $d\in M$ can be written uniquely as a linear combination $m=\sum_d m_d$ of its **homogeneous components**, where $\deg m_d=d$. Importantly, if M is a graded A-module and $m=\sum_d m_d\in M$ then $m_d\in M$, for all $d\in\mathbb{Z}$.

A homomorphism of graded A-modules M and N is an $\underline{A}$-module homomorphism $f\colon\underline{M}\longrightarrow\underline{N}$ such that $\deg f(m)=\deg m$, for all $m\in M$. That is, f is a degree preserving $\underline{A}$-module homomorphism. Let $\operatorname{Hom}_A(M,N)$ be the space of (degree preserving) homogeneous maps and set

$$\mathcal{H}om_A(M,N)=\bigoplus_{d\in\mathbb{Z}}\mathcal{H}om_A(M\langle d\rangle,N)\cong\bigoplus_{d\in\mathbb{Z}}\mathcal{H}om_A(M,N\langle -d\rangle).$$

The reader may check that $\mathcal{H}om_A(M,N)\cong\operatorname{Hom}_{\underline{A}}(\underline{M},\underline{N})$ as $\mathcal{Z}$-modules.

Let $\operatorname{Rep}(A)$ be the category of finitely generated graded A-modules together with degree preserving homomorphisms. Similarly, $\operatorname{Proj}(A)$ is the category of finitely generated projective A-modules with degree preserving maps.

If M is a graded $\mathcal{Z}$-module and $s\in\mathbb{Z}$ let $M\langle s\rangle$ be the graded $\mathcal{Z}$-module obtained by shifting the grading on M up by s; that is, $M\langle s\rangle_d=M_{d-s}$, for $d\in\mathbb{Z}$. Then $M\cong M\langle s\rangle$ as A-modules if and only if $s=0$. In contrast, $\underline{M}\cong\underline{M\langle s\rangle}$ as $\underline{A}$-modules, for all $s\in\mathbb{Z}$.

Suppose that q be an indeterminate and that M is a graded module. The **graded dimension** of M is the Laurent polynomial $\mathit{dim}_q M=\sum_{d\in\mathbb{Z}}(\dim M_d)q^d\in\mathbb{N}[q,q^{-1}]$. If M is a graded A-module and D is an irreducible graded A-module then the **graded decomposition number** is the Laurent polynomial

$$[M:D]_q=\sum_{s\in\mathbb{Z}}[M:D\langle s\rangle]\,q^s\in\mathbb{N}[q,q^{-1}].$$

By definition, the (ungraded) decomposition multiplicity $[\underline{M}:\underline{D}]$ is given by evaluating $[M:D]_q$ at $q=1$,

Suppose that A is a graded algebra and that $\underline{m}$ is an (ungraded) $\underline{A}$-module. A graded **lift** of $\underline{m}$ is a graded A-module M such that $\underline{M}\cong\underline{m}$ as $\underline{A}$-modules. If M is a graded lift of $\underline{m}$ then so it $M\langle s\rangle$, for any $s\in\mathbb{Z}$, so graded lifts are not unique in general. If $\underline{m}$ is indecomposable then its graded lift, if it exists, is unique up to grading shift [13, Lemma 2.5.3].

Following [49], the theory of cellular algebras from §1.3 extends to the graded setting in a natural way.

2.1.1. **Definition** ([49, §2]). *Suppose that A is $\mathbb{Z}$-graded $\mathcal{Z}$-algebra that is free of finite rank over $\mathcal{Z}$. A **graded cell datum** for A is a cell datum $(\mathcal{P},T,C)$ together with a degree function*

$$\deg:\coprod_{\lambda\in\mathcal{P}}T(\lambda)\longrightarrow\mathbb{Z}$$

such that

(GC$_d$) *the element c_{st} is homogeneous of degree $\deg c_{\mathsf{st}}=\deg(\mathsf{s})+\deg(\mathsf{t})$, for all $\lambda\in\mathcal{P}$ and $\mathsf{s},\mathsf{t}\in T(\lambda)$.*

*In this case, A is a **graded cellular algebra** with **graded cellular basis** $\{c_{\mathsf{st}}\}$.*

We use $\star$ for the *homogeneous* cellular algebra involution of A which is determined by $c_{\mathsf{st}}^\star=c_{\mathsf{ts}}$, for $\mathsf{s},\mathsf{t}\in T(\lambda)$.

2.1.2. **Example** (Toy example) The most basic example of a graded algebra is the truncated polynomial ring $A=F[x]/(x^{n+1})$, for some integer $n>0$, where $\deg x=2$. As an ungraded algebra, $\underline{A}$ has exactly one simple module, namely the field F with x acting as multiplication by zero. This algebra is a graded cellular algebra with $\mathcal{P}=\{0,1,\dots,n\}$, with its natural order, and $T(d)=\{d\}$ and $c_{dd}=x^d$. The irreducible graded A-modules are $F\langle d\rangle$, for $d\in\mathbb{Z}$, and $\mathit{dim}_q A=1+q^2+\dots+q^{2n}$. ♢

2.1.3. **Example** Let $A=\operatorname{Mat}_n(\mathcal{Z})$ be the $\mathcal{Z}$-algebra of $n\times n$-matrices. The basis of matrix units $\{\,e_{st}\mid 1\le s,t\le n\,\}$ is a cellular basis for A, where $\mathcal{P}=\{\heartsuit\}$ and $T(\heartsuit)=\{1,2,\dots,n\}$. We want to put a non-trivial grading on A. Let $\{d_1,\dots,d_n\}\subset\mathbb{Z}$ be a set of integers such that $d_s+d_{n-s+1}=0$, for $1\le s\le n$. Set $c_{st}=e_{s(n-t+1)}$ and define a degree function $\deg\colon T(\heartsuit)\longrightarrow\mathbb{Z}$ by $\deg s=d_s$. Then $\{\,c_{st}\mid 1\le s,t\le n\,\}$ is a graded cellular basis of A. We have $\mathit{dim}_q A=\sum_{s=1}^{s}q^{d_s}$. In particular, semisimple algebras can have non-trivial gradings. ♢

Exactly as in §1.3, for each $\lambda\in\mathcal{P}$ we obtain a *graded* cell module C^λ with homogeneous basis $\{\,c_{\mathsf{t}}\mid \mathsf{t}\in T(\lambda)\,\}$ and $\deg c_{\mathsf{t}}=\deg\mathsf{t}$. Generalizing (1.3.2), the graded cell module C^λ comes equipped with a *homogeneous* symmetric bilinear form $\langle\ ,\ \rangle_\lambda$ of degree zero. Therefore, if $x,y\in C^\lambda$ then $\langle x,y\rangle_\lambda\neq0$ only if $\deg x+$

$\deg y = 0$. Moreover, $\langle xa, y\rangle_\lambda = \langle x, ya^\star\rangle_\lambda$, for all $x, y \in C^\lambda$ and all $a \in A$. Consequently, $\operatorname{rad} C^\lambda = \{\, x \in C^\lambda \mid \langle x, y\rangle_\lambda = 0 \text{ for all } y \in C^\lambda \,\}$ is a graded submodule of C^λ so that $D^\lambda = C^\lambda / \operatorname{rad} C^\lambda$ is a graded A-module. Let $\mathcal{P}_0 = \{\, \mu \in \mathcal{P} \mid D^\mu \neq 0 \,\}$.

2.1.4. **Theorem** (Hu-Mathas [49, Theorem 2.10]). *Suppose that $\mathcal{Z}$ is a field and that A is a graded cellular algebra. Then:*

a) *If $D^\lambda \neq 0$, for $\lambda \in \mathcal{P}$, then D^λ is an absolutely irreducible graded A-module and $(D^\lambda)^\circledast \cong D^\lambda$.*
b) *$\{\, D^\lambda\langle s\rangle \mid \lambda \in \mathcal{P}_0 \text{ and } s \in \mathbb{Z} \,\}$ is a complete set of pairwise non-isomorphic irreducible (graded) A-modules.*
c) *If $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{P}_0$ then $[C^\lambda{:}D^\mu]_q \neq 0$ only if $\lambda \trianglerighteq \mu$. Moreover, $[C^\mu{:}D^\mu]_q = 1$.*

Forgetting the grading, the basis $\{c_{\mathsf{st}}\}$ is still a cellular basis of $\underline{A}$. Comparing Theorem 1.3.4 and Theorem 2.1.4 it follows that every (ungraded) irreducible $\underline{A}$-module has a graded lift that is unique up to shift. Conversely, if D is an irreducible graded A-module then $\underline{D}$ is an irreducible $\underline{A}$-module (this holds more generally for any finite dimensional graded algebra; see [107, Theorem 4.4.4]). It is an instructive exercise to prove that if A is a finite dimensional graded algebra then every simple $\underline{A}$-module has a graded lift and, up to shift, every graded simple A-module is of this form.

By [44, Theorems 3.2 and 3.3] every projective indecomposable $\mathscr{H}_n^\Lambda$-module has a graded lift. More generally, as shown in [107, §4], if M is a finitely generated graded A-module then the Jacobson radical of $\underline{M}$ has a graded lift.

The matrix $\mathbf{D}_A(q) = ([C^\lambda : D^\mu]_q)_{\lambda\in\mathcal{P},\mu\in\mathcal{P}_0}$ is the **graded decomposition matrix** of A. For each $\mu \in \mathcal{P}_0$ let P^μ be the projective cover of D^μ in $\operatorname{Rep}(A)$. The matrix $\mathbf{C}_A(q) = ([P^\lambda : D^\mu]_q)_{\lambda,\mu\in\mathcal{P}_0}$ is the **graded Cartan matrix** of A.

An A-module M has a **cell filtration** if it has a filtration $M = M_0 \supset M_1 \supset \cdots \supset M_z \supset 0$ such that all of the subquotients M_r/M_{r+1} are isomorphic to graded cell module, up to shift. Fixing isomorphisms $M_r/M_{r+1} \cong C^{\lambda_r}\langle d_r\rangle$, for some $\lambda_r \in \mathcal{P}$ and $d_r \in \mathbb{Z}$, define $(M : C^\lambda)_q = \sum_d m_d q^d$, where $m_d = \#\{\, 1 \le r \le z \mid \lambda_r = \lambda \text{ and } d_r = d \,\}$. In general, the multiplicities $(M : C^\lambda)_q$ depend upon the choice of filtration *and* the labelling of the isomorphisms $M_r/M_{r+1} \cong C^{\lambda_r}\langle d_r\rangle$ because the cell modules are not guaranteed to be pairwise non-isomorphic, even up to shift.

2.1.5. **Corollary** ([49, Theorem 2.17]). *Suppose that $\mathcal{Z} = F$ is a field. If $\mu \in \mathcal{P}_0$ then P^μ has a cell filtration such that $(P^\mu : C^\lambda)_q = [C^\lambda : D^{\boldsymbol{\mu}}]_q$, for all $\lambda \in \mathcal{P}$. Consequently, $\mathbf{C}_A(q) = \mathbf{D}_A(q)^{tr}\mathbf{D}_A(q)$ is a symmetric matrix.*

2.2. **Cyclotomic quiver Hecke algebras.** We are now ready to define cyclotomic quiver Hecke algebras. We start by defining the affine versions of these algebras and then pass to the cyclotomic quotients. Through this section we will make extensive use of the Lie theoretic data that is attached to the quiver Γ_e in §1.2.

2.2.1. **Definition** (Khovanov and Lauda [67, 68] and Rouquier [114]). *Suppose that $e \ge 2$ and $n \ge 0$. The **quiver Hecke algebra**, or **Khovanov-Lauda–Rouquier algebra**, of type Γ_e is the unital associative $\mathcal{Z}$-algebra $\mathscr{R}_n = \mathscr{R}_n(\mathcal{Z})$ with generators $\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$ and relations*

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}}e(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in I^n} e(\mathbf{i}) = 1,$$
$$y_r e(\mathbf{i}) = e(\mathbf{i})y_r, \qquad \psi_r e(\mathbf{i}) = e(s_r{\cdot}\mathbf{i})\psi_r, \qquad y_r y_s = y_s y_r,$$
$$\psi_r\psi_s = \psi_s\psi_r, \qquad \text{if } |r-s| > 1,$$
$$\psi_r y_s = y_s\psi_r, \qquad \text{if } s \neq r, r+1,$$
$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r\psi_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \qquad y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{i_r i_{r+1}})e(\mathbf{i}), \tag{2.2.2}$$
$$\psi_r^2 e(\mathbf{i}) = \begin{cases} (y_{r+1} - y_r)(y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \rightleftarrows i_{r+1},\\ (y_r - y_{r+1})e(\mathbf{i}), & \text{if } i_r \to i_{r+1},\\ (y_{r+1} - y_r)e(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1},\\ 0, & \text{if } i_r = i_{r+1},\\ e(\mathbf{i}), & \text{otherwise,} \end{cases} \tag{2.2.3}$$
$$(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i}) = \begin{cases} (y_r + y_{r+2} - 2y_{r+1})e(\mathbf{i}), & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ -e(\mathbf{i}), & \text{if } i_{r+2} = i_r \to i_{r+1},\\ e(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\ 0, & \text{otherwise,} \end{cases} \tag{2.2.4}$$

for $\mathbf{i}, \mathbf{j} \in I^n$ and all admissible r and s. Moreover, $\mathscr{R}_n^\Lambda$ is naturally $\mathbb{Z}$-graded with degree function determined by

$$\deg e(\mathbf{i}) = 0, \qquad \deg y_r = 2 \qquad \text{and} \qquad \deg \psi_s e(\mathbf{i}) = -c_{i_s, i_{s+1}},$$

for $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in I^n$.

Khovanov and Lauda [67,68] and Rouquier [114] define quiver Hecke algebras for quivers of arbitrary type. In the short time since their inception a lot has been discovered about these algebras. The first important result is that these algebras categorify the negative part of the corresponding quantum group [20,67,115,125].

2.2.5. *Remark.* We have defined only a special case of the quiver Hecke algebras defined in [67,114]. In addition to allowing arbitrary quivers, Khovanov and Lauda allow a more general choice of signs. Rouquier's definition, which is the most general, defines the quiver Hecke algebras in terms of a matrix $Q=(Q_{ij})_{i,j\in I}$ with entries in a polynomial ring $\mathcal{Z}[u,v]$ with the properties that $Q_{ii}=0$, Q_{ij} is not a zero divisor in $\mathcal{Z}[u,v]$ for $i\ne j$ and $Q_{ij}(u,v)=Q_{ji}(v,u)$, for $i,j\in I$. For an arbitrary quiver Γ, Rouquier [114, Definition 3.2.1] defines $\mathscr{R}_n(\Gamma)$ to be the algebra generated by $\psi_r, y_s, e(\mathbf{i})$ subject to the relations above except that the quadratic and braid relations are replaced with

$$\psi_r^2e(\mathbf{i}) = Q_{i_r,i_{r+1}}(y_r,y_{r+1})e(\mathbf{i}),$$

$$(\psi_r\psi_{r+1}\psi_r-\psi_{r+1}\psi_r\psi_{r+1})e(\mathbf{i}) = \begin{cases}\frac{Q_{i_r,i_{r+1}}(y_r,y_{r+1})-Q_{i_r,i_{r+1}}(y_r,y_{r+1})}{y_{r+2}-y_r}, & \text{if } i_{r+2}-i_r,\\ o, & \text{otherwise.}\end{cases}$$

The assumptions on Q ensure that the last expression is a polynomial in the generators. In general, $y_re(\mathbf{i})$ is homogeneous of degree $(\alpha_{i_r},\alpha_{i_r})$, for $1\le r\le n$ and $\mathbf{i}\in I^n$. Under some mild assumptions, $\mathscr{R}_n$ is independent of the choice of Q by [114, Proposition 3.12]. We leave it to the reader to find a suitable matrix Q for Definition 2.2.1.

For $\beta\in Q^+$ let $I^\beta=\{\,\mathbf{i}\in I^n \mid \beta=\alpha_{i_1}+\dots+\alpha_{i_n}\,\}$. Then $I^n=\bigsqcup_\beta I^\beta$ is the decomposition of I^n into a disjoint union of $\mathfrak{S}_n$-orbits. Define

$$\mathscr{R}_\beta=\mathscr{R}_ne_\beta, \qquad \text{where } e_\beta=\sum_{\mathbf{i}\in I^\beta}e(\mathbf{i}). \tag{2.2.6}$$

Then $\mathscr{R}_\beta=e_\beta\mathscr{R}_ne_\beta$ is a two-sided ideal of $\mathscr{R}_n$ and $\mathscr{R}_n=\bigoplus_{\beta\in Q^+}\mathscr{R}_\beta$ is the decomposition of $\mathscr{R}_n$ into blocks. That is, $\mathscr{R}_\beta$ is indecomposable for all $\beta\in Q^+$.

For the rest of these notes for $w\in\mathfrak{S}_n$ fix a reduced expression $w=s_{r_1}\dots s_{r_k}$, with $1\le r_j<n$. Using this fixed reduced expression for w define $\psi_w=\psi_{r_1}\dots\psi_{r_k}$.

2.2.7. **Example** As the ψ-generators of $\mathscr{R}_n$ do not satisfy the braid relations the element ψ_w will, in general, depend upon the choice of reduced expression for $w\in\mathfrak{S}_n$. For example, by (2.2.4) if $e\ne2$, $n=3$ and $w=s_1s_2s_1=s_2s_1s_2$ then $\psi_1\psi_2\psi_1e(0,2,0)=\psi_2\psi_1\psi_2e(0,2,0)+e(0,2,0)$, by (2.2.4). Therefore, the two different reduced expressions for w lead to different elements $\psi_w\in\mathscr{R}_n$. ◊

The (fixed) choice of reduced expression for each $w\in W$ is completely arbitrary. Even though ψ_w is not uniquely determined by w, these elements form part of a basis of $\mathscr{R}_n$.

2.2.8. **Theorem** (Khovanov-Lauda [67, Theorem 2.5], Rouquier [114, Theorem 3.7]). *Suppose that $\beta\in Q^+$. Then $\mathscr{R}_\beta(\mathcal{Z})$ is free as an $\mathcal{Z}$-algebra with homogeneous basis*

$$\{\,\psi_wy_1^{a_1}\dots y_n^{a_n}e(\mathbf{i}) \mid w\in\mathfrak{S}_n,\ a_1,\dots,a_n\in\mathbb{N} \text{ and } \mathbf{i}\in I^\beta\,\}.$$

We note that Li [85, Theorem 4.3.10] has given a graded cellular basis of $\mathscr{R}_n$ and, in the special case when $e=\infty$, that Kleshchev, Loubert and Miemietz [74] have given a graded affine cellular basis of $\mathscr{R}_n$, in the sense of Koenig and Xi [79].

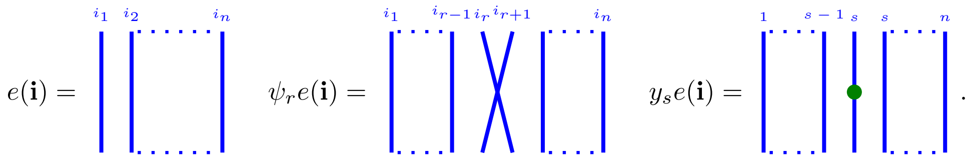
In these notes we are not directly concerned with the quiver Hecke algebras $\mathscr{R}_n$. Rather, we are more interested in *cyclotomic quotients* of these algebras.

2.2.9. **Definition** (Brundan-Kleshchev [19]). *Suppose that $\Lambda\in P^+$. The* ***cyclotomic quiver Hecke algebra*** *of type Γ_e and weight Λ is the quotient algebra $\mathscr{R}_n^\Lambda=\mathscr{R}_n/\langle y_1^{(\Lambda,\alpha_{i_1})}e(\mathbf{i}) \mid \mathbf{i}\in I^n\rangle$.*

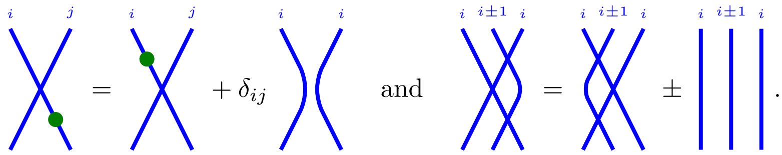
We abuse notation and identify the KLR generators of $\mathscr{R}_n$ with their images in $\mathscr{R}_n^\Lambda$. That is, we consider $\mathscr{R}_n$ to be generated by $\psi_1,\dots,\psi_{n-1},y_1,\dots,y_n$ and $e(\mathbf{i})$, for $\mathbf{i}\in I^n$, subject to the relations in Definition 2.2.1 and Definition 2.2.9.

When Λ is a weight of level 2, the algebras $\mathscr{R}_n^\Lambda$ first appeared in the work of Brundan and Stroppel [24] in their series of papers on the Khovanov diagram algebras. In full generality, the cyclotomic quotients of $\mathscr{R}_n$ were introduced by Khovanov-Lauda [67] and Rouquier [114]. Brundan and Kleshchev were the first to systematically study the cyclotomic quiver Hecke algebras $\mathscr{R}_n^\Lambda$, for any $\Lambda\in P^+$.

Although we will not need this here we note that, rather than working algebraically, it is often easier to work diagrammatically by identifying the elements of $\mathscr{R}_n^\Lambda$ with certain planar diagrams. In these diagrams, the end-points of the strings are labeled by $\{1,2,\dots,n,1',2',\dots,n'\}$ and the strings themselves are coloured by I^n. For example, following [67], the KLR generators can be identified with the diagrams:



Multiplication of diagrams is given by concatenation, read from top to bottom, subject to the relations above which are also interpreted diagrammatically. As an exercise, we leave it to the reader to identify the two relations in Definition 2.2.1 which correspond to the following 'local' relations on strings inside braid diagrams:



(For the second relation, $e\neq 2$.) For more rigorous definitions of such diagrams, and non-trivial examples of their application, we refer the reader to the works [48, 75, 85, 90] which, among others, use variations of these diagrams extensively.

2.2.10. **Example** (Rank one algebras) Suppose that $n=1$ and $\Lambda\in P^+$. Then

$$\mathscr{R}_1^\Lambda = \langle y_1, e(i) \mid y_1 e(i) = e(i) y_1 \text{ and } y_1^{\langle \Lambda,\alpha_i\rangle} e(\mathbf{i}) = 0, \text{ for } i\in I\rangle,$$

with $\deg y_1 = 2$ and $\deg e(i) = 0$, for $i\in I$. Therefore, there is an isomorphism of graded algebras

$$\mathscr{R}_1^\Lambda \cong \bigoplus_{\substack{i\in I\\ (\Lambda,\alpha_i)>0}} \mathcal{Z}[y]/y^{(\Lambda,\alpha_i)}\mathcal{Z}[y],$$

where $y=y_1$ is in degree 2. Armed with this description of $\mathscr{R}_n^\Lambda$ it is now straightforward to show that $\mathscr{H}_n^\Lambda \cong \mathscr{R}_n^\Lambda$ when $\mathcal{Z}$ is a field and $n=1$. $\Diamond$

2.3. Nilpotence and small representations.

In this section and the next we use the KLR relations to prove some results about the cyclotomic quiver Hecke algebras $\mathscr{R}_n^\Lambda$ for particular Λ and n.

By Theorem 2.2.8 the algebra $\mathscr{R}_n$ is infinite dimensional, so it is not obvious from the relations that the cyclotomic Hecke algebra $\mathscr{R}_n^\Lambda$ is finite dimensional — or even that $\mathscr{R}_n^\Lambda$ is non-zero. The following result shows that y_r is nilpotent, for $1\le r\le n$, which implies that $\mathscr{R}_n^\Lambda$ is finite dimensional.

2.3.1. **Lemma** (Brundan and Kleshchev [19, Lemma 2.1]). *Suppose that $1\le r\le n$ and $\mathbf{i}\in I^n$. Then $y_r^N e(\mathbf{i}) = 0$ for $N\gg 0$.*

Proof. We argue by induction on r. If $r=1$ then $y_1^{(\Lambda,\alpha_{i_1})}e(\mathbf{i})=0$ by Definition 2.2.9, proving the base step of the induction. Now consider $y_{r+1}e(\mathbf{i})$. By induction, we may assume that there exists $N\gg 0$ such that $y_r^N e(\mathbf{j}) = 0$, for all $\mathbf{j}\in I^n$. There are three cases to consider.

Case 1. $i_{r+1}\not\to i_r$.
By (2.2.3) and (2.2.2), $y_{r+1}^N e(\mathbf{i}) = y_{r+1}^N\psi_r^2 e(\mathbf{i}) = \psi_r y_r^N \psi_r e(\mathbf{i}) = \psi_r y_r^N e(s_r\cdot\mathbf{i})\psi_r = 0$, where the last equality follows by induction.

Case 2. $i_{r+1} = i_r\pm 1$.
Suppose first that $e\neq 2$. This is a variation on the previous case, with a twist. By (2.2.3) and (2.2.2), again

$$\begin{aligned}
y_{r+1}^{2N}e(\mathbf{i}) &= y_{r+1}^{2N-1}y_r e(\mathbf{i}) + y_{r+1}^{2N-1}(y_{r+1}-y_r)e(\mathbf{i})\\
&= y_r y_{r+1}^{2N-1}e(\mathbf{i}) \pm y_{r+1}^{2N-1}\psi_r^2 e(\mathbf{i})\\
&= y_r y_{r+1}^{2N-1}e(\mathbf{i}) \pm \psi_r y_r^{2N-1}e(s_r\cdot\mathbf{i})\psi_r\\
&= y_r y_{r+1}^{2N-1}e(\mathbf{i}) = \cdots = y_r^N y_{r+1}^N e(\mathbf{i}) = 0.
\end{aligned}$$

The case when $e=2$ is similar. First, observe that $y_{r+1}^2 e(\mathbf{i}) = (2y_r y_{r+1} - y_r^2 - \psi_r^2)e(\mathbf{i})$ by (2.2.3). Therefore, arguing as before, $y_{r+1}^{3N}e(\mathbf{i}) = y_r(2y_{r+1}-y_r)y_{r+1}^{3N-2}e(\mathbf{i}) = \cdots = y_r^N(2y_{r+1}-y_r)^N y_{r+1}^N e(\mathbf{i}) = 0$.

Case 3. $i_{r+1} = i_r$.
Let $\phi_r = \psi_r(y_r - y_{r+1})$. Then $\phi_r\psi_r e(\mathbf{i}) = -2\psi_r e(\mathbf{i})$ by (2.2.2), so that $(1+\phi_r)^2 e(\mathbf{i}) = e(\mathbf{i})$. Moreover,

$$(1+\phi_r)y_r(1+\phi_r)e(\mathbf{i}) = (y_r + \phi_r y_r + y_r\phi_r + \phi_r y_r\phi_r)e(\mathbf{i}) = y_{r+1}e(\mathbf{i}),$$

where the last equality is a small calculation using (2.2.2). Now we are done because

$$y_{r+1}^N e(\mathbf{i}) = \big((1+\phi_r)y_r(1+\phi_r)\big)^N e(\mathbf{i}) = (1+\phi_r)y_r^N(1+\phi_r)e(\mathbf{i}) = 0,$$

since ϕ_r commutes with $e(\mathbf{i})$ and $y_r^N e(\mathbf{i}) = 0$ by induction. □

We have marginally improved on Brundan and Kleshchev's original proof of Lemma 2.3.1 because, with a little more care, the argument gives an explicit bound for the nilpotency index of y_r. In general, this bound is far from sharp. For a better estimate of the nilpotency index of y_r see [52, Corollary 4.6] (and [48] when $e=\infty$). See [62, Lemma 4.4] for another argument which applies to cyclotomic quiver Hecke algebras of arbitrary type.

Combining Theorem 2.2.8 and Lemma 2.3.1 shows that $\mathscr{R}_n^\Lambda$ is a finite dimensional.

2.3.2. **Corollary** (Brundan and Kleshchev [19, Corollary 2.2])**.** *Suppose $\mathcal{Z}$ is a field. Then $\mathscr{R}_n^\Lambda$ is finite dimensional.*

As our next exercise we classify the one dimensional representations of $\mathscr{R}_n^\Lambda$ when $\mathcal{Z}=F$ is a field. For $i\in I$ let $\mathbf{i}_n^+ = (i, i+1, \dots, i+n-1)$ and $\mathbf{i}_n^- = (i, i-1, \dots, i-n+1)$. Then $\mathbf{i}_n^\pm \in I^n$. If $(\Lambda, \alpha_i) = 0$ then $e(\mathbf{i}_\pm^n) = 0$ by Definition 2.2.9. However, if $(\Lambda,\alpha_i)\ne 0$ then using the relations it is easy to see that $\mathscr{R}_n$ has unique one dimensional representations $D_{i,n}^+ = Fd_{i,n}^+$ and $D_{i,n}^- = Fd_{i,n}^-$ such that

$$d_{i,n}^\pm e(\mathbf{i}) = \delta_{\mathbf{i},\mathbf{i}_\pm^n} d_{i,n}^\pm \quad\text{ and }\quad d_{i,n}^+ y_r = 0 = d_{i,n}^\pm \psi_s,$$

for $\mathbf{i}\in I^n$, $1\le r\le n$ and $1\le s<n$ and such that $\deg d_{i,n}^\pm = 0$. In particular, this shows that $e(\mathbf{i}_\pm^n)\ne 0$. If $e\ne 2$ then $\{\, D_n^\pm(i) \mid i\in I \text{ and } (\Lambda,\alpha_i)\ne 0\,\}$ are pairwise non-isomorphic irreducible representations of $\mathscr{R}_n^\Lambda$. If $e=2$ then $\mathbf{i}_n^+ = \mathbf{i}_n^-$ so that $D_{i,n}^+ = D_{i,n}^-$.

2.3.3. **Proposition.** *Suppose that $\mathcal{Z}=F$ is a field and that D is a one dimensional graded $\mathscr{R}_n^\Lambda$-module. Then $D\cong D_{i,n}^\pm\langle k\rangle$, for some $k\in\mathbb{Z}$ and $i\in I$ such that $(\Lambda,\alpha_i)\ne 0$.*

Proof. Let d be a non-zero element of D so that $D=Fd$. Then $d=\sum_{\mathbf{j}\in I^n} de(\mathbf{j})$ so that $de(\mathbf{i})\ne 0$ for some $\mathbf{i}\in I^n$. Moreover, $de(\mathbf{j})=0$ if and only if $\mathbf{j}=\mathbf{i}$ since otherwise $de(\mathbf{i})$ and $de(\mathbf{j})$ are linearly independent elements of D, contradicting assumption that D is one dimensional. Now, $\deg dy_r = 2+\deg d$, so $dy_r=0$, for $1\le r\le n$, since D is one dimensional. Similarly, $d\psi_r = de(\mathbf{i})\psi_r = 0$ if $i_r = i_{r+1}$ or $i_r = i_{r+1}\pm 1$ since in these cases $\deg e(\mathbf{i})\psi_r\ne 0$.

It remains to show that $\mathbf{i}=\mathbf{i}_\pm^n$ and that $(\Lambda,\alpha_{i_1})\ne 0$. First, since $0\ne d = d\,e(\mathbf{i})$ we have that $e(\mathbf{i})\ne 0$ so that $(\Lambda,\alpha_{i_1})\ne 0$ by Definition 2.2.9. To complete the proof we show that if $\mathbf{i}\ne\mathbf{i}_\pm^n$ then $d=0$, which is a contradiction. First, suppose that $i_r = i_{r+1}$ for some r, with $1\le r<n$. Then $d = de(\mathbf{i}) = d\big(\psi_r y_{r+1} - y_r\psi_r\big) = 0$ by (2.2.2), which is not possible so $i_r\ne i_{r+1}$. Next, suppose that $i_{r+1}\ne i_r\pm 1$. Then $d = de(\mathbf{i}) = d\psi_r^2 e(\mathbf{i}) = d\psi_r e(s_r\cdot\mathbf{i})\psi_r = 0$ because D is one dimensional and $de(\mathbf{j})=0$ if $\mathbf{j}\ne\mathbf{i}$. This is another contradiction, so we must have $i_{r+1} = i_r\pm 1$ for $1\le r<n$. Therefore, if $\mathbf{i}\ne\mathbf{i}_\pm^n$ then $e\ne 2$, $n>2$ and $i_r = i_{r+2} = i_{r+1}\pm 1$ for some r. Applying the braid relation (2.2.4),

$$d = de(\mathbf{i}) = \pm de(\mathbf{i})\big(\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1}\big) = 0,$$

a contradiction. Setting $k=\deg d$ it follows that $D\cong D_{i,n}^\pm\langle k\rangle$, completing the proof. □

2.4. **Semisimple KLR algebras.** Now that we understand the one dimensional representations of $\mathscr{R}_n^\Lambda$ we consider the semisimple representation theory of the cyclotomic quiver Hecke algebras. These results do not appear in the literature, but there will be no surprises for the experts because everything here can be easily deduced from results which are known. The main idea is to show by example how to use the quiver Hecke algebra relations.

Recall from Corollary 1.6.11 that $\mathscr{H}_n^\Lambda$ is semisimple if and only if $(\Lambda,\alpha_{i,n})\le 1$, for all $i\in I$. In this section we use this criterion to study $\mathscr{R}_n^\Lambda$.

Recall from §1.8 that $\mathbf{i}^\mathsf{t} = (i_1^\mathsf{t},\dots,i_n^\mathsf{t})$ is the residue sequence of $\mathsf{t}\in\operatorname{Std}(\mathcal{P}_n)$, where $i_r^\mathsf{t} = c_r^\mathbb{Z}(\mathsf{t}) + e\mathbb{Z}$. In §1.4 we defined addable and removable nodes. If $i\in I$ then a node $A=(l,r,c)$ is an i**-node** if $i = \kappa_l + c - r + e\mathbb{Z}$.

2.4.1. **Lemma.** *Suppose that $(\Lambda,\alpha_{i,n})\le 1$, for all $i\in I$, where $\Lambda\in P^+$ has height ℓ. Then $e>n\ell$. Moreover, if $\mathsf{s},\mathsf{t}\in\operatorname{Std}(\mathcal{P}_n)$ then $\mathsf{s}=\mathsf{t}$ if and only if $\mathbf{i}^\mathsf{s} = \mathbf{i}^\mathsf{t}$.*

Proof. By definition, if $\Lambda = \Lambda(\boldsymbol{\kappa})\in P^+$ then $(\Lambda,\alpha_{\overline{\kappa}_l})\ge 1$, for $1\le l\le\ell$. Therefore, if $(\Lambda,\alpha_{i,n})\le 1$, for all $i\in I$, then $\kappa_l\ne\kappa_{l'}\pm d$, for $0\le d\le n$ and $1\le l<l'\le\ell$. This forces $e>n\ell$.

For the second statement, observe that if $i\in I$ and $\boldsymbol{\mu}\in\mathcal{P}_m$, where $0\le m<n$, then $\boldsymbol{\mu}$ has at most one addable i-node since $(\Lambda,\alpha_{i,n})\le 1$. Hence, it follows easily by induction on n that if $\mathsf{s},\mathsf{t}\in\operatorname{Std}(\mathcal{P}_n)$ then $\mathsf{s}=\mathsf{t}$ if and only if $\mathbf{i}^s = \mathbf{i}^\mathsf{t}$. □

We could have proved Lemma 2.4.1 by appealing toTheorem 1.6.10 and Corollary 1.6.11. We caution the reader that if t is a standard tableau then the contents $c_r^{\mathbb{Z}}(\mathsf{t}) \in \mathbb{Z}$ and the residues $i_r^{\mathsf{t}} \in I$ are in general different.

Let $I_\Lambda^n = \{\, \mathbf{i}^{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n)\,\}$ be the set of residue sequences of all of the standard tableaux in $\operatorname{Std}(\mathcal{P}_n)$. As a consequence of the proof of Lemma 2.4.1, if $\mathbf{i} = \mathbf{i}^{\mathsf{t}} \in I_\Lambda^n$ and $i_{r+1} = i_r \pm 1$ then r and $r+1$ must be in either in the same row or in the same column of t. Hence, we have the following useful fact.

2.4.2. **Corollary.** *Let $\Lambda \in P^+$ with $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$. Suppose that $\mathbf{i} \in I_\Lambda^n$ and that $i_{r+1} = i_r \pm 1$. Then $s_r \cdot \mathbf{i} \notin I_\Lambda^n$.*

When $\Lambda = \Lambda_0$ the next result is due to Brundan and Kleshchev [19, §5.5]. More generally, Kleshchev and Ram [77, Theorem 3.4] prove a similar result for quiver Hecke algebras of simply laced type.

2.4.3. **Proposition** (Seminormal representations of $\mathscr{R}_n^\Lambda$)**.** *Suppose that $\mathcal{Z} = F$ is a field, $\Lambda \in P^+$ and that $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$. Then for each $\boldsymbol{\lambda} \in \mathcal{P}_n$ there is a unique irreducible graded $\mathscr{R}_n^\Lambda$-module $S^{\boldsymbol{\lambda}}$ with homogeneous basis $\{\, \psi_{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\}$ such that $\deg \psi_{\mathsf{t}} = 0$, for all $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, and where the $\mathscr{R}_n^\Lambda$-action is given by*

$$\psi_{\mathsf{t}} e(\mathbf{i}) = \delta_{\mathbf{i},\mathbf{i}^{\mathsf{t}}} \psi_{\mathsf{t}}, \qquad \psi_{\mathsf{t}} y_r = 0 \qquad \textit{and} \qquad \psi_{\mathsf{t}} \psi_r = v_{\mathsf{t}(r,r+1)},$$

where we set $v_{\mathsf{t}(r,r+1)} = 0$ if $\mathsf{t}(r,r+1)$ is not standard.

Proof. By Lemma 2.4.1, if $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\mathsf{s} = \mathsf{t}$ if and only if $\mathbf{i}^{\mathsf{s}} = \mathbf{i}^{\mathsf{t}}$. Moreover, $i_{r+1}^{\mathsf{t}} = i_r^{\mathsf{t}} \pm 1$ if and only if r and $r+1$ are in the same row or in the same column of t. Similarly, $i_r^{\mathsf{t}} \ne i_{r+1}^{\mathsf{t}}$ for any r. Consequently, since $\psi_{\mathsf{t}} = \psi_{\mathsf{t}} e(\mathbf{i}^{\mathsf{t}})$ almost all of the relations in Definition 2.2.1 are trivially satisfied. In fact, all that we need to check is that $\psi_1, \dots, \psi_{n-1}$ satisfy the braid relations of the symmetric group $\mathfrak{S}_n$ with ψ_r^2 acting as zero when $i_{r+1}^{\mathsf{t}} = i_r^{\mathsf{t}} \pm 1$, which follows automatically by Corollary 2.4.2. By the same reasoning if $\mathsf{t}(r, r+1)$ is standard then $\deg e(\mathbf{i}^{\mathsf{t}})\psi_r = 0$. Hence, we can set $\deg \psi_{\mathsf{t}} = 0$, for all $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. This proves that $S^{\boldsymbol{\lambda}}$ is a graded $\mathscr{R}_n^\Lambda$-module.

It remains to show that $S^{\boldsymbol{\lambda}}$ is irreducible. If $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\mathsf{s} = \mathsf{t}^{\boldsymbol{\lambda}} d(\mathsf{s}) = \mathsf{t} d(\mathsf{t})^{-1} d(\mathsf{s})$, so $\psi_{\mathsf{s}} = \psi_{\mathsf{t}} \psi_{d(\mathsf{t})^{-1}} \psi_{d(\mathsf{s})}$. Suppose that $x = \sum_{\mathsf{t}} r_{\mathsf{t}} \psi_{\mathsf{t}}$ is a non-zero element of $S^{\boldsymbol{\lambda}}$. If $r_{\mathsf{t}} \ne 0$ then $\psi_{\mathsf{t}} = \frac{1}{r_{\mathsf{t}}} x e(\mathbf{i}^{\mathsf{t}})$, so it follows that $\psi_{\mathsf{s}} \in x \mathscr{R}_n^\Lambda$, for any $\mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda})$. Therefore, $S^{\boldsymbol{\lambda}} = x\mathscr{R}_n^\Lambda$ so that $S^{\boldsymbol{\lambda}}$ is irreducible as claimed. □

Consequently, $e(\mathbf{i}) \ne 0$ in $\mathscr{R}_n^\Lambda$, for all $\mathbf{i} \in I_n^\Lambda$. This was not clear until now.

We want to show that Proposition 2.4.3 describes all of the graded irreducible representations of $\mathscr{R}_n^\Lambda$, up to degree shift. To do this we need a better understanding of the set I_Λ^n. Okounkov and Vershik [110, Theorem 6.7] explicitly described the set of all *content sequences* $(c_1^{\mathbb{Z}}(\mathsf{t}), \dots, c_n^{\mathbb{Z}}(\mathsf{t}))$ when $\ell = 1$. This combinatorial result easily extends to higher levels and so suggests a description of I_Λ^n.

If $\mathbf{i} \in I^n$ and $1 \le m \le n$ let $\mathbf{i}_m = (i_1, \dots, i_m)$. Then $\mathbf{i}_m \in I^m$ and $I_\Lambda^m = \{\, \mathbf{i}_m \mid \mathbf{i} \in I_\Lambda^n\,\}$.

2.4.4. **Lemma** (*cf.* Ogievetsky-d'Andecy [109, Proposition 5])**.** *Suppose that $(\Lambda, \alpha_{i,n}) \le 1$, for $i \in I$, and that $\mathbf{i} \in I^n$. Then $\mathbf{i} \in I_\Lambda^n$ if and only if it satisfies the following three conditions:*

a) $(\Lambda, \alpha_{i_1}) \ne 0$.
b) *If $1 < r \le n$ and $(\Lambda, \alpha_{i_r}) = 0$ then $\{i_r - 1, i_r + 1\} \cap \{i_1, \dots, i_{r-1}\} \ne \emptyset$.*
c) *If $1 \le s < r \le n$ and $i_r = i_s$ then $\{i_r - 1, i_r + 1\} \subseteq \{i_{s+1}, \dots, i_{r-1}\}$.*

Proof. Suppose that $\mathsf{t} \in \operatorname{Std}(\mathcal{P}_n)$ and let $\mathbf{i} = \mathbf{i}^{\mathsf{t}}$. We prove by induction on r that $\mathbf{i}_r \in I_\Lambda^r$. By definition, $i_1 = \kappa_t + e\mathbb{Z}$ for some t with $1 \le t \le \ell$, so (a) holds. By induction we may assume that the subsequence $(i_1, \dots, i_{r-1})$ satisfies properties (a)–(c). If $(\Lambda, \alpha_{i_r}) = 0$ then r does not sit in the first row and first column of any component of t, so t has an entry in the row directly above r or in the column immediately to the left of r — or both! Hence, there exists an integer s with $1 \le s < r$ such that $i_s^{\mathsf{t}} = i_r^{\mathsf{t}} \pm 1$. Hence, (b) holds. Finally, suppose that $i_r = i_s$ as in (c). As the residues of the nodes in different components of t are disjoint it follows that s and r are in same component of t and on the same diagonal. In particular, r is not in the first row or in the first column of its component in t. As t is standard, the entries in t which are immediately above or to the left of r are both larger than s and smaller than r. Hence, (c) holds.

Conversely, suppose that $\mathbf{i} \in I^n$ satisfies properties (a)–(c). We show by induction on m that $\mathbf{i}_m \in I_\Lambda^m$, for $1 \le m \le n$. If $m = 1$ then $\mathbf{i}_1 \in I_\Lambda^1$ by property (a). Now suppose that $1 < m < n$ and that $\mathbf{i}_m \in I_\Lambda^m$. By induction $\mathbf{i}_m = \mathbf{i}^{\mathsf{s}}$, for some $\mathsf{s} \in \operatorname{Std}(\mathcal{P}_m)$. Let $\boldsymbol{\nu} = \operatorname{Shape}(\mathsf{s})$. If $i \in I$ then $(\Lambda, \alpha_{i,n}) \le 1$, so the multipartition $\boldsymbol{\nu}$ can have at most one addable i-node. On the other hand, reversing the argument of the last paragraph, using properties (b) and (c) with $r = m+1$, shows that $\boldsymbol{\nu}$ has at least one addable i_{m+1}-node. Let A be the unique addable i_{m+1}-node of $\boldsymbol{\nu}$. Then $\mathbf{i}_{m+1} = \mathbf{i}^{\mathsf{t}}$ where $\mathsf{t} \in \operatorname{Std}(\mathcal{P}_{m+1})$ is the unique standard tableau such that $\mathsf{t}_{\downarrow m} = \mathsf{s}$ and $\mathsf{t}(A) = m+1$. Hence, $\mathbf{i} \in I_\Lambda^{m+1}$ as required. □

By Proposition 2.4.3, if $\mathbf{i} \in I_\Lambda^n$ then $e(\mathbf{i}) \ne 0$. We use Lemma 2.4.4 to show that $e(\mathbf{i}) = 0$ if $\mathbf{i} \notin I_\Lambda^n$. First, a result that holds for all $\Lambda \in P^+$.

2.4.5. **Lemma.** *Suppose that $\Lambda\in P^+$, $\mathbf{i}\in I^n$ and $e(\mathbf{i})\ne 0$. Then $(\Lambda,\alpha_{i_1})\ne 0$. Moreover, if $(\Lambda,\alpha_{i_r})=0$, for $1<r\le n$, then $\{i_r-1,i_r+1\}\cap\{i_1,\dots,i_{r-1}\}\ne\emptyset$.*

Proof. By Definition 2.2.9, $e(\mathbf{i})=0$ whenever $(\Lambda,\alpha_{i_1})=0$. To prove the second claim suppose that $(\Lambda,\alpha_{i_r})=0$ and $i_r\pm1\notin\{i_1,\dots,i_{r-1}\}$. By induction on r, we may assume that $i_r\ne i_s$ for $1\le s<r$. Applying (2.2.3) r-times,

$$\begin{aligned}e(\mathbf{i})&=\psi_{r-1}^2e(\mathbf{i})=\psi_{r-1}e(i_1,\dots,i_r,i_{r-1},i_{r+1},\dots,i_n)\psi_{r-1}\\&=\dots=\psi_{r-1}\dots\psi_1e(i_r,i_1,\dots,i_{r-1},i_{r+1},\dots,i_n)\psi_1\dots\psi_{r-1}=0,\end{aligned}$$

where the last equality follows because $(\Lambda,\alpha_{i_r})=0$. □

2.4.6. **Proposition.** *Suppose that $1\le m\le n$ and that $(\Lambda,\alpha_{i,m})\le 1$, for all $i\in I$. Then $y_1=\dots=y_m=0$ and if $\mathbf{i}\in I^n$ then $e(\mathbf{i})\ne0$ only if $\mathbf{i}_m\in I^m_\Lambda$.*

Proof. We argue by induction on r to show that $y_r=0$ and $e(\mathbf{i})=0$ if $\mathbf{i}_r\notin I^r_\Lambda$, for $1\le r\le m$. If $r=1$ this is immediate because $y_1^{(\Lambda,\alpha_{i_1})}e(\mathbf{i})=0$ by Definition 2.2.9 and $(\Lambda,\alpha_{i_1})\le1$ by assumption. Suppose then that $1<r\le m$.

We first show that $e(\mathbf{i})=0$ if $\mathbf{i}_r\notin I^r_\Lambda$. By induction, Lemma 2.4.4 and Lemma 2.4.5, it is enough to show that $e(\mathbf{i})=0$ whenever there exists $s<r$ such that $i_s=i_r$ and $\{i_r-1,i_r+1\}\subseteq\{i_{s+1},\dots,i_{r-1}\}$. We may assume that s is maximal such that $i_s=i_r$ and $1\le s<r$. There are several cases to consider.

Case 1. $r=s+1$.
By (2.2.2), $e(\mathbf{i})=(y_{s+1}\psi_s-\psi_sy_s)e(\mathbf{i})=y_{s+1}\psi_se(\mathbf{i})$, since $y_s=0$ by induction. Using this identity twice, reveals that $e(\mathbf{i})=y_{s+1}\psi_se(\mathbf{i})=y_{s+1}e(\mathbf{i})\psi_s=y_{s+1}^2\psi_se(\mathbf{i})\psi_s=y_{s+1}^2\psi_s^2e(\mathbf{i})=0$, where the last equality comes from (2.2.3). Therefore, $e(\mathbf{i})=0$ as we wanted to show.

Case 2. *$s<r-1$ and $\{i_r-1,i_r+1\}\cap\{i_{s+1},\dots,i_{r-1}\}=\emptyset$.*
By the maximality of s, $i_r\notin\{i_{s+1},\dots,i_{r-1}\}$. Therefore, arguing as in the proof of Lemma 2.4.5, there exists a permutation $w\in\mathfrak{S}_r$ such that $e(\mathbf{i})=\psi_we(i_1,\dots,i_s,i_r,i_{s+1},\dots,i_{r-1},i_{r+1},\dots,i_n)\psi_w$. Hence, $e(\mathbf{i})=0$ by Case 1.

Case 3. *$s<r-1$ and $\{i_r-1,i_r+1\}\cap\{i_{s+1},\dots,i_{r-1}\}=\{j\}$, where $j=i_r\pm1$.*
Let t be an index such that $i_t=j=i_r\pm1$ and $s<t<r$. Note that if there exists an integer t' such that $i_t=i_{t'}$ and $s<t<t'<r$ then we may assume that $i_s\in\{i_{t+1},\dots,i_{t'-1}\}$ by Lemma 2.4.4(c) and induction. Therefore, since s was chosen to be maximal, t is the unique integer such that $i_t=j$ and $s<t<r$. Hence, arguing as in Case 2, there exists a permutation $w\in\mathfrak{S}_r$ such that

$$e(\mathbf{i})=\psi_we(i_1,\dots,i_{s-1},i_{s+1},\dots,i_{t-1},i_s,i_t,i_r,i_{t+1},\dots,i_{r-1},i_{r+1},\dots,i_n)\psi_w.$$

For convenience, we identify $e(i_1,\dots,i_s,i_t,i_r,\dots,i_n)$ with $e(i,j,i)$, where $i=i_s=i_r$ and $j=i\pm1$. Then we are reduced to showing that $e(i,j,i)=0$. Since we have a sequence of length 3 we may assume that $e>3$ by Lemma 2.4.1. By (2.2.4),

$$\begin{aligned}e(i,j,i)&=\pm\big(\psi_1\psi_2\psi_1-\psi_2\psi_1\psi_2\big)e(i,j,i)\\&=\pm\psi_1\psi_2e(j,i,i)\psi_1\mp\psi_2\psi_1e(i,i,j)\psi_2\\&=\pm\psi_1\psi_2(y_3\psi_2-\psi_2y_2)e(j,i,i)\psi_1\mp\psi_2\psi_1(y_2\psi_1-\psi_1y_1)e(i,i,j)\psi_2,\end{aligned}$$

where for the last equality we have used (2.2.2) twice. Translating back to our previous notation, y_1 and y_2 correspond to y_{t-1} and y_t, respectively. By induction, if $t<r-1$ then $y_1=y_2=y_3=0$, so the displayed equation becomes $e(\mathbf{i})=0$. If $t=r-1$ then we only know that $y_1=y_2=0$, so $e(i,j,i)=\pm\psi_1\psi_2y_3\psi_2(j,i,i)\psi_1$. Hence, by (2.2.2),

$$e(i,j,i)=\pm\psi_1\psi_2(\psi_2y_2+1)e(j,i,i)\psi_1=\pm\psi_1\psi_2e(j,i,i)\psi_1=\pm\psi_1\psi_2\psi_1e(i,j,i).$$

Applying the last equation twice, and then using (2.2.3),

$$\begin{aligned}e(i,j,i)&=\pm\psi_1\psi_2\psi_1e(i,j,i)=\psi_1\psi_2\psi_1^2\psi_2\psi_1e(i,j,i)\\&=\pm\psi_1\psi_2(y_2-y_1)\psi_2\psi_1e(i,j,i)=0,\end{aligned}$$

where last equation follows because $y_1=y_2=0$ by induction. Consequently, $e(\mathbf{i})=0$ as we wanted.

Combining Cases 1-3 shows that $e(\mathbf{i})\ne0$ whenever $\{i_r-1,i_r+1\}\subsetneq\{i_{s+1},\dots,i_{r-1}\}$. Hence, $\mathbf{i}_r\in I^r_\Lambda$ as required.

To complete the proof of the inductive step (and of the proposition), it remains to show that $y_r=0$. Using what we have just proved, it is enough to show that $y_re(\mathbf{i})=0$ whenever $\mathbf{i}_r\in I^r_\Lambda$. If $i_{r-1}=i_r\pm1$ then, by induction and (2.2.3),

$$y_re(\mathbf{i})=(y_r-y_{r-1})e(\mathbf{i})=\pm\psi_{r-1}^2e(\mathbf{i})=\pm\psi_{r-1}e(s_{r-1}\cdot\mathbf{i})\psi_{r-1}=0,$$

where the last equality follows because $(s_r\cdot\mathbf{i})_r\notin I^r_\Lambda$ by Corollary 2.4.2. If $i_{r-1}\ne i_r\pm1$ then $i_{r-1}\not\to i_r$ by Lemma 2.4.4 since $\mathbf{i}_r\in I^r_\Lambda$. Therefore, $y_re(\mathbf{i})=y_r\psi_{r-1}^2e(\mathbf{i})=\psi_{r-1}y_{r-1}\psi_{r-1}e(\mathbf{i})=0$ since $y_{r-1}=0$ by induction. This completes the proof. $\square$

Before giving our main application of Proposition 2.4.6 we consider what this result means for the cyclotomic quiver Hecke algebra of the symmetric group.

2.4.7. **Example** (Symmetric groups) Suppose that $\Lambda=\Lambda_0$ and that $1<e\le n$. Then $(\Lambda,\alpha_{i,e-1})\le 1$ for all $i\in I$. Therefore, Proposition 2.4.6 shows that $y_r=0$ for $1\le r\le e-1$ and that $e(\mathbf{i})\ne0$ only if $\mathbf{i}_{e-1}\in I^{e-1}_\Lambda$. In addition, we also have $\psi_1=0$ because if $\mathbf{i}\in I^n$ then $\psi_1e(\mathbf{i})=e(s_1\cdot\mathbf{i})\psi_1=0$ because if $\mathbf{i}_{e-1}\in I^{e-1}_\Lambda$ then $(s_1\cdot\mathbf{i})_{e-1}\notin I^{e-1}_\Lambda$.

Translating the proof of Proposition 2.4.6 back to Lemma 2.4.1, the reason why $\psi_1=0$ is that if $\mathbf{i}=\mathbf{i}^{\mathsf{t}}$ is the residue sequence of some standard tableau $\mathsf{t}\in\mathrm{Std}(\mathcal{P}_n)$ then $i_1=0$ and $i_2\ne0$, so $s_1\cdot\mathbf{i}$ can never be a residue sequence. By the same reasoning, ψ_1 is not necessarily zero if Λ has level $\ell>1$. $\Diamond$

We now completely describe the KLR algebras $\mathscr{R}^\Lambda_n$ when $\Lambda\in P^+$ and $(\Lambda,\alpha_{i,n})\le1$, for $i\in I$. For $(\mathsf{s},\mathsf{t})\in\mathrm{Std}^2(\mathcal{P}_n)$ define $e_{\mathsf{st}}=\psi_{d(\mathsf{s})^{-1}}e(\mathbf{i}^{\boldsymbol\lambda})\psi_{d(\mathsf{t})}$, where $\mathbf{i}^{\boldsymbol\lambda}=\mathbf{i}^{\mathsf{t}^{\boldsymbol\lambda}}$.

2.4.8. **Theorem.** *Suppose that $\Lambda\in P^+$ and $(\Lambda,\alpha_{i,n})\le1$, for all $i\in I$. Then $\mathscr{R}^\Lambda_n$ is a graded cellular algebra with graded cellular basis $\{\,e_{\mathsf{st}}\mid(\mathsf{s},\mathsf{t})\in\mathrm{Std}^2(\mathcal{P}_n)\,\}$ with $\deg e_{\mathsf{st}}=0$ for all $(\mathsf{s},\mathsf{t})\in\mathrm{Std}^2(\mathcal{P}_n)$.*

Proof. By Proposition 2.4.6, $y_r=0$ for $1\le r\le n$ and $e(\mathbf{i})=0$ if $\mathbf{i}\notin I^n_\Lambda$. In particular, this implies that $\psi_1,\dots,\psi_{n-1}$ satisfy the braid relations for the symmetric group $\mathfrak{S}_n$ because, by Lemma 2.4.4, if $\mathbf{i}\in I^n_\Lambda$ then $(i,i\pm1,i)$ is not a subsequence of $\mathbf{i}$, for any $i\in I$. Therefore, $\mathscr{R}^\Lambda_n$ is spanned by the elements $\psi_ve(\mathbf{i})\psi_w$, where $v,w\in\mathfrak{S}_w$ and $\mathbf{i}\in I^n_\Lambda$. Moreover, if $\mathbf{j}\in I^n$ then $e(\mathbf{j})\psi_ve(\mathbf{i})\psi_w=0$ unless $\mathbf{j}=v\cdot\mathbf{i}\in I^n_\Lambda$. Therefore, $\mathscr{R}^\Lambda_n$ is spanned by the elements $\{\,e_{\mathsf{st}}\mid(\mathsf{s},\mathsf{t})\in\mathrm{Std}^2(\mathcal{P}_n)\,\}$ as required by the statement of the theorem. Hence, $\mathscr{R}^\Lambda_n$ has rank at most $\ell^nn!$ by Theorem 1.6.7.

Let K be the algebraic closure of the field of fractions of $\mathcal{Z}$. Then $\mathscr{R}^\Lambda_n(K)\cong\mathscr{R}^\Lambda_n(\mathcal{Z})\otimes_{\mathcal{Z}}K$. By the last paragraph, the dimension of $\mathscr{R}^\Lambda_n$ is at most $\ell^nn!$. Let $\mathrm{rad}\,\mathscr{R}^\Lambda_n(K)$ be the Jacobson radical of $\mathscr{R}^\Lambda_n(K)$. For each $\boldsymbol\lambda\in\mathcal{P}_n$, Proposition 2.4.3 constructs an irreducible graded Specht module $S^{\boldsymbol\lambda}$. By Lemma 2.4.1, if $\boldsymbol\lambda,\boldsymbol\mu\in\mathcal{P}_n$ and $d\in\mathbb{Z}$ then $S^{\boldsymbol\lambda}\cong S^{\boldsymbol\mu}\langle d\rangle$ if and only if $\boldsymbol\lambda=\boldsymbol\mu$ and $d=0$. Therefore, by the Wedderburn theorem,

$$\ell^nn!\ge\dim\mathscr{R}^\Lambda_n(K)/\,\mathrm{rad}\,\mathscr{R}^\Lambda_n(K)\ge\sum_{\boldsymbol\lambda\in\mathcal{P}_n}(\dim S^{\boldsymbol\lambda})^2=\sum_{\boldsymbol\lambda\in\mathcal{P}_n}|\,\mathrm{Std}(\boldsymbol\lambda)|^2=\ell^nn!.$$

Hence, we have equality throughout so that $\{\,e_{\mathsf{st}}\mid(\mathsf{s},\mathsf{t})\in\mathrm{Std}^2(\mathcal{P}_n)\,\}$ is a basis of $\mathscr{R}^\Lambda_n(K)$. As the elements $\{e_{\mathsf{st}}\}$ span $\mathscr{R}^\Lambda_n(\mathcal{Z})$, and their images in $\mathscr{R}^\Lambda_n(K)$ are linearly independent, it follows that $\{e_{\mathsf{st}}\}$ is also a basis of $\mathscr{R}^\Lambda_n(\mathcal{Z})$.

It remains to prove that $\{e_{\mathsf{st}}\}$ is a graded cellular basis of $\mathscr{R}^\Lambda_n$. The orthogonality of the KLR idempotents implies that $e_{\mathsf{st}}e_{\mathsf{uv}}=\delta_{\mathsf{tu}}e_{\mathsf{sv}}$. Therefore, $\{e_{\mathsf{st}}\}$ is a basis of matrix units for $\mathscr{R}^\Lambda_n$. Consequently, $\mathscr{R}^\Lambda_n$ is a direct sum of matrix rings, for any integral domain $\mathcal{Z}$, and $\{e_{\mathsf{st}}\}$ is a cellular basis of $\mathscr{R}^\Lambda_n$.

Finally, we need to show that e_{st} is homogeneous of degree zero. This will follow if we show that $\deg\psi_re(\mathbf{i})=0$, for $1\le r<n$ and $\mathbf{i}\in I^n_\Lambda$. In fact, this is already clear because if $\mathbf{i}\in I^n_\Lambda$ then $i_r\ne i_{r+1}$, by Lemma 2.4.4, and if $i_{r+1}=i_r\pm1$ then $\psi_re(\mathbf{i})=0$ by Corollary 2.4.2 and Proposition 2.4.6. $\square$

By definition, $e_{\mathsf{st}}e_{\mathsf{uv}}=\delta_{\mathsf{tv}}e_{\mathsf{sv}}$. Let $\mathrm{Mat}_d(\mathcal{Z})$ be the ring of $d\times d$ matrices over $\mathcal{Z}$. Hence, the proof of Theorem 2.4.8 also yields the following.

2.4.9. **Corollary.** *Suppose that $\mathcal{Z}$ is an integral domain and that $(\Lambda,\alpha_{i,n})\le1$, for all $i\in I$. Then*

$$\mathscr{R}^\Lambda_n(\mathcal{Z})\cong\bigoplus_{\boldsymbol\lambda\in\mathcal{P}_n}\mathrm{Mat}_{s_{\boldsymbol\lambda}}(\mathcal{Z}),$$

where $s_{\boldsymbol\lambda}=\#\,\mathrm{Std}(\boldsymbol\lambda)$ for $\boldsymbol\lambda\in\mathcal{P}_n$.

Another consequence of Theorem 2.4.8 is that the KLR relations simplify dramatically in the semisimple case.

2.4.10. **Corollary.** *Suppose that $\mathcal{Z}$ is an integral domain and that $\Lambda\in P^+$ with $(\Lambda,\alpha_{i,n})\le1$, for all $i\in I$. Then $\mathscr{R}^\Lambda_n$ is the unital associative $\mathbb{Z}$-graded algebra generated by $\psi_1,\dots,\psi_{n-1}$ and $e(\mathbf{i})$, for $\mathbf{i}\in I^n_\Lambda$, subject to the relations*

$$\textstyle\sum_{\mathbf{i}\in I^n_\Lambda}e(\mathbf{i})=1,\qquad e(\mathbf{i})e(\mathbf{j})=\delta_{\mathbf{ij}}e(\mathbf{i}),$$

$$\psi_r\psi_{r+1}\psi_r=\psi_{r+1}\psi_r\psi_{r+1},\qquad \psi_r\psi_s=\psi_s\psi_r,\qquad \text{if } |r-s|>1,$$

$$\psi_re(\mathbf{i})=\begin{cases}e(s_r\cdot\mathbf{i})\psi_r, & \text{if } i_{r+1}\ne i_r\pm1,\\ 0, & \text{if } i_{r+1}=i_r\pm1,\end{cases}$$

for $\mathbf{i},\mathbf{j}\in I^n$ *and all admissible* r *and* s*. Moreover,* $\mathscr{R}_n^\Lambda$ *is concentrated in degree zero.*

As a final application, we prove Brundan and Kleshchev's graded isomorphism theorem in this special case.

2.4.11. **Corollary.** *Suppose that* $\mathcal{Z}=K$ *is a field and that* $(\Lambda,\alpha_{i,n})\le 1$*, for all* $i\in I$*. Then* $\mathscr{R}_n^\Lambda\cong\mathscr{H}_n^\Lambda$*.*

Proof. By Corollary 2.4.10 and Theorem 1.6.7, there is a well-defined homomorphism $\Theta:\mathscr{R}_n^\Lambda\longrightarrow\mathscr{H}_n^\Lambda$ determined by

$$e(\mathbf{i}^\mathsf{s})\mapsto F_\mathsf{s}\qquad\text{and}\qquad \psi_r e(\mathbf{i}^\mathsf{s})\mapsto\frac{1}{\alpha_r(\mathsf{s})}\Big(T_r+\frac{1}{\rho^Q(\mathsf{s})}\Big)F_\mathsf{s},$$

for $\mathsf{s}\in\operatorname{Std}(\mathcal{P}_n)$ and $1\le r<n$. By definition, Θ is injective so it is an isomorphism by Theorem 2.4.8. □

We emphasize that it is essential to work over a field in Corollary 2.4.11 because Corollary 2.4.9 says that $\mathscr{R}_n^\Lambda$ is always a direct sum of matrix rings whereas if $n>1$ this is only true of $\mathscr{H}_n^\Lambda$ when it is defined over a field.

These results suggest that $\mathscr{R}_n^\Lambda$ should be considered as the "idempotent completion" of the algebra $\mathscr{H}_n^\Lambda$ obtained by adjoining idempotents $e(\mathbf{i})$, for $\mathbf{i}\in I^n$. We will see how to make sense of the idempotents $e(\mathbf{i})\in\mathscr{H}_n^\Lambda$ for any $\mathbf{i}\in I^n$ in Theorem 3.1.1 and Lemma 4.2.2 below.

2.5. **The nil-Hecke algebra.** Still working just with the relations we now consider the shadow of the nil-Hecke algebra in the cyclotomic KLR setting. For the affine KLR algebras the nil-Hecke algebras case has been well-studied [67,114]. For the cyclotomic quotients (in type A) the story is similar.

For this section fix $i\in I$ and set $\beta=n\alpha_i$ and $\Lambda=n\Lambda_i$. Following (2.2.6), set $\mathscr{R}_\beta^\Lambda=e(\mathbf{i})\mathscr{R}_n^\Lambda e(\mathbf{i})$, where $\mathbf{i}=\mathbf{i}^\beta=(i^n)$. Then $\mathscr{R}_\beta^\Lambda$ is a direct summand of $\mathscr{R}_n^\Lambda$ and, moreover, it is a non-unital subalgebra with identity element $e(\mathbf{i})$. As $\mathscr{R}_\beta^\Lambda$ contains only one idempotent, $\psi_r=\psi_r e(\mathbf{i})$ and $y_s=y_s e(\mathbf{i})$. Therefore, $\mathscr{R}_\beta^\Lambda$ is the unital associative graded algebra generated by ψ_r and y_s, for $1\le r<n$ and $1\le s\le n$, with relations

$$\begin{gathered}
y_1^n=0,\qquad \psi_r^2=0,\qquad y_ry_s=y_sy_r,\\
\psi_ry_{r+1}=y_r\psi_r+1,\qquad y_{r+1}\psi_r=\psi_ry_r+1,\\
\psi_r\psi_s=\psi_s\psi_r\quad\text{if } |r-s|>1,\qquad \psi_ry_s=y_s\psi_r\quad\text{if } s\ne r,r+1,\\
\psi_r\psi_{r+1}\psi_r=\psi_{r+1}\psi_r\psi_{r+1}.
\end{gathered}$$

The grading on $\mathscr{R}_\beta^\Lambda$ is determined by $\deg\psi_r=-2$ and $\deg y_s=2$. Some readers will recognize this presentation as defining as a cyclotomic quotient of the **nil-Hecke algebra** of type A [81]. Note that the argument from Case 3 of Lemma 2.3.1 shows that $y_r^\ell=0$ for $1\le r\le\ell$.

Let $\boldsymbol{\lambda}=(1|1|\dots|1)\in\mathcal{P}_\beta$. Then the map $\mathsf{t}\mapsto d(\mathsf{t})$ defines a bijection between the set of standard $\boldsymbol{\lambda}$-tableaux and the symmetric group $\mathfrak{S}_n$. For convenience, we identify the standard $\boldsymbol{\lambda}$-tableaux with the set of (non-standard) tableaux of partition shape (n) by concatenating their components. In other words, if $d=d(\mathsf{t})$ then $\mathsf{t}=\boxed{d_1}\boxed{d_2}\boxed{\cdot\cdot}\boxed{d_n}$, where $d=d_1\dots d_n$ is the permutation written in one-line notation.

If $\mathsf{v},\mathsf{s}\in\operatorname{Std}(\boldsymbol{\lambda})$ then write $\mathsf{s}\mathrel{\dot\rhd}\mathsf{v}$ if $\mathsf{s}\rhd\mathsf{v}$ and $\ell(d(\mathsf{v}))=\ell(d(\mathsf{s}))+1$. To make this more explicit write $t\prec_\mathsf{v} m$ if t is in an earlier component of v than m – that is, t is to the left of m in v. Then the reader can check that $\mathsf{s}\mathrel{\dot\rhd}\mathsf{v}$ if and only if there exist integers $1\le m<t\le n$ such that $\mathsf{s}=\mathsf{v}(m,t)$, $m\prec_\mathsf{v} t$ and if $m<l<t$ then either $l\prec_\mathsf{v} m$ or $t\prec_\mathsf{v} l$.

2.5.1. **Example** Suppose that $n=6$. Let $\mathsf{v}=\boxed{4}\boxed{6}\boxed{5}\boxed{3}\boxed{1}\boxed{2}$ and take $t=3$. Then

$$\Big\{\ \boxed{3}\boxed{6}\boxed{5}\boxed{4}\boxed{1}\boxed{2},\ \boxed{4}\boxed{6}\boxed{3}\boxed{5}\boxed{1}\boxed{2},\ \boxed{4}\boxed{6}\boxed{5}\boxed{2}\boxed{1}\boxed{3},\ \boxed{4}\boxed{6}\boxed{5}\boxed{1}\boxed{3}\boxed{2}\ \Big\}$$

is the set of $\boldsymbol{\lambda}$-tableaux $\{\,\mathsf{s}\mid\mathsf{s}=\mathsf{v}(3,r)\mathrel{\dot\rhd}\mathsf{v}\text{ for }1\le r\le n\,\}$. ◊

We can now state the main result of the section.

2.5.2. **Proposition.** *Suppose that* $\beta=n\alpha_i$ *and* $\Lambda=n\Lambda_i$*, for* $i\in I$*. Then there is a unique graded* $\mathscr{R}_\beta^\Lambda$*-module* $S^{\boldsymbol{\lambda}}$ *with homogeneous basis* $\{\,\psi_\mathsf{s}\mid\mathsf{s}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$ *such that* $\deg\psi_\mathsf{s}=\binom{n}{2}-2\ell(d(\mathsf{s}))$ *and*

$$\begin{aligned}
\psi_\mathsf{s}\psi_r&=\begin{cases}\psi_{\mathsf{s}(r,r+1)}, & \text{if } \mathsf{s}\rhd\mathsf{s}(r,r+1)\in\operatorname{Std}(\boldsymbol{\lambda}),\\ 0, & \text{otherwise},\end{cases}\\
\psi_\mathsf{v}y_t&=\sum_{\substack{1\le k<t\\ \mathsf{u}=\mathsf{v}(k,t)\mathrel{\dot\rhd}\mathsf{u}}}\psi_\mathsf{u}\;-\;\sum_{\substack{t<s\le n\\ \mathsf{u}=\mathsf{v}(k,t)\mathrel{\dot\rhd}\mathsf{v}}}\psi_\mathsf{u},
\end{aligned}$$

for $\mathsf{s},\mathsf{v}\in\operatorname{Std}(\boldsymbol{\lambda})$*,* $1\le r<n$ *and* $1\le t\le n$*. Moreover, if* $\mathcal{Z}$ *is a field then* $S^{\boldsymbol{\lambda}}$ *is irreducible.*

Proof. The uniqueness is clear. To show that $S^{\boldsymbol\lambda}$ is an $\mathscr{R}^\Lambda_\beta$-module we check that the action respects the relations of $\mathscr{R}^\Lambda_\beta$. By definition, if $\mathsf{v}\in\operatorname{Std}(\boldsymbol\lambda)$ then $\psi_{\mathsf{v}}=\psi_{\mathsf{t}^{\boldsymbol\lambda}}\psi_{d(\mathsf{v})}$ and $\psi_{\mathsf{v}}\psi_r^2=0$ since $\psi_{\mathsf{v}}\psi_r=0$ if $\mathsf{v}(r,r+1)\rhd\mathsf{v}$. In particular, this implies that the action of $\psi_1\dots,\psi_{n-1}$ on $S^{\boldsymbol\lambda}$ respects the braid relations of $\mathfrak{S}_n$ and that ψ_{v} has the specified degree. Further, note that if $\mathsf{u}\rhd\mathsf{v}$ then $\ell(d(\mathsf{v}))=\ell(d(\mathsf{u}))+1$ so that $\deg\psi_{\mathsf{u}}=\deg\psi_{\mathsf{v}}+2$.

By the last paragraph, the action of $\mathscr{R}^\Lambda_\beta$ is compatible with the grading on $S^{\boldsymbol\lambda}$, but we still need to check the relations involving $y_1,\dots,y_n$. First consider $\psi_{\mathsf{v}}y_ry_t=\psi_{\mathsf{v}}y_ty_r$, for $1\le r,t\le n$ and $\mathsf{v}\in\operatorname{Std}(\boldsymbol\lambda)$. If $r=t$ there is nothing to prove so suppose $r\ne t$. By definition,

$$\psi_{\mathsf{v}}y_ty_r=\sum_{\mathsf{u}\rhd\mathsf{v}}\sum_{\mathsf{s}\rhd\mathsf{u}}\varepsilon_t(\mathsf{v},\mathsf{u})\varepsilon_r(\mathsf{u},\mathsf{s})\psi_{\mathsf{s}},$$

for appropriate choices of the signs $\varepsilon_t(\mathsf{v},\mathsf{u})$ and $\varepsilon_r(\mathsf{u},\mathsf{s})$. Suppose that ψ_{s} appears with non-zero coefficient in this sum. Then we can write $\mathsf{u}=\mathsf{v}(m,t)$ and $\mathsf{u}=\mathsf{v}(l,r)$, for some l,m such that $\mathsf{s}\rhd\mathsf{u}\rhd\mathsf{v}$. If $l\ne m$ then the permutations (m,t) and (l,r) commute. As the lengths add, we also have that $\mathsf{s}\rhd\mathsf{v}(l,r)\rhd\mathsf{v}$. Therefore, ψ_{s} appears with the same coefficient in $\psi_{\mathsf{v}}y_ty_s$ and $\psi_{\mathsf{v}}y_ry_t$. If $l=m$ then $\mathsf{s}\rhd\mathsf{u}\trianglerighteq\mathsf{v}$ only if m is in between r and t in v. That is, either $r\prec_{\mathsf{v}}m\prec_{\mathsf{v}}t$ or $t\prec_{\mathsf{v}}m\prec_{\mathsf{v}}r$. However, this implies that either $\mathsf{s}\not\rhd\mathsf{u}$ or $\mathsf{u}\not\rhd\mathsf{v}$, so that ψ_{s} does not appear in either $\psi_{\mathsf{v}}y_ry_t$ or in $\psi_{\mathsf{v}}y_ty_r$. Hence, the actions y_r and y_t on $S^{\boldsymbol\lambda}$ commute.

Similar, but easier, calculations with tableaux show that the action on $S^{\boldsymbol\lambda}$ respects the three relations $\psi_ry_t=y_t\psi_r$, $\psi_ry_{r+1}=y_r\psi_r+1$ and $y_{r+1}\psi_r=\psi_ry_r+1$. To complete the verification of the relations in $\mathscr{R}^\Lambda_\beta$ it remains to show that $\psi_{\mathsf{v}}y_1^n=0$, for all $\mathsf{v}\in\operatorname{Std}(\boldsymbol\lambda)$. This is clear, however, because $\psi_{\mathsf{v}}y_1$ is equal to a linear combination of terms ψ_{s} where 1 appears in an earlier component of s than it does in v.

Finally, it remains to prove that $S^{\boldsymbol\lambda}$ is irreducible over a field. First we need some more notation. Let $\mathsf{t}_{\boldsymbol\lambda}=\boxed{n}\boxed{\cdots}\boxed{\cdots}\boxed{2}\boxed{1}$ and set $w_{\boldsymbol\lambda}=d(\mathsf{t}_{\boldsymbol\lambda})$. Then $w_{\boldsymbol\lambda}$ is the unique element of longest length in $\mathfrak{S}_n$. Recall from §1.4, that $d'(\mathsf{t})$ is the unique permutation such that $\mathsf{t}=\mathsf{t}_{\boldsymbol\lambda}d'(\mathsf{t})$ and, moreover, $d(\mathsf{t})d'(\mathsf{t})^{-1}=w_{\boldsymbol\lambda}$ with the lengths adding. Therefore, if $\ell(d(\mathsf{s}))\ge\ell(d(\mathsf{t}))$ then $\psi_{\mathsf{s}}\psi^\star_{d'(\mathsf{t})}=\delta_{\mathsf{st}}\psi_{\mathsf{t}_{\boldsymbol\lambda}}$.

We are now ready to show that $S^{\boldsymbol\lambda}$ is irreducible. Suppose that $x=\sum_{\mathsf{s}}r_{\mathsf{s}}\psi_{\mathsf{s}}$ is a non-zero element of $S^{\boldsymbol\lambda}$. Let t be any tableau such that $r_{\mathsf{t}}\ne0$ and $\ell(d(\mathsf{t}))$ is minimal. Then, by the last paragraph, $x\psi^\star_{d'(\mathsf{t})}=r_{\mathsf{t}}\psi_{\mathsf{t}_{\boldsymbol\lambda}}$, so $\psi_{\mathsf{t}_{\boldsymbol\lambda}}\in x\mathscr{R}^\Lambda_\beta$. We have already observed that y_1 acts by moving 1 to an earlier component. Therefore, $\psi_{\mathsf{t}_{\boldsymbol\lambda}}y_1^{n-1}=(-1)^{n-1}\psi_{\mathsf{t}_{\boldsymbol\lambda,1}}$, where $\mathsf{t}_{\boldsymbol\lambda,1}=\boxed{1}\boxed{n}\boxed{\cdots}\boxed{3}\boxed{2}$. Similarly, $\psi_{\mathsf{t}_{\boldsymbol\lambda}}y_1^{n-1}y_2^{n-2}=(-1)^{2n-3}\psi_{\mathsf{t}_{\boldsymbol\lambda,2}}$, where $\mathsf{t}_{\boldsymbol\lambda,2}=\boxed{1}\boxed{2}\boxed{n}\boxed{\cdots}\boxed{3}$. Continuing in this way shows that $\psi_{\mathsf{t}_{\boldsymbol\lambda}}y_1^{n-1}y_2^{n-2}\dots y_{n-1}=(-1)^{\frac12n(n-1)}\psi_{\mathsf{t}^{\boldsymbol\lambda}}$. Hence, $x\mathscr{R}^\Lambda_\beta=S^{\boldsymbol\lambda}$, so that $S^{\boldsymbol\lambda}$ is irreducible as claimed. □

The proof of Proposition 2.5.2 shows that $y_1^{n-1}y_2^{n-2}\dots y_{n-1}$ is a non-zero element of $S^{\boldsymbol\lambda}$. Using the relations, and a bit of ingenuity, it is possible to show that $\{\,\psi_wy_1^{a_1}\dots y_n^{a_n}\mid w\in\mathfrak{S}_n\text{ and }0\le a_r\le n-r,\text{ for }1\le r\le n\,\}$ is a basis of $\mathscr{R}^\Lambda_\beta$. Alternatively, it follows from [20, Theorem 4.20] that $\dim\mathscr{R}^\Lambda_\beta=(n!)^2$. Hence, we obtain the following.

2.5.3. **Corollary.** *Suppose that $\beta=n\alpha_i$ and $\Lambda=n\Lambda_i$, for $i\in I$. Let $\boldsymbol\lambda=(1|1|\dots|1)$ and for $\mathsf{s},\mathsf{t}\in\operatorname{Std}(\boldsymbol\lambda)$ define $\psi_{\mathsf{st}}=\psi^\star_{d(\mathsf{s})}e(\mathbf{i}^{\boldsymbol\lambda})y^{\boldsymbol\lambda}\psi_{d(\mathsf{t})}$, where $\mathbf{i}^{\boldsymbol\lambda}=\mathbf{i}^{\mathsf{t}^{\boldsymbol\lambda}}$ and $y^{\boldsymbol\lambda}=y_1^{n-1}y_2^{n-2}\dots y_{n-1}$. Then $\{\,\psi_{\mathsf{st}}\mid\mathsf{s},\mathsf{t}\in\operatorname{Std}(\boldsymbol\lambda)\,\}$ is a graded cellular basis of $\mathscr{R}^\Lambda_\beta$.*

The basis of the Specht module $S^{\boldsymbol\lambda}$ in Proposition 2.5.2 is well-known because it is really a disguised version of the basis of *Schubert polynomials* of the coinvariant algebra of the symmetric group $\mathfrak{S}_n$ [83, 94]. The **coinvariant algebra** $\mathscr{C}_n$ is the quotient of the polynomial ring $\mathbb{Z}[\mathbf{x}]=\mathbb{Z}[x_1,\dots,x_n]$ by the symmetric polynomials in $x_1,\dots,x_n$ of positive degree. Identify x_r with its image in $\mathscr{C}_n$, for $1\le r\le n$. Then $\mathscr{C}_n$ is free of rank $n!$. As we have quotiented out by a homogeneous ideal, $\mathscr{C}_n$ inherits a grading from $\mathbb{Z}[\mathbf{x}]$, where we set $\deg x_r=2$ for $1\le r\le n$. There is a well-defined action of $\mathscr{R}^\Lambda_\beta$ on $\mathscr{C}_n$ where y_r acts as multiplication by x_r, and ψ_r acts as a *divided difference operator*:

$$f(\mathbf{x})\psi_r=\partial_rf(\mathbf{x})=\frac{f(\mathbf{x})-f(s_r\cdot\mathbf{x})}{x_r-x_{r+1}},$$

where $\mathbf{x}=(x_1,\dots,x_n)$ and $s_r\cdot\mathbf{x}=(x_1,\dots,x_{r+1},x_r,\dots,x_n)$ for $1\le r<n$. Here we are secretly thinking of $\mathscr{R}^\Lambda_\beta$ as being a quotient of the nil-Hecke algebra, where this action is well-known.

For $d\in\mathfrak{S}_n$ define $\sigma_d=(x_1^{n-1}x_2^{n-2}\dots x_{n-1})\psi_{w_0d}$. Then $\{\,\sigma_d\mid d\in\mathfrak{S}_n\,\}$ is the basis of **Schubert polynomials** of $\mathscr{C}_n$. The Specht module is isomorphic to $\mathscr{C}_n$ as an $\mathscr{R}^\Lambda_\beta$-module, where an isomorphism is given by $\psi_{\mathsf{t}}\mapsto\sigma_{d'(\mathsf{t})}$. To see this it is enough to know that the Schubert polynomials satisfy the identity

$$\partial_r\sigma_d=\begin{cases}\sigma_{s_rd}, & \text{if }\ell(s_rd)=\ell(d)-1,\\ 0, & \text{otherwise.}\end{cases}$$

Now observe that by the last paragraph of the proof of Proposition 2.5.2, if $\mathsf{t}\in\mathfrak{S}_n$ then

$$\psi_{\mathsf{t}}=\psi_{\mathsf{t}^{\boldsymbol\lambda}}\psi_{d(\mathsf{t})}=\psi_{\mathsf{t}_{\boldsymbol\lambda}}y_1^{n-1}y_2^{n-1}\dots y_{n-1}\psi_{d(\mathsf{t})}.$$

Hence, our claim follows by identifying $\psi_{\mathfrak{t}_\lambda}$ with the polynomial $1 \in \mathscr{C}_n$.

Finally, we remark that the formula for the action of $y_1, \dots, y_n$ in Proposition 2.5.2 is a well-known corollary of Monk's rule; for example see [94, Exercise 2.7.3].

3. Isomorphisms, Specht modules and categorification

In the last section we proved that the algebras $\mathscr{R}_n^\Lambda$ and $\mathscr{H}_n^\Lambda$ are isomorphic when $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$. This section starts with Brundan and Kleshchev's Graded Isomorphism Theorem: $\mathscr{R}_n^\Lambda \cong \mathscr{H}_n^\Lambda$, for all $\Lambda \in P^+$. Then we start to investigate the consequences of this result for both algebras.

3.1. Brundan and Kleshchev's Graded Isomorphism Theorem.

One of the most fundamental results for the cyclotomic Hecke algebras $\mathscr{H}_n^\Lambda$ is Brundan and Kleshchev's spectacular isomorphism theorem.

3.1.1. **Theorem** (Graded Isomorphism Theorem [19,114])**.** *Suppose that $\mathcal{Z} = F$ is a field, $v \in K$ has quantum characteristic e and that $\Lambda \in P^+$. Then there is an isomorphism of algebras $\mathscr{R}_n^\Lambda \cong \mathscr{H}_n^\Lambda$.*

Suppose that F is a field of characteristic $p > 0$ and that $e = pf$, where $f > 1$. Then F cannot contain an element v of quantum characteristic e, so Theorem 3.1.1 says nothing about the quiver Hecke algebra $\mathscr{R}_n^\Lambda(F)$.

As a first consequence of Theorem 3.1.1, by identifying $\mathscr{H}_n^\Lambda$ and $\mathscr{R}_n^\Lambda$ we can consider $\mathscr{H}_n^\Lambda$ as a graded algebra.

3.1.2. **Corollary.** *Suppose that $\Lambda \in P^+$ and $\mathcal{Z} = F$ is a field. Then there is a unique grading on $\mathscr{H}_n^\Lambda$ such that $\deg e(\mathbf{i}) = 0$, $\deg y_r = 2$ and $\deg \psi_s e(\mathbf{i}) = -c_{i_s, i_{s+1}}$, for $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in I^n$.*

Brundan and Kleshchev prove Theorem 3.1.1 by constructing family of isomorphisms $\mathscr{R}_n^\Lambda \longrightarrow \mathscr{H}_n^\Lambda$, together with their inverses, and then painstakingly checking that these isomorphisms respect the relations of both algebras. Their argument starts with the well-known fact that $\mathscr{H}_n^\Lambda$ decomposes into a direct sum of simultaneous generalized eigenspaces for the Jucys-Murphy elements $L_1, \dots, L_n$. These eigenspaces are indexed by I^n, so for each $\mathbf{i} \in I^n$ there is an element $e(\mathbf{i}) \in \mathscr{H}_n^\Lambda$, possibly zero, such that $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i})$. We describe these idempotents explicitly in Lemma 4.2.2 below.

Translating through Definition 1.1.1, Brundan and Kleshchev's isomorphism is given by $e(\mathbf{i}) \mapsto e(\mathbf{i})$ and

$$y_r \mapsto \sum_{\mathbf{i} \in I^n} v^{-i_r}\big(L_r - [i_r]_v\big)e(\mathbf{i}), \quad \text{and} \quad \psi_s \mapsto \sum_{\mathbf{i} \in I^n} \big(T_s + P_s(\mathbf{i})\big)\frac{1}{Q_s(\mathbf{i})}e(\mathbf{i}),$$

for $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in I^n$. We are abusing notation by identifying the KLR generators with their images in $\mathscr{H}_n^\Lambda$. Here, $P_r(\mathbf{i})$ and $Q_r(\mathbf{i})$ are certain rational functions in y_r and y_{r+1} which are well-defined because $(L_t - [i_t]_v)e(\mathbf{i})$ is nilpotent in $\mathscr{H}_n^\Lambda$, for $1 \le t \le n$. The inverse isomorphism is given by $e(\mathbf{i}) \mapsto e(\mathbf{i})$,

$$L_r \mapsto \sum_{\mathbf{i} \in I^n} \big(v^{i_r} y_r + [i_r]_v\big)e(\mathbf{i}) \quad \text{and} \quad T_s \mapsto \sum_{\mathbf{i} \in I^n} \big(\psi_s Q_s(\mathbf{i}) - P_s(\mathbf{i})\big)e(\mathbf{i}),$$

for $1 \le r \le n$, $1 \le s < n$ and $\mathbf{i} \in I^n$.

Rouquier [114, Corollary 3.20] has given a quicker proof of Theorem 3.1.1 by first showing that the (non-cyclotomic) quiver Hecke algebra $\mathscr{R}_n$ is isomorphic to the (extended) affine Hecke algebra of type A. Following [52] we sketch another approach to Theorem 3.1.1 in §4.2 below.

The following easy but important application of Theorem 3.1.1 was a surprise (at least to the author!).

3.1.3. **Corollary.** *Suppose that $\mathcal{Z} = F$ is a field and that $v, v' \in F$ are two elements of quantum characteristic e. Let $\Lambda \in P^+$. Then $\mathscr{H}_n^\Lambda(F, v) \cong \mathscr{H}_n^\Lambda(F, v')$.*

Proof. By Theorem 3.1.1, $\mathscr{H}_n^\Lambda(F, v) \cong \mathscr{R}_n^\Lambda(F) \cong \mathscr{H}_n^\Lambda(F, v')$. □

Consequently, up to isomorphism, the algebra $\mathscr{H}_n^\Lambda$ depends only on e, Λ and the field F. Therefore, because $\mathscr{H}_n^\Lambda$ is cellular, the decomposition matrices of $\mathscr{H}_n^\Lambda$ depend only on e, Λ and p, where p is the characteristic of F. In the special case of the symmetric group, when $\Lambda = \Lambda_0$, this weaker statement for the decomposition matrices was conjectured in [97, Conjecture 6.38].

When $F = \mathbb{C}$ it is easy to prove Corollary 3.1.3 because there is a Galois automorphism of $\mathbb{Q}(v)$, as an extension of $\mathbb{Q}$, which interchanges v and v'. It is not difficult to see that this automorphism induces an isomorphism $\mathscr{H}_n^\Lambda(F, v) \cong \mathscr{H}_n^\Lambda(F, v')$. This argument fails for fields of positive characteristic because such fields have fewer automorphisms.

3.2. **Graded Specht modules.** As we noted in §2.1, if we impose a grading on an algebra $\underline{A}$ then it is not true that every (ungraded) $\underline{A}$-module has a graded lift, so there is no reason to expect that graded lifts of the Specht modules $\underline{S}^\lambda$ exist in general. Of course, graded Specht modules do exist and this section describes one way to define them.

Recall from §1.5 that the ungraded Specht module $\underline{S}^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n$, has basis $\{\, m_{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\}$. By construction, $\underline{S}^{\boldsymbol{\lambda}} = m_{\mathsf{t}^{\boldsymbol{\lambda}}}\mathscr{H}_n^\Lambda$. Brundan, Kleshchev and Wang [23] proved that $\underline{S}^{\boldsymbol{\lambda}}$ has a graded lift essentially by declaring that $m_{\mathsf{t}^{\boldsymbol{\lambda}}}$ should be homogeneous and then showing that this induces a grading on the Specht module $\underline{S}^{\boldsymbol{\lambda}} = m_{\mathsf{t}^{\boldsymbol{\lambda}}}\mathscr{R}_n^\Lambda$.

Partly inspired by [23], Jun Hu and the author [49] showed that $\mathscr{H}_n^\Lambda$ is a graded cellular algebra. The graded cell modules constructed from this cellular basis coincide exactly with those of [23]. Perhaps most significantly, the construction of the graded Specht modules using cellular algebra techniques endows the graded Specht modules with a homogeneous bilinear form of degree zero.

Following Brundan, Kleshchev and Wang [23, §3.5] we now define the degree of a standard tableau. Suppose that $\boldsymbol{\mu} \in \mathcal{P}_n$. For $i \in I$ let $\operatorname{Add}_i(\boldsymbol{\mu})$ be the set of addable i-nodes of $\boldsymbol{\mu}$ and let $\operatorname{Rem}_i(\boldsymbol{\mu})$ be its set of removable i-nodes. If A is an addable or removable i-node of $\boldsymbol{\mu}$ define

$$\begin{aligned} d_A(\boldsymbol{\mu}) &= \#\{\, B \in \operatorname{Add}_i(\boldsymbol{\mu}) \mid A < B\,\} - \#\{\, B \in \operatorname{Rem}_i(\boldsymbol{\mu}) \mid A < B\,\},\\ d^A(\boldsymbol{\mu}) &= \#\{\, B \in \operatorname{Add}_i(\boldsymbol{\mu}) \mid A > B\,\} - \#\{\, B \in \operatorname{Rem}_i(\boldsymbol{\mu}) \mid A > B\,\},\\ d_i(\boldsymbol{\mu}) &= \#\operatorname{Add}_i(\boldsymbol{\mu}) - \#\operatorname{Rem}_i(\boldsymbol{\mu}). \end{aligned} \tag{3.2.1}$$

If t is a standard $\boldsymbol{\mu}$-tableau then its **codegree** and **degree** are defined inductively by setting $\operatorname{codeg}_e \mathsf{t} = 0 = \deg_e \mathsf{t}$ if $n = 0$ and if $n > 0$ defining

$$\operatorname{codeg}_e \mathsf{t} = \operatorname{codeg}_e \mathsf{t}_{\downarrow(n-1)} + d^A(\boldsymbol{\mu}) \quad \text{and} \quad \deg_e \mathsf{t} = \deg_e \mathsf{t}_{\downarrow(n-1)} + d_A(\boldsymbol{\mu}),$$

where $A = \mathsf{t}^{-1}(n)$. When e is fixed write $\operatorname{codeg} \mathsf{t} = \operatorname{codeg}_e \mathsf{t}$ and $\deg \mathsf{t} = \deg_e \mathsf{t}$.

Implicitly, all of these definitions depend on the choice of multicharge $\boldsymbol{\kappa}$. The definition of the (co)degree of standard tableaux due to Brundan, Kleshchev and Wang [23], however, the underlying combinatorics dates back to Misra and Miwa [104] and their work on the crystal graph and Fock space representations of $U_q(\widehat{\mathfrak{sl}}_e)$.

Recall that we fixed a reduced expression for each permutation $w \in \mathfrak{S}_n$. In §1.4 for each tableau $\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ we have defined permutations $d'(\mathsf{t}), d(\mathsf{t}) \in \mathfrak{S}_n$ by $\mathsf{t}_{\boldsymbol{\lambda}} d'(\mathsf{t}) = \mathsf{t} = \mathsf{t}^{\boldsymbol{\lambda}} d(\mathsf{t})$.

3.2.2. **Definition** ([49, Definitions 4.9 and 5.1])**.** *Suppose that $\boldsymbol{\mu} \in \mathcal{P}_n$. Define non-negative integers $d_1^{\boldsymbol{\mu}}, \dots, d_n^{\boldsymbol{\mu}}$ and $d^1_{\boldsymbol{\mu}}, \dots, d^n_{\boldsymbol{\mu}}$ recursively by requiring that*

$$d^1_{\boldsymbol{\mu}} + \dots + d^k_{\boldsymbol{\mu}} = \operatorname{codeg}(\mathsf{t}^{\boldsymbol{\mu}}_{\downarrow k}) \quad \textit{and} \quad d_1^{\boldsymbol{\mu}} + \dots + d_k^{\boldsymbol{\mu}} = \deg(\mathsf{t}^{\boldsymbol{\mu}}_{\downarrow k}),$$

for $1 \le k \le n$. Now set $\mathbf{i}_{\boldsymbol{\mu}} = \mathbf{i}^{\mathsf{t}_{\boldsymbol{\mu}}}$, $\mathbf{i}^{\boldsymbol{\mu}} = \mathbf{i}^{\mathsf{t}^{\boldsymbol{\mu}}}$, $y^{\boldsymbol{\mu}} = y_1^{d^1_{\boldsymbol{\mu}}} \dots y_n^{d^n_{\boldsymbol{\mu}}}$ and $y^{\boldsymbol{\mu}} = y_1^{d_1^{\boldsymbol{\mu}}} \dots y_n^{d_n^{\boldsymbol{\mu}}}$. For $(\mathsf{s}, \mathsf{t}) \in \operatorname{Std}^2(\boldsymbol{\mu})$ define

$$\psi'_{\mathsf{st}} = \psi^\star_{d'(\mathsf{s})} e(\mathbf{i}_{\boldsymbol{\mu}}) y_{\boldsymbol{\mu}} \psi_{d'(\mathsf{t})} \quad \textit{and} \quad \psi_{\mathsf{st}} = \psi^\star_{d(\mathsf{s})} e(\mathbf{i}^{\boldsymbol{\mu}}) y^{\boldsymbol{\mu}} \psi_{d(\mathsf{t})},$$

where $\star$ is the unique (homogeneous) anti-isomorphism of $\mathscr{R}_n^\Lambda$ which fixes the KLR generators.

3.2.3. **Example** Suppose that $e = 3$, $\Lambda = \Lambda_0 + \Lambda_2$ and $\boldsymbol{\mu} = (7,6,3,2|4,3,1)$, with multicharge $\boldsymbol{\kappa} = (0,2)$. Then

$$\mathsf{t}^{\boldsymbol{\mu}} = \left(\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & \\ 14 & 15 & 16 & & & & \\ 17 & 18 & & & & & \end{array} \middle| \begin{array}{cccc} 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & \\ 26 & & & \\ & & & \end{array} \right) \quad \text{and} \quad \mathsf{t}_{\boldsymbol{\mu}} = \left(\begin{array}{ccccccc} 9 & 13 & 17 & 20 & 22 & 24 & 26 \\ 10 & 14 & 18 & 21 & 23 & 25 & \\ 11 & 15 & 19 & & & & \\ 12 & 16 & & & & & \end{array} \middle| \begin{array}{cccc} 1 & 4 & 6 & 8 \\ 2 & 5 & 7 & \\ 3 & & & \\ & & & \end{array} \right).$$

The reader may check that $e(\mathbf{i}^{\boldsymbol{\mu}}) = e(01201202012011200120121200)$. We have coloured the nodes in $\mathsf{t}^{\boldsymbol{\mu}}$ which have column index divisible by e or have residue 2, which is the residue of 19 in$=\mathsf{t}^{\boldsymbol{\mu}}_{19}$. This should convince the reader that $y^{\boldsymbol{\mu}} = y_3^2 y_6^2 y_8 y_{10} y_{11} y_{13} y_{15} y_{16} y_{21} y_{25}$. Using similar colourings for $\mathsf{t}_{\boldsymbol{\mu}}$, and reading right to left, $y_{\boldsymbol{\mu}} = y_3^2 y_4 y_7 y_{11} y_{15} y_{19}$. ◊

3.2.4. **Example** Let $\beta = n\alpha_i$ and $\Lambda = n\Lambda_i$, for some $i \in I$, so that $\mathscr{R}_\beta^\Lambda$ is the nil-Hecke algebra $\mathscr{R}_\beta^\Lambda$ of §2.5. Let $\boldsymbol{\lambda} = (1|1|\dots|1)$. Then $y^{\boldsymbol{\lambda}} = y_1^{n-1} \dots y_{n-2}^2 y_{n-1}$. Hence, the basis $\{\psi_{\mathsf{st}}\}$ of $\mathscr{R}_\beta^\Lambda$ coincides with that of Corollary 2.5.3. ◊

3.2.5. **Example** As in Example 2.2.7, in general, the basis element ψ_{st} depends on the choices of reduced expressions that we have fixed for the permutations $d(\mathsf{s})$ and $d(\mathsf{t})$. For example, suppose that $\Lambda = 2\Lambda_0 + \Lambda_1$, $\boldsymbol{\kappa} = (0,1,0)$ and $\boldsymbol{\mu} = (1|1|1)$ and consider the standard $\boldsymbol{\mu}$-tableaux $\mathsf{t}^{\boldsymbol{\mu}} = \left(\boxed{1} \,\middle|\, \boxed{2} \,\middle|\, \boxed{3}\right)$ and $\mathsf{t}_{\boldsymbol{\mu}} = \left(\boxed{3} \,\middle|\, \boxed{2} \,\middle|\, \boxed{1}\right)$. Then $d(\mathsf{t}^{\boldsymbol{\mu}}) = 1$ and $d(\mathsf{t}_{\boldsymbol{\mu}}) = (1,3) = s_1 s_2 s_1 = s_2 s_1 s_2$ has two different reduced expressions. Let $\psi_{\mathsf{t}_{\boldsymbol{\mu}}\mathsf{t}_{\boldsymbol{\mu}}} =$

$\psi_1\psi_2\psi_1 e(\mathbf{i}^{\boldsymbol{\mu}})y^{\boldsymbol{\mu}}\psi_1\psi_2\psi_1$ and $\hat\psi_{\mathsf{t}_\mu\mathsf{t}_\mu} = \psi_2\psi_1\psi_2 e(\mathbf{i}^{\boldsymbol{\mu}})y^{\boldsymbol{\mu}}\psi_2\psi_1\psi_2$. Then the calculation in Example 2.2.7 implies that

$$\hat\psi_{\mathsf{t}_\mu\mathsf{t}_\mu} = \psi_{\mathsf{t}_\mu\mathsf{t}_\mu} + \psi_{\mathsf{t}_\mu\mathsf{t}^\mu} + \psi_{\mathsf{t}^\mu\mathsf{t}_\mu} + \psi_{\mathsf{t}^\mu\mathsf{t}^\mu}.$$

This is probably the simplest example where different reduced expressions leads to different ψ-basis elements, but examples occur for almost all $\mathscr{R}_n^\Lambda$. This said, in view of Proposition 2.4.3, ψ_{st} is independent of the choice of reduced expressions for $d(\mathsf{s})$ and $d(\mathsf{t})$ whenever $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$. The ψ-basis can be independent of the choice of reduced expressions even when $\mathscr{R}_n^\Lambda$ is not semisimple. For example, this is always the case when $e > n$ and $\ell = 2$ by [50, Appendix]. These algebras are typically not semisimple. ♢

3.2.6. **Theorem** (Hu-Mathas [49, Theorem 5.8]). *Suppose that $\mathcal{Z} = F$ is a field. Then*

$$\{\, \psi_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n) \,\}$$

is a graded cellular basis of $\mathscr{R}_n^\Lambda$ with $\psi_{\mathsf{st}}^\star = \psi_{\mathsf{ts}}$ and $\deg \psi_{\mathsf{st}} = \deg \mathsf{s} + \deg \mathsf{t}$, for $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$.

Using the theory of graded cellular algebras from §2.1, we obtain graded Specht modules from Theorem 3.2.6. By [49, Corollary 5.10] the graded Specht modules $\{\, S^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n \,\}$ attached to the ψ-basis coincide with those constructed by Brundan, Kleshchev and Wang [23]. When $(\Lambda, \alpha_{i,n}) \le 1$ it is not hard to show that these Specht modules coincide with those we constructed in Proposition 2.4.3 above. Similarly, for the nil-Hecke algebra considered in §2.5, the graded Specht module $S^{\boldsymbol{\lambda}}$, with $\boldsymbol{\lambda} = (1|1|\dots|1)$, is isomorphic to the graded module constructed in Proposition 2.5.2. Moreover, on forgetting the grading $S^{\boldsymbol{\lambda}}$ coincides exactly with the ungraded Specht module $\underline{S}^{\boldsymbol{\lambda}}$ constructed in §1.5, for $\boldsymbol{\lambda} \in \mathcal{P}_n$.

If $\boldsymbol{\lambda} \in \mathcal{P}_n$ he graded Specht module $S^{\boldsymbol{\lambda}}$ has basis $\{\, \psi_{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$, with $\deg \psi_{\mathsf{t}} = \deg \mathsf{t}$. The reader should be careful not to confuse $\psi_{\mathsf{t}} \in S^{\boldsymbol{\lambda}}$ with $\psi_{d(\mathsf{t})} \in \mathscr{R}_n^\Lambda$! Hence, using Theorem 3.2.6 we recover [20, Theorem 4.20]:

$$\dim_q S^{\boldsymbol{\lambda}} = \sum_{\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})} q^{\deg \mathsf{t}} \qquad\Longrightarrow\qquad \dim_q \mathscr{H}_n^\Lambda = \sum_{(\mathsf{s},\mathsf{t}) \in \operatorname{Std}(\boldsymbol{\lambda})} q^{\deg \mathsf{s} + \deg \mathsf{t}} = \sum_{\boldsymbol{\lambda} \in \mathcal{P}_n} \big(\dim_q S^{\boldsymbol{\lambda}}\big)^2.$$

In essence, Theorem 3.2.6 is proved in much the same way that Brundan, Kleshchev and Wang [23] constructed a grading on the Specht modules: we proved that the transition matrix between the ψ-basis and the Murphy basis of Theorem 1.5.1 is triangular. In order to do this we needed the correct definition of the elements $y^{\boldsymbol{\mu}}$, which we discovered by first looking at the one dimensional two-sided ideals of $\mathscr{H}_n^\Lambda$ (which are necessarily homogeneous). We then used Brundan and Kleshchev's Graded Isomorphism Theorem 3.1.1, together with the seminormal forms (Theorem 1.6.7), to show that $e(\mathbf{i}^{\boldsymbol{\mu}})y^{\boldsymbol{\mu}} \ne 0$. This established that the basis of Theorem 3.2.6 is a graded cellular basis. Finally, the combinatorial results of [23] are used to determine the degree of ψ-basis elements.

Following the recipe in §2.1, for $\boldsymbol{\mu} \in \mathcal{P}_n$ define $D^{\boldsymbol{\mu}} = S^{\boldsymbol{\mu}}/\operatorname{rad} S^{\boldsymbol{\mu}}$, where $\operatorname{rad} S^{\boldsymbol{\mu}}$ is the radical of the homogeneous bilinear form on $S^{\boldsymbol{\mu}}$. This yields the classification of the graded irreducible $\mathscr{H}_n^\Lambda$-modules. The main point of the next result is that the labelling of the graded irreducible $\mathscr{H}_n^\Lambda$-modules agrees with Corollary 1.5.2.

3.2.7. **Corollary** ([20, Theorem 5.13], [49, Corollary 5.11]). *Suppose that $\Lambda \in P^+$ and that $\mathcal{Z} = F$ is a field. Then $\{\, D^{\boldsymbol{\mu}}\langle d\rangle \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$ and $d \in \mathbb{Z} \,\}$ is a complete set of pairwise non-isomorphic graded $\mathscr{H}_n^\Lambda$-modules. Moreover, $(D^{\boldsymbol{\mu}})^\circledast \cong D^{\boldsymbol{\mu}}$ and $D^{\boldsymbol{\mu}}$ is absolutely irreducible, for all $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$.*

The KLR algebra $\mathscr{R}_n$ is always $\mathbb{Z}$-free, however, it is not clear whether the same is true for the cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$. To prove this you cannot use the Graded Isomorphism Theorem 3.1.1 because this result holds only over a field. Using some extremely sophisticated diagram calculus calculations, Li [85] proved the following.

3.2.8. **Theorem** (Li [85]). *Suppose that $\Lambda \in P^+$. Then the quiver Hecke algebra $\mathscr{R}_n^\Lambda(\mathbb{Z})$ is free as a $\mathbb{Z}$-module of rank $\ell^n n!$. Moreover, $\mathscr{R}_n^\Lambda(\mathbb{Z})$ is a graded cellular algebra with graded cellular basis $\{\, \psi_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n) \,\}$.*

Therefore, $\mathscr{R}_n^\Lambda$ is free over any commutative ring and any field is a splitting field for $\mathscr{R}_n^\Lambda$. Moreover, the graded Specht modules, together with their homogeneous bilinear forms, are defined over $\mathbb{Z}$. The integrality of the graded Specht modules can also be proved using Theorem 3.6.2 below.

The next result lists some important properties of the ψ-basis.

3.2.9. **Proposition.** *Suppose that $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$ and that $\mathcal{Z}$ is an integral domain. Then:*

a) [49, Lemma 5.2] *If $\mathbf{i},\mathbf{j} \in I^n$ then $\psi_{\mathsf{st}} = \delta_{\mathbf{i},\mathbf{i}^{\mathsf{s}}}\delta_{\mathbf{j},\mathbf{i}^{\mathsf{t}}} e(\mathbf{i})\psi_{\mathsf{st}} e(\mathbf{j})$.*

b) [50, Lemma 3.17] *Suppose that ψ_{st} and $\hat\psi_{\mathsf{st}}$ are defined using different reduced expressions for the permutations $d(\mathsf{s}), d(\mathsf{t}) \in \mathfrak{S}_n$. Then there exist $a_{\mathsf{uv}} \in \mathcal{Z}$ such that*

$$\hat\psi_{\mathsf{st}} = \psi_{\mathsf{st}} + \sum_{(\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})} a_{\mathsf{uv}}\psi_{\mathsf{uv}},$$

where $a_{\mathsf{uv}} \ne 0$ only if $\mathbf{i}^{\mathsf{u}} = \mathbf{i}^{\mathsf{s}}$, $\mathbf{i}^{\mathsf{v}} = \mathbf{i}^{\mathsf{t}}$ and $\deg \mathsf{u} + \deg \mathsf{v} = \deg \mathsf{s} + \deg \mathsf{t}$.

c) [51, Corollary 3.11] *If* $1 \le r \le n$ *then there exist* $b_{\mathsf{uv}} \in \mathcal{Z}$ *such that*

$$\psi_{\mathsf{st}} y_r = \sum_{(\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})} b_{\mathsf{uv}} \psi_{\mathsf{uv}},$$

where $b_{\mathsf{uv}} \neq 0$ *only if* $\mathbf{i}^{\mathsf{u}} = \mathbf{i}^{\mathsf{s}}$, $\mathbf{i}^{\mathsf{v}} = \mathbf{i}^{\mathsf{t}}$ *and* $\deg \mathsf{u} + \deg \mathsf{v} = \deg \mathsf{s} + \deg \mathsf{t} + 2$.

Part (a) follows quickly using the relations in Definition 2.2.1 and the definition of the ψ-basis. In contrast, parts (b) and (c) are proved by using Theorem 3.1.1 to reduce an analogous properties of seminormal bases. With part (c), it is fairly easy to show that $b_{\mathsf{uv}} \neq 0$ only if $\mathsf{u} \trianglerighteq \mathsf{s}$. The difficult part is showing that $b_{\mathsf{uv}} \neq 0$ only if $\mathsf{v} \trianglerighteq \mathsf{t}$. Again, this is done using seminormal bases.

Finally, we note that Theorem 3.2.8 implies that $e(\mathbf{i}) \neq 0$ in $\mathscr{R}_n^\Lambda$ if and only if $\mathbf{i} \in I_\Lambda^n = \{\, \mathbf{i}^{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n) \,\}$, generalizing Proposition 2.4.6. In fact, if F is a field and $\mathscr{H}_n^\Lambda(F) \cong \mathscr{R}_n^\Lambda(F)$ then it is shown in [49, Lemma 4.1] that the non-zero KLR idempotents are a complete set of primitive (central) idempotents in the Gelfand-Zetlin algebra $\mathscr{L}_n(F)$ and that $\mathscr{L}_n(F) = \langle y_1, \dots, y_n, e(\mathbf{i}) \mid \mathbf{i} \in I^n \rangle$. It follows that $\mathscr{L}_n(F)$ is a positively graded commutative algebra with one dimensional irreducible modules indexed by I_Λ^n, up to shift. It would be interesting to find a (homogeneous) basis of $\mathscr{L}_n(F)$. The author would also like to know whether $\mathscr{R}_n^\Lambda$ is projective as an $\mathscr{L}_n$-module.

3.3. **Blocks and dual Specht modules.** This section shows that the blocks of $\mathscr{H}_n^\Lambda$ are graded symmetric algebras and it proves a corresponding statement relating the graded Specht modules and their graded duals.

Theorem 1.8.1 describes the block decomposition of $\mathscr{H}_n^\Lambda$ so, by Theorem 3.1.1, it also describes the block decomposition of $\mathscr{R}_n^\Lambda$. As in (2.2.6), let

$$\mathscr{R}_\beta^\Lambda = \mathscr{R}_n^\Lambda e_\beta, \qquad \text{where } e_\beta = \sum_{\mathbf{i} \in I^\beta} e(\mathbf{i}).$$

It follows from Definition 2.2.1 that e_β is central in $\mathscr{R}_n^\Lambda$, so $\mathscr{R}_\beta^\Lambda = e_\beta \mathscr{R}_n^\Lambda e_\beta$ is a two-sided ideal of $\mathscr{R}_n^\Lambda$. Let $Q_n^+ = Q_n^+(\Lambda) = \{\, \beta \in Q^+ \mid e_\beta \neq 0 \,\}$ in $\mathscr{R}_n^\Lambda$. Similarly, let $\mathcal{P}_\beta = \{\, \boldsymbol{\lambda} \in \mathcal{P}_n \mid \mathbf{i}^{\boldsymbol{\lambda}} \in I^\beta \,\} = \{\, \boldsymbol{\lambda} \in \mathcal{P}_n \mid \beta^{\boldsymbol{\lambda}} = \beta \,\}$.

Combining Theorem 3.2.8, Theorem 3.1.1 and Corollary 1.8.2 we obtain the following.

3.3.1. **Theorem.** *Suppose that* $\Lambda \in P^+$. *Then* $\mathscr{R}_n^\Lambda = \bigoplus_{\beta \in Q_n^+} \mathscr{R}_\beta^\Lambda$ *is the decomposition of* $\mathscr{R}_n^\Lambda$ *into indecomposable two-sided ideals. Moreover,* $\mathscr{R}_\beta^\Lambda$ *is a graded cellular algebra with cellular basis* $\{\, \psi_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_\beta) \,\}$ *and weight poset* $\mathcal{P}_\beta$.

By virtue of Theorem 3.2.8, the block decomposition of $\mathscr{R}_n^\Lambda$ holds over $\mathbb{Z}$, even though we cannot talk about the blocks as linkage classes of simple modules in this case. Compare with Theorem 2.4.8 in the semisimple case.

Suppose that A is a graded $\mathcal{Z}$-algebra. Then A is a **graded symmetric algebra** if there exists a homogeneous non-degenerate trace form $\tau\colon A \longrightarrow \mathcal{Z}$. That is, $\tau(ab) = \tau(ba)$ and if $0 \neq a \in A$ then there exists $b \in A$ such that $\tau(ab) \neq 0$. The map τ is **homogeneous of degree** d if $\tau(a) \neq 0$ only if $\deg a = -d$.

Fix $\beta \in Q^+$. The **defect** of β is the non-negative integer

$$\operatorname{def} \beta = (\Lambda, \beta) - \frac{1}{2}(\beta, \beta) = \frac{1}{2}\Big((\Lambda, \Lambda) - (\Lambda - \beta, \Lambda - \beta)\Big).$$

If $\boldsymbol{\lambda} \in \mathcal{P}_n$ set $\operatorname{def} \boldsymbol{\lambda} = \operatorname{def} \beta^{\boldsymbol{\lambda}}$ (see Corollary 1.8.2). If $\lambda \in \mathcal{P}_{1,n}$ is a partition then $\operatorname{def} \lambda$ is equal to its e-weight; see, for example, [34, Proposition 2.1] or the proof of [82, Lemma 7.6].

The definitions readily imply the following fundamental relationship connecting degrees, codegrees and defects.

3.3.2. **Lemma.** *Suppose that* $\boldsymbol{\lambda} \in \mathcal{P}_n$.

a) [23, Lemma 3.11] *If* $A \in \operatorname{Add}_i(\boldsymbol{\lambda})$ *then* $d_A(\boldsymbol{\lambda}) + 1 + d^A(\boldsymbol{\lambda}) = d_i(\boldsymbol{\lambda})$ *and* $\operatorname{def}(\boldsymbol{\lambda}+A) = \operatorname{def} \boldsymbol{\lambda} + d_i(\boldsymbol{\lambda}) - 1$.
b) [23, Lemma 3.12] *If* $\mathsf{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ *then* $\deg \mathsf{s} + \operatorname{codeg} \mathsf{s} = \operatorname{def} \boldsymbol{\lambda}$.

In Definition 3.2.2 we defined two sets of elements $\{\psi_{\mathsf{st}}\}$ and $\{\psi'_{\mathsf{st}}\}$ in $\mathscr{R}_n^\Lambda$. Just as there are two versions of the Murphy basis $\{m_{\mathsf{st}}\}$ which are built from the trivial and sign representations of $\mathscr{H}_n^\Lambda$ [99], respectively, there are two versions of the ψ-basis. By [49, Theorem 6.17], $\{\, \psi'_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n) \,\}$ is also a graded cellular basis of $\mathscr{H}_n^\Lambda$ with weight poset $(\mathcal{P}_n, \trianglelefteq)$ and with $\deg \psi'_{\mathsf{st}} = \operatorname{codeg} \mathsf{s} + \operatorname{codeg} \mathsf{t}$. We warn the reader that we are following the conventions of [50], rather than the notation of [49]. See [50, Lemma 3.15 and Remark 3.12] for the translation.

The bases $\{\psi_{\mathsf{st}}\}$ and $\{\psi'_{\mathsf{uv}}\}$ of $\mathscr{R}_n^\Lambda$ are dual in the sense that if $(\mathsf{s},\mathsf{t}), (\mathsf{u},\mathsf{v}) \in \operatorname{Std}^2(\mathcal{P}_\beta)$ then, by [51, Theorem 6.17],

$$\psi_{\mathsf{st}} \psi'_{\mathsf{ts}} \neq 0 \quad \text{and} \quad \psi_{\mathsf{st}} \psi'_{\mathsf{uv}} \neq 0 \qquad \text{only if} \quad \mathbf{i}^{\mathsf{t}} = \mathbf{i}^{\mathsf{u}} \text{ and } \mathsf{u} \trianglerighteq \mathsf{t}. \tag{3.3.3}$$

Let τ be the usual non-degenerate trace form on $\mathscr{H}_n^\Lambda$ [18, 93]. In general, τ is not homogeneous, however, it can be written as a sum of homogeneous components. Let τ_β be the homogeneous component of τ of degree $-2\operatorname{def}\beta$. By [51, Theorem 6.17], $\tau_\beta(\psi_{\mathsf{st}}\psi'_{\mathsf{st}})\neq 0$. Therefore, τ_β is non-degenerate and we obtain the following.

3.3.4. **Theorem** (Hu-Mathas [49, Corollary 6.18]). *Suppose that $\beta\in Q_n^+$. Then $\mathscr{R}_\beta^\Lambda$ a graded symmetric algebra with homogeneous trace form of degree $-2\operatorname{def}\beta$.*

It would be better to have an intrinsic definition of τ_β for $\mathscr{R}_n^\Lambda(\mathbb{Z})$. Webster [127, Remark 2.27] has given a diagrammatic description of a trace form on an arbitrary cyclotomic KLR algebra. It is unclear to the author how these two forms on $\mathscr{R}_n^\Lambda$ are related.

The basis $\{\psi'_{\mathsf{st}}\}$ is a graded cellular basis of $\mathscr{H}_n^\Lambda$ so it defines another collection of graded cell modules. The **dual graded Specht module** $S_{\boldsymbol\lambda}$ is the graded cell module indexed by $\boldsymbol\lambda\in\mathcal{P}_\beta$ and determined by the ψ'-basis. The dual Specht module $S_{\boldsymbol\lambda}$ has basis $\{\,\psi'_{\mathsf{t}}\mid \mathsf{t}\in\operatorname{Std}(\boldsymbol\lambda)\,\}$, with $\deg\psi'_{\mathsf{t}}=\operatorname{codeg}\mathsf{t}$, so

$$\dim_q S_{\boldsymbol\lambda}=\sum_{\mathsf{t}\in\operatorname{Std}(\boldsymbol\lambda)} q^{\operatorname{codeg}\mathsf{t}}.$$

We can identify $S_{\boldsymbol\lambda}\langle\operatorname{codeg}\mathsf{t}_{\boldsymbol\lambda}\rangle$ with $(\psi'_{\mathsf{t}_{\boldsymbol\lambda}\mathsf{t}_{\boldsymbol\lambda}}+\mathscr{H}_n'^{\lhd\boldsymbol\lambda})\mathscr{H}_n^\Lambda$, where $\mathscr{H}_n'^{\lhd\boldsymbol\lambda}$ is the two-sided ideal of $\mathscr{H}_n^\Lambda$ spanned by ψ'_{st} where $(\mathsf{s},\mathsf{t})\in\operatorname{Std}^2(\boldsymbol\mu)$ for some multipartition $\boldsymbol\mu$ such that $\boldsymbol\lambda\rhd\boldsymbol\mu$. Similarly, we can identify $S^{\boldsymbol\lambda}\langle\deg\mathsf{t}^{\boldsymbol\lambda}\rangle$ with $(\psi_{\mathsf{t}^{\boldsymbol\lambda}\mathsf{t}^{\boldsymbol\lambda}}+\mathscr{H}_n^{\rhd\boldsymbol\lambda})\mathscr{H}_n^\Lambda$. Therefore, by (3.3.3) there is a non-degenerate pairing

$$\{\ ,\ \}: S^{\boldsymbol\lambda}\langle\deg\mathsf{t}^{\boldsymbol\lambda}\rangle\times S_{\boldsymbol\lambda}\langle\operatorname{codeg}\mathsf{t}_{\boldsymbol\lambda}\rangle\longrightarrow\mathbb{Z}$$

given by $\{a+\mathscr{H}_n^{\rhd\boldsymbol\lambda}, b+\mathscr{H}_n'^{\lhd\boldsymbol\lambda}\}=\tau_\beta(ab^\star)$. Hence, using Lemma 3.3.2, we obtain:

3.3.5. **Corollary** (Hu-Mathas [49, Proposition 6.19]).
Suppose that $\boldsymbol\lambda\in\mathcal{P}_n$. Then $S^{\boldsymbol\lambda}\cong S_{\boldsymbol\lambda}^{\circledast}\langle\operatorname{def}\boldsymbol\lambda\rangle$ and $S_{\boldsymbol\lambda}=(S^{\boldsymbol\lambda})^{\circledast}\langle\operatorname{def}\boldsymbol\lambda\rangle$.

This result holds for the Specht modules defined over $\mathbb{Z}$ by Theorem 3.2.8 or by [75, Theorem 7.25].

There is an interesting a byproduct of the proof of Corollary 3.3.5. In the ungraded setting the Specht module $\underline{S}^{\boldsymbol\lambda}$ is isomorphic to the submodule of $\mathscr{H}_n^\Lambda$ generated by an element $m_{\boldsymbol\lambda}T_{w_{\boldsymbol\lambda}}m'_{\boldsymbol\lambda}$; see [31, Definition 2.1 and Theorem 2.9]. By [49, Corollary 6.21], $m_{\boldsymbol\lambda}T_{w_{\boldsymbol\lambda}}m'_{\boldsymbol\lambda}$ is homogeneous and, in fact, $\psi_{\mathsf{t}^{\boldsymbol\lambda}\mathsf{t}^{\boldsymbol\lambda}}\psi_{w_{\boldsymbol\lambda}}\psi'_{\mathsf{t}_{\boldsymbol\lambda}\mathsf{t}_{\boldsymbol\lambda}}=m_{\boldsymbol\lambda}T_{w_{\boldsymbol\lambda}}m'_{\boldsymbol\lambda}$. Moreover, $\psi_{\mathsf{t}^{\boldsymbol\lambda}\mathsf{t}^{\boldsymbol\lambda}}\psi_{w_{\boldsymbol\lambda}}\psi'_{\mathsf{t}_{\boldsymbol\lambda}\mathsf{t}_{\boldsymbol\lambda}}\mathscr{R}_n^\Lambda\cong S^{\boldsymbol\lambda}\langle\operatorname{def}\boldsymbol\lambda+\operatorname{codeg}\mathsf{t}_{\boldsymbol\lambda}\rangle$.

3.4. **Induction and restriction.** The cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda$ is naturally a subalgebra of $\mathscr{H}_{n+1}^\Lambda$, and $\mathscr{H}_{n+1}^\Lambda$ is free as an $\mathscr{H}_n^\Lambda$-module, by (1.1.2). This gives rise to the usual induction and restriction functors. These functors can be decomposed into the i-induction and i-restriction functors, for $i\in I$, by projecting onto the blocks of these two algebras. As we will see, these functors are implicitly built into the graded setting.

Recall that $I=\mathbb{Z}/e\mathbb{Z}$ and $\Lambda\in P^+$. For each $i\in I$ define

$$e_{n,i}=\sum_{\mathbf{j}\in I^n} e(\mathbf{j}\vee i)\in\mathscr{R}_{n+1}^\Lambda.$$

The relations for $\mathscr{R}_{n+1}^\Lambda$ in Definition 2.2.1 imply that $e_{n,i}$ is an idempotent and that $\sum_{i\in I}e_{n,i}=\sum_{\mathbf{i}\in I^{n+1}}e(\mathbf{i})$ is the identity element of $\mathscr{R}_{n+1}^\Lambda$.

3.4.1. **Lemma.** *Suppose that $i\in I$ and that $\mathcal{Z}$ is an integral domain. Then there is a (non-unital) embedding of graded algebras $\mathscr{R}_n^\Lambda\hookrightarrow\mathscr{R}_{n+1}^\Lambda$ given by*

$$e(\mathbf{j})\mapsto e(\mathbf{j}\vee i),\qquad y_r\mapsto e_{n,i}y_r\quad\text{and}\quad\psi_s\mapsto e_{n,i}\psi_s,$$

for $\mathbf{j}\in I^n$, $1\le r\le n$ and $1\le s<n$. This map induces an exact functor

$$i\text{-Ind}:\operatorname{Rep}(\mathscr{R}_n^\Lambda)\longrightarrow\operatorname{Rep}(\mathscr{R}_{n+1}^\Lambda);M\mapsto M\otimes_{\mathscr{R}_n^\Lambda}e_{n,i}\mathscr{R}_{n+1}^\Lambda.$$

Moreover, $\operatorname{Ind}=\bigoplus_{i\in I}i\text{-Ind}$ *is the graded induction functor from* $\operatorname{Rep}(\mathscr{R}_n^\Lambda)$ *to* $\operatorname{Rep}(\mathscr{R}_{n+1}^\Lambda)$.

Proof. The images of the homogeneous generators of $\mathscr{R}_n^\Lambda$ under this embedding commute with $e_{n,i}$, which implies that this map defines a non-unital degree preserving homomorphism from $\mathscr{R}_n^\Lambda$ to $\mathscr{R}_{n+1}^\Lambda$. This map is an embedding by Theorem 3.2.8. The remaining claims follow because, by definition, $e_{n,i}$ is an idempotent and $\sum_{i\in I}e_{n,i}$ is the identity element of $\mathscr{R}_{n+1}^\Lambda$. □

The ***i*-induction functor** i-Ind functor is obviously a left adjoint to the ***i*-restriction** i-Res functor which sends an $\mathscr{R}_{n+1}^\Lambda$-module M to $e_{n,i}M=M\otimes_{\mathscr{R}_{n+1}^\Lambda}\mathscr{R}_{n+1}^\Lambda e_{n,i}$. A much harder fact is that these functors are two-sided adjoints.

3.4.2. **Theorem** (Kashiwara [64, Theorem 3.5]). *Suppose $i\in I$. Then (E_i,F_i) is a biadjoint pair.*

Kashiwara proves this theorem for all cyclotomic quiver Hecke algebras such that the associated Cartan matrix is symmetrizable. We are cheating by stating this result now because its proof builds upon Kang and Kashiwara's proof that the cyclotomic quiver Hecke algebras of arbitrary type categorify the integrable highest weight modules of the corresponding quantum group [62]. Theorem 3.4.2 was conjectured by Khovanov-Lauda [67] and Rouquier [114].

We want to describe these functors on the graded Specht modules. This result generalizes the well-known (ungraded) branching rules for the symmetric group [54, Example 17.16] and the cyclotomic Hecke algebras [10, 102, 118].

Recall the definition of the integers $d^A(\boldsymbol{\lambda})$ and $d_A(\boldsymbol{\lambda})$ from (3.2.1).

3.4.3. **Theorem.** *Suppose that $\mathcal{Z}$ is an integral domain and $\boldsymbol{\lambda} \in \mathcal{P}_n$.*

a) [51, Main theorem] *Let $A_1 < A_2 \dots < A_z$ be the addable i-nodes of $\boldsymbol{\lambda}$. Then i-$\operatorname{Ind} S^{\boldsymbol{\lambda}}$ has a graded Specht filtration*
$$0 = I_0 \subset I_1 \subset \dots \subset I_z = i\text{-}\operatorname{Ind} S^{\boldsymbol{\lambda}},$$
such that $I_j/I_{j-1} \cong S^{\boldsymbol{\lambda}+A_j}\langle d^{A_j}(\boldsymbol{\lambda})\rangle$,

b) [23, Theorem 4.11] *Let $B_y < \dots < B_2 < B_1$ be the removable i-nodes of $\boldsymbol{\lambda}$. Then i-$\operatorname{Res} S^{\boldsymbol{\lambda}}$ has a graded Specht filtration*
$$0 = R_0 \subset R_1 \subset \dots \subset R_y = i\text{-}\operatorname{Res} S^{\boldsymbol{\lambda}},$$
such that $R_j/R_{j-1} \cong S^{\boldsymbol{\lambda}-B_j}\langle d^{B_j}(\boldsymbol{\lambda})\rangle$, for $1 \le j \le y$.

c) [51, Corollary 4.6] *Let $A_z < \dots < A_2 < A_1$ be the addable i-nodes of $\boldsymbol{\lambda}$. Then i-$\operatorname{Ind} S_{\boldsymbol{\lambda}}$ has a graded Specht filtration*
$$0 = I_0 \subset I_1 \subset \dots \subset I_z = i\text{-}\operatorname{Ind} S_{\boldsymbol{\lambda}},$$
such that $I_j/I_{j-1} \cong S^{\boldsymbol{\lambda}+A_j}\langle d_{A_j}(\boldsymbol{\lambda})\rangle$, for $1 \le j \le z$.

The corresponding statement for the restriction of the dual graded Specht modules follows easily using Corollary 3.3.5. As we do not need this we leave it as an exercise for the reader.

Part (b) was proved first using a standard argument based on properties of the graded cellular basis of $S^{\boldsymbol{\lambda}}$. Part (a), which was conjectured by Brundan, Kleshchev and Wang [23, Remark 4.12], is proved by extending some elegant ideas of Ryom-Hansen [118] to the graded setting using results from [49].

3.5. **Grading Ariki's Categorification Theorem.** We now relate the graded representation theory of the Hecke algebras $\mathscr{H}_n^\Lambda$ with the representation theory of the quantum group $U_q(\widehat{\mathfrak{sl}}_e)$ by lifting Ariki's Categorification Theorem to the graded setting. This allows us to give a new proof of Brundan and Kleshchev's theorem that the cyclotomic KLR algebras categorify the integrable highest weight modules of $U_q(\widehat{\mathfrak{sl}}_e)$. Our main tools are Ariki's categorification theorem, the graded branching rules of Theorem 3.4.3 and the Fock space and canonical basis combinatorics.

Throughout this section we assume that the Hecke algebras $\mathscr{H}_n^\Lambda$ are defined over a field F, for $n \ge 0$. In the end we will assume that F is a field of characteristic zero, however, almost all of the results in this section hold over any field F.

Recall that $U_q(\widehat{\mathfrak{sl}}_e)$ is the quantum group over $\mathbb{Q}(q)$ associated with the quiver Γ_e. Therefore, $U_q(\widehat{\mathfrak{sl}}_e)$ is generated by elements E_i, F_i and $K_i^{\pm}$, for $i \in I$, subject to the quantum Serre relations [88, §3.1].

Let $\mathcal{P} = \bigcup_{n\ge0}\mathcal{P}_n$, $\mathcal{K}^\Lambda = \bigcup_{n\ge0}\mathcal{K}_n^\Lambda$ and set $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$. The **combinatorial Fock space** $\mathscr{F}_{\mathcal{A}}^\Lambda$ is the free $\mathcal{A}$-module with basis the set of symbols $\{\,|\boldsymbol{\lambda}\rangle \mid \boldsymbol{\lambda} \in \mathcal{P}\,\}$. Let $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda = \mathscr{F}_{\mathcal{A}}^\Lambda \otimes_{\mathcal{A}} \mathbb{Q}(q)$. Then, $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$ is an infinite dimensional $\mathbb{Q}(q)$-vector space. We consider $\{\,|\boldsymbol{\lambda}\rangle \mid \boldsymbol{\lambda} \in \mathcal{P}\,\}$ as a basis of $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$ by identifying $|\boldsymbol{\lambda}\rangle$ and $|\boldsymbol{\lambda}\rangle \otimes 1_{\mathbb{Q}(q)}$.

3.5.1. **Theorem** (Hayashi [47])**.** *Suppose that $\Lambda \in P^+$. Then $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$ is an integrable $U_q(\widehat{\mathfrak{sl}}_e)$-module with $U_q(\widehat{\mathfrak{sl}}_e)$-action determined by*
$$E_i|\boldsymbol{\lambda}\rangle = \sum_{B\in\operatorname{Rem}_i(\boldsymbol{\lambda})} q^{d_B(\boldsymbol{\lambda})}|\boldsymbol{\lambda}-B\rangle \quad \text{and} \quad F_i|\boldsymbol{\lambda}\rangle = \sum_{A\in\operatorname{Add}_i(\boldsymbol{\lambda})} q^{-d^A(\boldsymbol{\lambda})}|\boldsymbol{\lambda}+A\rangle,$$
and $K_i|\boldsymbol{\lambda}\rangle = q^{d_i(\boldsymbol{\lambda})}|\boldsymbol{\lambda}\rangle$, for all $i \in I$ and $\boldsymbol{\lambda} \in \mathcal{P}_n$.

Hayashi [47] considered only the special case when $\Lambda = \Lambda_0$. The general case follows easily from this using the coproduct of $U_q(\widehat{\mathfrak{sl}}_e)$ because the definitions imply that $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda \cong \mathscr{F}_{\mathbb{Q}(q)}^{\Lambda_{\overline{\kappa_1}}} \otimes \dots \otimes \mathscr{F}_{\mathbb{Q}(q)}^{\Lambda_{\overline{\kappa_\ell}}}$ as a $U_q(\widehat{\mathfrak{sl}}_e)$-module. The crystal and canonical bases of the higher level Fock spaces were studied in [60, 104, 124].

For each dominant weight $\Lambda \in P^+$ let $L(\Lambda) = U_q(\widehat{\mathfrak{sl}}_e)v_\Lambda$ be the integrable highest weight module of high weight Λ, where v_Λ is a highest weight vector of weight Λ. It follows from Theorem 3.5.1 that $L(\Lambda) \cong U_q(\widehat{\mathfrak{sl}}_e)|\underline{\mathbf{0}}\rangle$ as $U_q(\widehat{\mathfrak{sl}}_e)$-modules, where $\underline{\mathbf{0}} = (0|0|\dots|0) \in \mathcal{P}_0$ is the empty multipartition of level ℓ.

Let $\mathrm{Rep}(\mathscr{H}_n^\Lambda)$ be the category of finitely generated graded $\mathscr{H}_n^\Lambda$-modules and let $\mathrm{Proj}_{\mathcal{A}}(\mathscr{H}_n^\Lambda)$ be the category of finitely generated projective graded $\mathscr{H}_n^\Lambda$-modules. Let $[\mathrm{Rep}(\mathscr{H}_n^\Lambda)]$ and $[\mathrm{Proj}(\mathscr{H}_n^\Lambda)]$ be the Grothendieck groups of these categories. If M is a finitely generated $\mathscr{H}_n^\Lambda$-module let $[M]$ be its image in $[\mathrm{Rep}(\mathscr{H}_n^\Lambda)]$ or, abusing notation slightly, in $[\mathrm{Proj}(\mathscr{H}_n^\Lambda)]$ if M is projective.

3.5.2. **Definition.** *Suppose that $\boldsymbol{\mu}\in\mathcal{K}_n^\Lambda$. Let $Y^{\boldsymbol{\mu}}$ be the projective cover of $D^{\boldsymbol{\mu}}$ in* $\mathrm{Rep}(\mathscr{H}_n^\Lambda)$.

Then $\{\,[D^{\boldsymbol{\mu}}]\mid\boldsymbol{\mu}\in\mathcal{K}_n^\Lambda\,\}$ is a basis of $[\mathrm{Rep}(\mathscr{H}_n^\Lambda)$ and $\{\,[Y^{\boldsymbol{\mu}}]\mid\boldsymbol{\mu}\in\mathcal{K}_n^\Lambda\,\}$ is a basis for $\mathrm{Proj}(\mathscr{H}_n^\Lambda)$. We use the notation $Y^{\boldsymbol{\mu}}$ because these modules are special cases of the graded lifts of the *Young modules* constructed in [98]; see [50, §5.1] and [92, §2.6]. By Corollary 2.1.5, $[P^{\boldsymbol{\mu}}]=\sum_{\boldsymbol{\lambda}}d_{\boldsymbol{\lambda\mu}}(q)[S^{\boldsymbol{\lambda}}]$ in $[\mathrm{Rep}(\mathscr{H}_n^\Lambda)]$.

Consider $[\mathrm{Rep}(\mathscr{H}_n^\Lambda)]$ and $[\mathrm{Proj}(\mathscr{H}_n^\Lambda)]$ as $\mathcal{A}$-modules by letting q act as the grading shift functor: $[M\langle d\rangle]=q^d[M]$, for $d\in\mathbb{Z}$. Set

$$[\mathrm{Rep}_{\mathcal{A}}^\Lambda]=\bigoplus_{n\ge0}[\mathrm{Rep}(\mathscr{H}_n^\Lambda)]\quad\text{and}\quad[\mathrm{Proj}_{\mathcal{A}}^\Lambda]=\bigoplus_{n\ge0}[\mathrm{Proj}(\mathscr{H}_n^\Lambda)].$$

Extending scalars, let $[\mathrm{Rep}_{\mathbb{Q}(q)}^\Lambda]=[\mathrm{Rep}_{\mathcal{A}}^\Lambda]\otimes_{\mathcal{A}}\mathbb{Q}(q)$ and $[\mathrm{Proj}_{\mathbb{Q}(q)}^\Lambda]=[\mathrm{Proj}_{\mathcal{A}}^\Lambda]\otimes_{\mathcal{A}}\mathbb{Q}(q)$.

3.5.3. **Proposition.** *Suppose that $\Lambda\in P^+$. Then the i-induction and i-restriction functors of $[\mathrm{Rep}_{\mathbb{Q}(q)}^\Lambda](\mathscr{H}_n^\Lambda)$ induce isomorphisms $[\mathrm{Proj}_{\mathbb{Q}(q)}^\Lambda]\cong L(\Lambda)\cong[\mathrm{Rep}_{\mathbb{Q}(q)}^\Lambda]$ of $U_q(\widehat{\mathfrak{sl}}_e)$-modules.*

Proof. Recall that $\mathbf{d}_q$ is the graded decomposition matrix of $\mathscr{H}_n^\Lambda$ and $\mathbf{d}_q^T$ is its transpose. Define linear maps

$$\begin{tikzcd}[\mathrm{Proj}_{\mathbb{Q}(q)}^\Lambda] \arrow[r, hook, "\mathbf{d}_q^T"] \arrow[dr, "\mathbf{c}_q"'] & \mathscr{F}_{\mathbb{Q}(q)}^\Lambda \arrow[d, "\mathbf{d}_q"] \\ & [\mathrm{Rep}_{\mathbb{Q}(q)}^\Lambda]\end{tikzcd}$$

where $\mathbf{d}_q^T([Y^{\boldsymbol{\mu}}])=\sum_{\boldsymbol{\lambda}}d_{\boldsymbol{\lambda\mu}}(q)|\boldsymbol{\lambda}\rangle$, $\mathbf{d}_q(|\boldsymbol{\lambda}\rangle)=\sum_{\boldsymbol{\mu}}d_{\boldsymbol{\lambda\mu}}(q)[D^{\boldsymbol{\mu}}]$ and where $\mathbf{c}_q=\mathbf{d}_q^T\circ\mathbf{d}_q$ is the Cartan map. We claim that that these maps can be made into $U_q(\widehat{\mathfrak{sl}}_e)$-module homomorphisms.

The i-induction and i-restriction functors are exact, for $i\in I$. Therefore, they send projective modules to projectives and they induce vector space endomorphisms of the Grothendieck groups $[\mathrm{Rep}_{\mathbb{Q}(q)}^\Lambda]$ and $[\mathrm{Proj}_{\mathbb{Q}(q)}^\Lambda]$. By Theorem 3.4.3, and Lemma 3.3.2(a) for the first formula,

$$\begin{aligned}[][i\text{-}\mathrm{Ind}\,S^{\boldsymbol{\lambda}}\langle1-d_i(\boldsymbol{\lambda})\rangle]&=\sum_{A\in\mathrm{Add}_i(\boldsymbol{\lambda})}q^{d_A(\boldsymbol{\lambda})+1-d_i(\boldsymbol{\lambda})}[S^{\boldsymbol{\lambda}+A}]=\sum_{A\in\mathrm{Add}_i(\boldsymbol{\lambda})}q^{-d^A(\boldsymbol{\lambda})}[S^{\boldsymbol{\lambda}+A}],\\ [i\text{-}\mathrm{Res}\,S^{\boldsymbol{\lambda}}]&=\sum_{B\in\mathrm{Rem}_i(\boldsymbol{\lambda})}q^{d_B(\boldsymbol{\lambda})}[S^{\boldsymbol{\lambda}-B}].\end{aligned}$$

Identifying E_i with i-Res and $qF_iK_i^{-1}$ with i-Ind, the vector space maps $\mathbf{d}_q$ and $\mathbf{d}_q^T$ become well-defined $U_q(\widehat{\mathfrak{sl}}_e)$-module homomorphisms by Theorem 3.5.1. By construction, the $U_q(\widehat{\mathfrak{sl}}_e)$-modules $[\mathrm{Rep}_{\mathbb{Q}(q)}^\Lambda]$ and $[\mathrm{Proj}_{\mathbb{Q}(q)}^\Lambda]$ are both cyclic, being generated by $[P^{\underline{\mathbf{0}}}]=[S^{\underline{\mathbf{0}}}]=[D^{\underline{\mathbf{0}}}]$. As $L(\Lambda)\cong U_q(\widehat{\mathfrak{sl}}_e)|\underline{\mathbf{0}}\rangle$ is irreducible, the proposition follows. □

By Theorem 2.1.4(c), the graded decomposition matrix $\mathbf{d}_q=\big(d_{\boldsymbol{\lambda\mu}}(q)\big)$ is invertible over $\mathcal{A}$ with inverse $\mathbf{e}_q=\big(e_{\boldsymbol{\lambda\mu}}(q)\big)$. Therefore, we can consider $\{\,[S^{\boldsymbol{\lambda}}]\mid\boldsymbol{\lambda}\in\mathcal{K}^\Lambda\,\}$ to be an $\mathcal{A}$-basis of either Grothendieck group, where we abuse notation by identifying $[S^{\boldsymbol{\mu}}]$ in $[\mathrm{Proj}_{\mathbb{Q}(q)}^\Lambda]$ with $(\mathbf{d}_q^T)^{-1}|\boldsymbol{\mu}\rangle)=\sum_{\boldsymbol{\lambda}}e_{\boldsymbol{\lambda\mu}}(q)[Y^{\boldsymbol{\lambda}}]$, for $\boldsymbol{\mu}\in\mathcal{K}^\Lambda$.

Let $U_q(\widehat{\mathfrak{sl}}_e)_{\mathcal{A}}$ be Lusztig's $\mathcal{A}$-form of $U_q(\widehat{\mathfrak{sl}}_e)$. Theorem 3.5.1 implies that $U_q(\widehat{\mathfrak{sl}}_e)_{\mathcal{A}}$ acts on the $\mathcal{A}$-submodule of $\mathscr{F}_{\mathcal{A}}^\Lambda$ of $\mathscr{F}_{\mathbb{Q}(q)}^\Lambda$. In particular, there are well-defined actions of the divided powers $E_i^{(k)}$ and $F_i^{(k)}$ on $\mathscr{F}_{\mathcal{A}}^\Lambda$, for $i\in I$ an $k\ge0$. By Proposition 3.5.3, $[\mathrm{Rep}_{\mathcal{A}}^\Lambda]$ and $[\mathrm{Proj}_{\mathcal{A}}^\Lambda]$ are both $U_q(\widehat{\mathfrak{sl}}_e)_{\mathcal{A}}$-modules.

The **bar involution** on $\mathcal{A}=\mathbb{Z}[q,q^{-1}]$ is the unique $\mathbb{Z}$-linear map such that $\overline{q}=q^{-1}$. A **semilinear** map of $\mathcal{A}$-modules is a $\mathbb{Z}$-linear map $\theta\colon M\longrightarrow N$ such that $\theta(f(q)m)=\overline{f(q)}\theta(m)$, for all $f(q)\in\mathcal{A}$ and $m\in M$.

There is a natural pairing $(\ ,\ )\colon[\mathrm{Proj}_{\mathcal{A}}^\Lambda]\times[\mathrm{Rep}_{\mathcal{A}}^\Lambda]\longrightarrow\mathcal{A}$ that is determined by

$$([P],[M])=\mathit{dim}_q\,\mathcal{H}om_{\mathscr{H}_n^\Lambda}(P,M),$$

for graded $\mathscr{H}_n^\Lambda$-modules P and M with P projective. This pairing is **sesquilinear** in the sense that it is semilinear in the first variable and $\mathcal{A}$-linear in the second. By definition, if $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}^\Lambda$ then $([P^{\boldsymbol{\lambda}}],[D^{\boldsymbol{\mu}}])=\delta_{\boldsymbol{\lambda\mu}}$. By biadjointness (Theorem 3.4.2), or a direct calculation using Theorem 3.4.3,

$$(i\text{-}\mathrm{Ind}[P],[M])=([P],i\text{-}\mathrm{Res}[M])\qquad\text{and}\qquad(i\text{-}\mathrm{Res}[P],[M])=([P],i\text{-}\mathrm{Ind}[M]).$$

Therefore, $(\ ,\ )$ is a **Shapovalov form** in the sense of [20, (3.39)].

Recall that $M^{\circledast} = \operatorname{Hom}_F(M,F)$ is the contragredient dual of the graded $\mathscr{H}_n^\Lambda$-module M. Similarly, if P is a graded projective $\mathscr{H}_n^\Lambda$-module define $P^{\#} = \operatorname{Hom}_{\mathscr{H}_n^\Lambda}(P, \mathscr{H}_n^\Lambda)$. In both cases the $\mathscr{H}_n^\Lambda$-action is given by $(f\cdot h)(m) = f(mh^\star)$, for $h\in\mathscr{H}_n^\Lambda$, $m\in M$ and $f\in M^{\circledast}$ or $f\in P^{\#}$. These dualities induce semilinear linear involutions on $[\operatorname{Rep}^\Lambda_{\mathbb{Q}(q)}]$ and $[\operatorname{Proj}^\Lambda_{\mathbb{Q}(q)}]$, which are given by

$$[P]^{\#} = [P^{\#}], \quad \text{and} \quad [M]^{\circledast} = [M^{\circledast}]$$

for $M\in\operatorname{Rep}(\mathscr{H}_n^\Lambda)$ and $P\in\operatorname{Proj}(\mathscr{H}_n^\Lambda)$.

We can now show that the Specht modules and dual Specht modules are dual bases with respect to the Shapovalov form. if $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{K}^\Lambda$ then

$$\begin{aligned}([S^{\boldsymbol{\lambda}}],[S^{\boldsymbol{\mu}\circledast}]) &= \sum_{\boldsymbol{\sigma}} \overline{e_{\boldsymbol{\sigma\lambda}}(q)}([P^{\boldsymbol{\sigma}}],[S^{\boldsymbol{\mu}}]^{\circledast}) = \sum_{\boldsymbol{\sigma},\boldsymbol{\tau}} \overline{e_{\boldsymbol{\sigma\lambda}}(q)}\,\overline{d_{\boldsymbol{\mu\tau}}(q)}([P^{\boldsymbol{\sigma}}],[D^{\boldsymbol{\tau}}]) \\ &= \sum_{\boldsymbol{\sigma}} \overline{d_{\boldsymbol{\mu\sigma}}(q)}\,\overline{e_{\boldsymbol{\sigma\lambda}}(q)} = \delta_{\boldsymbol{\lambda\mu}}.\end{aligned} \tag{3.5.4}$$

Equivalently, $([S^{\boldsymbol{\lambda}}],[S_{\boldsymbol{\mu}}]) = \delta_{\boldsymbol{\lambda\mu}}q^{-\operatorname{def}\boldsymbol{\mu}}$ by Corollary 3.3.5. In particular, the form $(\ ,\ )$ is non-degenerate. The Shapovalov form justifies our making the identifications $[\operatorname{Proj}^\Lambda_{\mathcal{A}}] = L(\Lambda)_{\mathcal{A}}$ and $[\operatorname{Rep}^\Lambda_{\mathcal{A}}] = L(\Lambda)^*_{\mathcal{A}}$.

Importantly, the involutions $\circledast$ and $\#$ commute with the action of $U_q(\widehat{\mathfrak{sl}}_e)^-_{\mathcal{A}}$.

3.5.5. **Lemma.** *The involutions $\#$ and $\circledast$ on $[\operatorname{Proj}^\Lambda_{\mathcal{A}}]$ and $[\operatorname{Rep}^\Lambda_{\mathcal{A}}]$, respectively, commute with the action of F_i, for $i\in I$.*

Proof. It is enough to check that F_i commutes with $\#$ and $\circledast$ on the Specht modules in $[\operatorname{Proj}^\Lambda_{\mathcal{A}}]$ and $[\operatorname{Rep}^\Lambda_{\mathcal{A}}]$. Now,

$$\begin{aligned}
F_i[S^{\boldsymbol{\lambda}}]^{\circledast} &= q^{-\operatorname{def}\boldsymbol{\lambda}}F_i[S_{\boldsymbol{\lambda}}], && \text{by Corollary 3.3.5,}\\
&= q^{-\operatorname{def}\boldsymbol{\lambda}} \sum_{A\in\operatorname{Add}_i(\boldsymbol{\lambda})} q^{-d^A(\boldsymbol{\lambda})}[S_{\boldsymbol{\lambda}+A}], && \text{by Theorem 3.4.3(c),}\\
&= \sum_{A\in\operatorname{Add}_i(\boldsymbol{\lambda})} q^{-d^A(\boldsymbol{\lambda})+d_i(\boldsymbol{\lambda})-1-\operatorname{def}(\boldsymbol{\lambda}+A)}[S_{\boldsymbol{\lambda}+A}], && \text{by Lemma 3.3.2(a),}\\
&= \sum_{A\in\operatorname{Add}_i(\boldsymbol{\lambda})} q^{d_A(\boldsymbol{\lambda})-\operatorname{def}(\boldsymbol{\lambda}+A)}[S_{\boldsymbol{\lambda}+A}], && \text{by Lemma 3.3.2(a),}\\
&= \Big(\sum_{A\in\operatorname{Add}_i(\boldsymbol{\lambda})} q^{-d^A(\boldsymbol{\lambda})}[S^{\boldsymbol{\lambda}+A}]\Big)^{\circledast}, && \text{by Corollary 3.3.5,}\\
&= (F_i[S^{\boldsymbol{\lambda}}])^{\circledast}.
\end{aligned}$$

Essentially the same argument shows that $F_i[S^{\boldsymbol{\lambda}}]^{\#} = (F_i[S^{\boldsymbol{\lambda}}])^{\#}$. □

Similarly, $\circledast$ and $\#$ commute with E_i, for $i\in I$.

The following result is well-known and easily verified. See, for example, [20, Lemma 2.5].

3.5.6. **Lemma.** *Suppose $[P]\in[\operatorname{Proj}^\Lambda_{\mathcal{A}}]$ and $[M]\in[\operatorname{Rep}^\Lambda_{\mathcal{A}}]$. Then $([P]^{\#},[M]) = \overline{([P],[M]^{\circledast})}$.*

The effect of the involutions $\#$ and $\circledast$ on $[\operatorname{Proj}^\Lambda_{\mathcal{A}}]$ and $[\operatorname{Rep}^\Lambda_{\mathcal{A}}]$ is particularly nice.

3.5.7. **Lemma.** *Suppose that $\boldsymbol{\lambda}\in\mathcal{K}^\Lambda$. Then $[Y^{\boldsymbol{\lambda}}]^{\#} = [Y^{\boldsymbol{\lambda}}]$, $[D^{\boldsymbol{\lambda}}]^{\circledast} = [D^{\boldsymbol{\lambda}}]$,*

$$[S^{\boldsymbol{\lambda}}]^{\#} = [S^{\boldsymbol{\lambda}}] + \sum_{\substack{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_n\\ \boldsymbol{\mu}\rhd\boldsymbol{\lambda}}} a^{\boldsymbol{\lambda\mu}}(q)[S^{\boldsymbol{\mu}}] \quad \text{and} \quad [S^{\boldsymbol{\lambda}}]^{\circledast} = [S^{\boldsymbol{\lambda}}] + \sum_{\substack{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_n\\ \boldsymbol{\lambda}\rhd\boldsymbol{\mu}}} a_{\boldsymbol{\lambda\mu}}(q)[S^{\boldsymbol{\mu}}],$$

for some Laurent polynomials $a_{\boldsymbol{\lambda\mu}}(q), a^{\boldsymbol{\lambda\mu}}(q)\in\mathcal{A}$.

Proof. That $[D^{\boldsymbol{\mu}}]^{\circledast} = [D^{\boldsymbol{\mu}}]$ is immediate by Corollary 3.2.7. Using Lemma 3.5.6, this implies that $[Y^{\boldsymbol{\mu}}]^{\#} = [Y^{\boldsymbol{\mu}}]$. Finally, by Theorem 2.1.4,

$$\begin{aligned}
[S^{\boldsymbol{\lambda}}]^{\circledast} &= \Big(\sum_{\boldsymbol{\lambda}\unrhd\boldsymbol{\mu}} d_{\boldsymbol{\lambda\mu}}(q)[D^{\boldsymbol{\mu}}]\Big)^{\circledast} = \sum_{\substack{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_n\\ \boldsymbol{\lambda}\unrhd\boldsymbol{\mu}}} d_{\boldsymbol{\lambda\mu}}(q^{-1})[D^{\boldsymbol{\mu}}]\\
&= \sum_{\substack{\boldsymbol{\nu}\in\mathcal{K}^\Lambda_n\\ \boldsymbol{\lambda}\unrhd\boldsymbol{\nu}}} \Big(\sum_{\substack{\boldsymbol{\mu}\in\mathcal{K}^\Lambda_n\\ \boldsymbol{\nu}\unrhd\boldsymbol{\mu}\unrhd\boldsymbol{\lambda}}} d_{\boldsymbol{\lambda\mu}}(q^{-1})e_{\boldsymbol{\mu\nu}}(q)\Big)[S^{\boldsymbol{\nu}}]
\end{aligned}$$

as claimed. Writing $[S^{\boldsymbol\lambda}]=\sum_{\boldsymbol\mu} e_{\boldsymbol\mu\boldsymbol\lambda}(q)[P^{\boldsymbol\mu}]$, essentially the same argument shows that $[S^{\boldsymbol\mu}]^{\#}$ can be written in the required form. Alternatively, use Lemma 3.5.6. ∎

The triangularity of the action of $\circledast$ and $\#$ on $[\mathrm{Rep}_{\mathcal{A}}^{\Lambda}]$ and $[\mathrm{Proj}_{\mathcal{A}}^{\Lambda}]$, respectively, has the following easy but important consequence.

3.5.8. **Proposition.** *There exist bases* $\{\,B^{\boldsymbol\lambda}\mid \boldsymbol\lambda\in\mathcal{K}^\Lambda\,\}$ *and* $\{\,B_{\boldsymbol\lambda}\mid \boldsymbol\lambda\in\mathcal{K}^\Lambda\,\}$ *of* $[\mathrm{Proj}_{\mathcal{A}}^{\Lambda}]$ *and* $[\mathrm{Rep}_{\mathcal{A}}^{\Lambda}]$*, respectively, which are uniquely determined by the properties* $(B^{\boldsymbol\lambda})^{\#}=B^{\boldsymbol\lambda}$ *and* $(B_{\boldsymbol\lambda})^{\circledast}=B_{\boldsymbol\lambda}$

$$B^{\boldsymbol\lambda}=[S^{\boldsymbol\lambda}]+\sum_{\substack{\boldsymbol\mu\in\mathcal{K}_n^\Lambda\\ \boldsymbol\mu\rhd\boldsymbol\lambda}} b^{\boldsymbol\lambda\boldsymbol\mu}(q)[S^{\boldsymbol\mu}]\quad\textit{and}\quad B_{\boldsymbol\lambda}=[S^{\boldsymbol\lambda}]+\sum_{\substack{\boldsymbol\mu\in\mathcal{K}_n^\Lambda\\ \boldsymbol\lambda\rhd\boldsymbol\mu}} b_{\boldsymbol\lambda\boldsymbol\mu}(q)[S^{\boldsymbol\mu}]$$

for <u>*polynomials*</u> $b_{\boldsymbol\lambda\boldsymbol\mu}(q),b^{\boldsymbol\lambda\boldsymbol\mu}(q)\in\delta_{\boldsymbol\lambda\boldsymbol\mu}+q\mathbb{Z}[q]$*. Moreover, if* $\boldsymbol\lambda,\boldsymbol\mu\in\mathcal{K}^\Lambda$ *then*

$$(B^{\boldsymbol\lambda},B_{\boldsymbol\mu})=\sum_{\substack{\boldsymbol\sigma\in\mathcal{K}^\Lambda\\ \boldsymbol\lambda\unrhd\boldsymbol\sigma\unrhd\boldsymbol\mu}} b^{\boldsymbol\lambda\boldsymbol\sigma}(q)b_{\boldsymbol\sigma\boldsymbol\mu}(q)=\delta_{\boldsymbol\lambda\boldsymbol\mu}.$$

Proof. The existence and uniqueness of these two bases follows immediately from Lemma 3.5.7 by a standard argument known as *Lusztig's Lemma* [88, Lemma 24.2.1]. For completeness, we quickly sketch a common variation on this argument for the basis $\{B_{\boldsymbol\mu}\}$.

Fix a multipartition $\boldsymbol\mu\in\mathcal{K}_n^\Lambda$, for some $n\ge 0$, and suppose that $B_{\boldsymbol\mu}$ and $B'_{\boldsymbol\mu}$ are two elements of $[\mathrm{Proj}_{\mathcal{A}}^{\Lambda}]$ with the required properties. Then $B_{\boldsymbol\mu}-B'_{\boldsymbol\mu}$ is $\circledast$-invariant. By assumption we can write $B_{\boldsymbol\mu}-B'_{\boldsymbol\mu}=\sum_{\boldsymbol\lambda\rhd\boldsymbol\mu} b'_{\boldsymbol\lambda\boldsymbol\mu}(q)[S^{\boldsymbol\lambda}]$, for some polynomials $b'_{\boldsymbol\lambda\boldsymbol\mu}(q)\in\mathbb{Z}[q]$. Since these coefficients are polynomials, Lemma 3.5.7 forces $B_{\boldsymbol\mu}-B'_{\boldsymbol\mu}=0$, proving uniqueness.

To prove existence, we argue by induction on dominance. If $\boldsymbol\mu$ is minimal in $\mathcal{K}_n^\Lambda$ then we can set $B_{\boldsymbol\mu}=[S^{\boldsymbol\mu}]=D^{\boldsymbol\mu}$ by Lemma 3.5.7. If $\boldsymbol\mu\in\mathcal{K}_n^\Lambda$ is not minimal with respect to dominance then set $B'_{\boldsymbol\mu}=[D^{\boldsymbol\mu}]$. Then $(B'_{\boldsymbol\mu})^{\circledast}=B'_{\boldsymbol\mu}$ and $B'_{\boldsymbol\mu}=[S^{\boldsymbol\mu}]+\sum_{\boldsymbol\mu\rhd\boldsymbol\nu} b'_{\boldsymbol\mu\boldsymbol\nu}(q)[S^{\boldsymbol\nu}]$, for some Laurent polynomials $b'_{\boldsymbol\mu\boldsymbol\nu}(q)\in\mathbb{Z}[q,q^{-1}]$. If $b'_{\boldsymbol\mu\boldsymbol\nu}(q)\in q\mathbb{Z}[q]$ for all $\boldsymbol\mu\rhd\boldsymbol\nu$ then we can set $B_{\boldsymbol\mu}=B'_{\boldsymbol\mu}$. Otherwise, find $\boldsymbol\mu\rhd\boldsymbol\nu$ minimal with respect to dominance such that $b'_{\boldsymbol\mu\boldsymbol\nu}(q)\notin q\mathbb{Z}[q]$. Using induction, define $B''_{\boldsymbol\mu}=B'_{\boldsymbol\mu}-p_{\boldsymbol\mu\boldsymbol\nu}(q)B_{\boldsymbol\nu}$, where $p_{\boldsymbol\mu\boldsymbol\nu}(q)$ is the unique Laurent polynomial such that $\overline{p_{\boldsymbol\mu\boldsymbol\nu}(q)}=p_{\boldsymbol\mu\boldsymbol\nu}(q)$ and $b'_{\boldsymbol\mu\boldsymbol\nu}(q)-p_{\boldsymbol\mu\boldsymbol\nu}(q)\in q\mathbb{Z}[q]$. Then $(B''_{\boldsymbol\mu})^{\circledast}=B''_{\boldsymbol\mu}$ and the coefficient of $[S^{\boldsymbol\nu}]$ in $B''_{\boldsymbol\mu}$ belongs to $q\mathbb{Z}[q]$. Continuing in this way, a finite number of steps will construct an element $B_{\boldsymbol\mu}$ with the required properties.

Turning to the inner products, if $\boldsymbol\lambda,\boldsymbol\mu\in\mathcal{K}^\Lambda$ then, since $([S^{\boldsymbol\sigma}],[S^{\boldsymbol\tau}]^{\circledast})=\delta_{\boldsymbol\sigma\boldsymbol\tau}$ by (3.5.4),

$$\begin{aligned}(B^{\boldsymbol\lambda},B_{\boldsymbol\mu})=(B^{\boldsymbol\lambda},B_{\boldsymbol\mu}^{\circledast})&=\sum_{\boldsymbol\sigma\unrhd\boldsymbol\lambda}\sum_{\boldsymbol\tau\unlhd\boldsymbol\mu}\overline{b^{\boldsymbol\lambda\boldsymbol\sigma}(q)}\,\overline{b_{\boldsymbol\tau\boldsymbol\mu}(q)}([S^{\boldsymbol\sigma}],[S^{\boldsymbol\tau}]^{\circledast})\\ &=\sum_{\boldsymbol\lambda\unrhd\boldsymbol\sigma\rhd\boldsymbol\mu}\overline{b^{\boldsymbol\lambda\boldsymbol\sigma}(q)}\,\overline{b_{\boldsymbol\sigma\boldsymbol\mu}(q)}.\end{aligned}$$

In particular, $(B^{\boldsymbol\lambda},B_{\boldsymbol\mu})\in\delta_{\boldsymbol\lambda\boldsymbol\mu}+q^{-1}\mathbb{Z}[q^{-1}]$. On the other hand, $(B^{\boldsymbol\lambda},B_{\boldsymbol\mu})=(B^{\boldsymbol\lambda\#},B_{\boldsymbol\mu})=\overline{(B^{\boldsymbol\lambda},B_{\boldsymbol\mu}^{\circledast})}=\overline{(B^{\boldsymbol\lambda},B_{\boldsymbol\mu})}$ by Lemma 3.5.6, Therefore, $(B^{\boldsymbol\lambda},B_{\boldsymbol\mu})=\delta_{\boldsymbol\lambda\boldsymbol\mu}$ as this is the only bar invariant polynomial in $\delta_{\boldsymbol\lambda\boldsymbol\mu}+q^{-1}\mathbb{Z}[q^{-1}]$. ∎

By Lemma 3.5.5, the action of F_i on $[\mathrm{Rep}_{\mathcal{A}}^{\Lambda}]$ and $[\mathrm{Proj}_{\mathcal{A}}^{\Lambda}]$, for $i\in I$, commutes with $\#$ and with $\circledast$. (In the language of [20, §3.1], $\#$ and $\circledast$ are *compatible bar-involutions*). It follows that the basis $\{B^{\boldsymbol\mu}\}$ is Lusztig's **canonical basis** [87, §14.4], or Kashiwara's **upper global basis** [63], of $L(\Lambda)$ and $\{B_{\boldsymbol\mu}\}$ is the **dual canonical basis**, or the **lower global basis**.

3.5.9. **Proposition.** *Suppose that F is an arbitrary field and that $n\ge 0$. Then the following are equivalent:*

a) $B^{\boldsymbol\mu}=[Y^{\boldsymbol\mu}]$*, for all* $\boldsymbol\mu\in\mathcal{K}_n^\Lambda$.
b) $B_{\boldsymbol\mu}=[D^{\boldsymbol\mu}]$*, for all* $\boldsymbol\mu\in\mathcal{K}_n^\Lambda$.
c) $d_{\boldsymbol\lambda\boldsymbol\mu}(q)\in\delta_{\boldsymbol\lambda\boldsymbol\mu}+q\mathbb{N}[q]$*, for all* $\boldsymbol\lambda\in\mathcal{P}_n$ *and* $\boldsymbol\mu\in\mathcal{K}_n^\Lambda$.

Proof. In the Grothendieck groups, $[D^{\boldsymbol\lambda}]=[S^{\boldsymbol\lambda}]+\sum_{\boldsymbol\lambda\rhd\boldsymbol\mu}e_{\boldsymbol\lambda\boldsymbol\mu}(q)[S^{\boldsymbol\mu}]$ and $[Y^{\boldsymbol\lambda}]=[S^{\boldsymbol\lambda}]+\sum_{\boldsymbol\mu\rhd\boldsymbol\lambda}d_{\boldsymbol\lambda\boldsymbol\mu}(q)[S^{\boldsymbol\mu}]$. Moreover, by Lemma 3.5.7, $[Y^{\boldsymbol\mu}]^{\#}=[Y^{\boldsymbol\mu}]$ and $[D^{\boldsymbol\mu}]^{\circledast}=[D^{\boldsymbol\mu}]$, for $\boldsymbol\mu\in\mathcal{K}_n^\Lambda$. By definition, $d_{\boldsymbol\lambda\boldsymbol\mu}(q)\in\mathbb{N}[q,q^{-1}]$ and $\mathbf{e}_q=\mathbf{d}_q^{-1}$. Therefore, $d_{\boldsymbol\lambda\boldsymbol\mu}(q)\in\delta_{\boldsymbol\lambda\boldsymbol\mu}+q\mathbb{N}[q]$ for all $\boldsymbol\lambda,\boldsymbol\mu$, if and only if $e_{\boldsymbol\lambda\boldsymbol\mu}(q)\in\delta_{\boldsymbol\lambda\boldsymbol\mu}+q\mathbb{Z}[q]$ for all $\boldsymbol\lambda,\boldsymbol\mu$. Hence, the proposition is immediate from Proposition 3.5.8. ∎

We can now state Ariki's celebrated Categorification Theorem. By specializing $q=1$ the quantum group $U_q(\widehat{\mathfrak{sl}}_e)_{\mathcal{A}}\otimes\mathbb{Q}$ becomes the Kac-Moody algebra $U(\widehat{\mathfrak{sl}}_e)$. Let $L_1(\Lambda)$ be the irreducible integrable highest weight $U(\widehat{\mathfrak{sl}}_e)$-module of high weight Λ. The canonical bases of $L_1(\Lambda)$ are obtained by specializing $q=1$ in the canonical bases of $L(\Lambda)_{\mathcal{A}}$. Forgetting the grading in the results above, $\underline{\operatorname{Rep}}^\Lambda_{\mathbb{Q}} \cong L_1(\Lambda) \cong \underline{\operatorname{Proj}}^\Lambda_{\mathbb{Q}}$, where $\underline{\operatorname{Rep}}^\Lambda_{\mathbb{Q}} = \bigoplus_n \operatorname{Rep}(\underline{\mathscr{H}}^\Lambda_n) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\underline{\operatorname{Proj}}^\Lambda_{\mathbb{Q}} = \bigoplus_n \operatorname{Proj}(\underline{\mathscr{H}}^\Lambda_n) \otimes_{\mathbb{Z}} \mathbb{Q}$.

3.5.10. **Theorem** (Ariki's Categorification Theorem [2, Theorem 4.4])**.**
Suppose that F is a field of characteristic zero. Then the canonical basis of $L_1(\Lambda)$ coincides with the basis of (ungraded) projective indecomposable $\mathscr{H}^\Lambda_n$-modules $\{\,[\underline{Y}^{\boldsymbol{\lambda}}] \mid \boldsymbol{\lambda} \in \mathcal{K}^\Lambda\,\}$ of $\underline{\operatorname{Proj}}^\Lambda_{\mathbb{Q}}$.

For a detailed proof of this important result see [4, Theorem 12.5]. For a overview and historical account of Ariki's theorem see [41]. For a proof in the degenerate case see [21, Theorem 3.10].

Combining Theorem 3.5.10 with Proposition 3.5.9 we obtain the main result of this section.

3.5.11. **Corollary** (Brundan and Kleshchev [20, Theorem 5.14])**.** *Suppose that F is a field of characteristic zero. Then the canonical basis of $L(\Lambda)$ coincides with the basis $\{\,[Y^{\boldsymbol{\lambda}}] \mid \boldsymbol{\lambda} \in \mathcal{K}^\Lambda\,\}$ of $[\operatorname{Proj}^\Lambda_{\mathbb{Q}(q)}]$. In particular, $d_{\boldsymbol{\lambda\mu}}(q) \in \delta_{\boldsymbol{\lambda\mu}} + q\mathbb{N}[q]$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}^\Lambda$.*

When Λ is a weight of level 2 and $e = \infty$ this was first proved by Brundan and Stroppel [24, Theorem 9.2]. For extensions of this result to cyclotomic quiver Hecke algebras of arbitrary type see [62, 84, 115, 127].

Corollary 3.5.11 implies that the graded decomposition numbers $d_{\boldsymbol{\lambda\mu}}(q) = [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_q = b^{\boldsymbol{\lambda\mu}}(q)$ are *parabolic Kazhdan-Lusztig polynomials*. Explicit formulas are given in [92, Lemma 2.46]. When $e = \infty$ see also [50, Theorem 7.8] and [18, Theorem 3.1].

For the canonical basis $\{B^{\boldsymbol{\mu}}\}$ it is immediate that $b^{\boldsymbol{\lambda\mu}}(q) \in \mathbb{Z}[q]$ are polynomials, however, it is a deep fact that their coefficients are non-negative integers. In contrast, it is immediate that $d_{\boldsymbol{\lambda\mu}}(q)\mathbb{N}[q, q^{-1}]$ but it is a deep fact that they are polynomials rather than *Laurent polynomials*. Thus, the difficult result changes from positivity of coefficients to positivity of exponents in the graded setting. In fact, it is also true when $F = \mathbb{C}$ that the inverse graded decomposition numbers $e_{\boldsymbol{\lambda\mu}}(-q) = b_{\boldsymbol{\lambda\mu}}(-q)$ are polynomials with non-negative integer coefficients. This is perhaps best explained by passing to the Koszul dual of the corresponding graded cyclotomic Schur algebras [6, 50, 121] using [50, 92].

Brundan and Kleshchev's proof of Corollary 3.5.11 is quite different to the one given here. They have to work quite hard to define triangular bar involutions on $L(\Lambda)$ whereas we have done this by exploiting the representation theory of $\mathscr{H}^\Lambda_n$. The catch is that Brundan and Kleshchev have an explicit description of their bar involutions, which they can compute with, whereas we have no hope of working with our bar involution unless we already know the graded decomposition matrices. On the other hand, our approach works for any multicharge $\boldsymbol{\kappa}$.

To complete the proof of Corollary 3.5.11, Brundan and Kleshchev lift Grojnowski's elegant approach [46] to the representation theory of $\mathscr{H}^\Lambda_n$ to the graded setting. As a result they obtain graded analogues of Kleshchev's modular branching rules [16, 70, 71] which, under categorification, correspond to the action of the crystal operators on the crystal graph of $L(\Lambda)$; see [20, Theorem 4.12]. By invoking Ariki's theorem they deduce an analogue of Corollary 3.5.11, although possibly with different labelling of the simple modules. Finally, they then prove that the labelling of the irreducible $\mathscr{H}^\Lambda_n$-modules coming from the branching rules agrees with the labelling in Corollary 1.5.2; compare with [5, 7].

We have not yet given an explicit description of the labelling of the (graded) irreducible $\mathscr{H}^\Lambda_n$-modules because, by definition, $\mathcal{K}^\Lambda_n = \{\,\boldsymbol{\mu} \in \mathcal{P}_n \mid \underline{D}^{\boldsymbol{\mu}} \neq 0\,\}$. Extending (3.2.1), given nodes $A, C \in \operatorname{Rem}_i(\boldsymbol{\lambda})$ define

$$d_A^C = \#\{\,B \in \operatorname{Add}_i(\boldsymbol{\lambda}) \mid A < B < C\,\} - \#\{\,B \in \operatorname{Rem}_i(\boldsymbol{\lambda}) \mid A < B < C\,\}.$$

Following Misra and Miwa [104] (and Kleshchev [69]), a removable i-node A is **normal** if $d_A \le 0$ and $d_A^C < 0$ whenever $C \in \operatorname{Rem}_i(\boldsymbol{\lambda})$ and $A < C$. A normal i-node A is **good** if $A \le B$ whenever B is a normal i-node. Write $\boldsymbol{\lambda} \xrightarrow{\text{good}} \boldsymbol{\mu}$ if $\mu = \lambda + A$ for some good node A. Misra and Miwa [104, Theorem 3.2] show that the crystal graph of $L(\Lambda)_{\mathcal{A}}$, considered as a submodule $\mathscr{F}^\Lambda_{\mathcal{A}}$, is the graph with vertex set

$$\mathscr{L}^\Lambda_0 = \{\,\boldsymbol{\mu} \in \mathcal{P} \mid \boldsymbol{\mu} = \underline{\mathbf{0}} \text{ or } \boldsymbol{\lambda} \xrightarrow{\text{good}} \boldsymbol{\mu} \text{ for some } \boldsymbol{\lambda} \in \mathscr{L}^\Lambda_0\,\},$$

and with labelled edges $\boldsymbol{\lambda} \xrightarrow{\text{i}} \boldsymbol{\mu}$ whenever $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by adding a good i-node, for some $i \in I$.

3.5.12. **Corollary** (Ariki [3])**.** *Suppose that F is an arbitrary field and that $\boldsymbol{\mu} \in \mathcal{P}_n$. Then $\mathcal{K}^\Lambda = \mathscr{L}^\Lambda_0$. That is, if $\boldsymbol{\mu} \in \mathcal{P}_n$ then $D^{\boldsymbol{\mu}}_F \neq 0$ if and only if $\boldsymbol{\mu} \in \mathscr{L}^\Lambda_0$.*

Proof. If F is a field of characteristic zero then $\mathcal{K}^\Lambda = \mathscr{L}^\Lambda_0$ by Corollary 3.5.11, Proposition 3.5.9 and the definition of crystal graphs. If F is a field of positive characteristic then a straightforward modular reduction argument shows that $\underline{D}^{\boldsymbol{\mu}}_F \neq 0$ only if $\underline{D}^{\boldsymbol{\mu}}_{\mathbb{C}} \neq 0$, for $\boldsymbol{\mu} \in \mathcal{P}_n$ (compare with §3.7 below). So, $\mathcal{K}^\Lambda \subseteq \mathscr{L}^\Lambda_0$. By Proposition 3.5.3, the number of irreducible $\mathscr{H}^\Lambda_n$-modules depends only on e, and in particular not on F, so $\mathcal{K}^\Lambda = \mathscr{L}^\Lambda_0$ as required. □

3.6. **Homogeneous Garnir relations.** We have now seen that $\mathscr{R}_n^\Lambda$ is a graded cellular algebra and, as a consequence, that there exist graded lifts of the Specht modules for arbitrary $\Lambda \in P^+$. However, at this point we cannot really compute inside the graded Specht modules because we do not know how to write basis elements indexed by non-standard tableaux in terms of standard ones. This section shows how to do this. First we need some combinatorics.

Fix a multipartition $\boldsymbol{\lambda}$ and a node $A = (l, r, c) \in \boldsymbol{\lambda}$. A (row) **Garnir node** of $\boldsymbol{\lambda}$ is any node $A = (l, r, c)$ such that $(l, r + 1, c) \in \boldsymbol{\lambda}$. The (e, A)**-Garnir belt** is the set of nodes

$$\begin{aligned}\mathbf{B}_A = &\{\, (l, r, c) \in \boldsymbol{\lambda} \mid r \geq c \text{ and } e\lceil \tfrac{r-c+1}{e} \rceil \leq \lambda_r^{(l)} - c + 1 \,\}\\ \cup \quad &\{\, (l, r + 1, c) \in \boldsymbol{\lambda} \mid r \leq c \text{ and } c \geq e\lceil \tfrac{c-r+1}{e} \rceil \,\}.\end{aligned}$$

Let $b_A = \#\mathbf{B}_A/e$ and write $b_A = a_A + c_A$ where ea_A is the number of nodes in $\mathbf{B}_A$ in row (l, r). Let $\mathscr{D}_A$ be the set of minimal length right coset representatives of $\mathfrak{S}_{a_A} \times \mathfrak{S}_{c_A}$ in $\mathfrak{S}_{b_A}$; see, for example, [97, Proposition 3.3]. When $e = \infty$ these definitions should be interpreted as $\mathbf{B}_A = \emptyset$, $b_A = 0 = a_A = c_A$ and $\mathscr{D}_A = 1$.

Suppose A is a Garnir node of $\boldsymbol{\lambda}$. The rows of $\boldsymbol{\lambda}$ are indexed by pairs (l, r), corresponding to row r in $\mu^{(l)}$ where $1 \leq l \leq \ell$ and $r \geq 1$. Order the row indices lexicographically. Let t_A be the $\boldsymbol{\lambda}$-tableau which agrees with $\mathsf{t}^{\boldsymbol{\mu}}$ for all numbers $k < \mathsf{t}^{\boldsymbol{\mu}}(A) = \mathsf{t}^{\boldsymbol{\mu}}(l, r, c)$ and $k > \mathsf{t}^{\boldsymbol{\mu}}(l, r + 1, c)$ and where the remaining entries in rows (l, r) and $(l, r + 1)$ are filled in increasing order from left to right first along the nodes in row $(l, r + 1)$ which are in the first c columns but not in $\mathbf{B}_A$, then along the nodes in row (l, r) of $\mathbf{B}_A$ followed by the nodes in row $(l, r + 1)$ of $\mathbf{B}_A$, and then along the remaining nodes in row (l, r).

3.6.1. **Example** As Garnir belts are contained in consecutive rows of the same component, the general case can be understood by looking at a two-rowed partition (of level one), so we consider the case $e = 3$, $\lambda = (14, 6)$ and $A = (1, 1, 4)$. Then

$$\mathsf{t}_A = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&5&6&7&8&9&10&11&12&13&17&18\\\hline 4&14&15&16&19&20\\\cline{1-6}\end{array}$$

The lines in t_A show how the $(3, A)$-Garnir belt decomposes into a disjoint union of "e-bricks". In general, b_A is equal to the number of e-bricks in the Garnir belt and a_A is the number of e-bricks in its first row. In this case, $b_A = 4$ and $a_A = 3$. Therefore, $\mathscr{D}_A = \{1, s_3, s_3s_2, s_3s_2s_1\}$. $\Diamond$

Let $k_A = \mathsf{t}_A(A)$ be the number occupying A in t_A. For $1 \leq r < b_A$ define

$$w_r^A = \prod_{a=k_A+e(r-1)}^{k_A+re-1} (a, a + e).$$

The elements $\{\, w_r^A \mid 1 \leq r < b_A \,\}$ generate a subgroup of $\mathfrak{S}_n$ that is isomorphic to $\mathfrak{S}_{b_A}$ via the map $w_r^A \mapsto s_r$, for $1 \leq r < b_A$. Set $\mathbf{i}^A = \mathbf{i}^{\mathsf{t}_A}$. If $d \in \mathscr{D}_A$ choose a reduced expression $d = s_{r_1} \dots s_{r_k}$ for d and define

$$\tau_d^A = e(\mathbf{i}^A)(\psi_{w_{r_1}^A} + 1) \dots (\psi_{w_{r_k}^A} + 1) \in \mathscr{R}_n^\Lambda.$$

The elements τ_d^A of $\mathscr{R}_n^\Lambda$ seem to be very special and deserving of further study. They are homogeneous elements in $\mathscr{R}_n^\Lambda$ of degree zero which are independent of all choices of reduced expressions. Moreover, by [75, Theorem 4.13], the elements $\{\, \tau_r^A \mid 1 \leq r < b_A \,\}$ satisfy the braid relations and they generate a copy of $\mathfrak{S}_{b_A}$ inside $\mathscr{R}_n^\Lambda$!

3.6.2. **Theorem** (Kleshchev, Mathas and Ram [75, Theorem 6.23]). *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n$ and that $\mathcal{Z}$ is an integral domain. The graded Specht module $S_{\mathcal{Z}}^{\boldsymbol{\lambda}}$ of $\mathscr{R}_n^\Lambda(\mathcal{Z})$ is isomorphic to the graded $\mathscr{R}_n^\Lambda$-module generated by a homogeneous element $v_{\mathsf{t}^{\boldsymbol{\mu}}}$ of degree $\deg \mathsf{t}^{\boldsymbol{\mu}}$ subject to the relations:*

a) $v_{\mathsf{t}^{\boldsymbol{\mu}}} e(\mathbf{i}) = \delta_{\mathbf{i}\mathbf{i}^{\boldsymbol{\lambda}}} v_{\mathsf{t}^{\boldsymbol{\mu}}}$.
b) $v_{\mathsf{t}^{\boldsymbol{\mu}}} y_s = 0$, *for* $1 \leq s \leq n$.
c) $v_{\mathsf{t}^{\boldsymbol{\mu}}} \psi_r = 0$ *whenever r and $r + 1$ are in the same row of $\mathsf{t}^{\boldsymbol{\mu}}$, for $1 \leq r < n$.*
d) $\sum_{d \in \mathscr{D}_A} v_{\mathsf{t}^{\boldsymbol{\mu}}} \psi_{\mathsf{t}_A} \tau_d^A = 0$, *for all Garnir nodes $A \in \boldsymbol{\lambda}$.*

There is an analogous description of the dual Specht modules $S_{\boldsymbol{\mu}}$ in terms of *column Garnir relations* [75, §7].

The relations in part (d) are the **homogeneous Garnir relations**. These relations are a homogeneous form of the well-known Garnir relations of the symmetric group [54, Theorem 7.2]. Relations (a)–(c) already appear in [23] and, in terms of the cellular basis machinery, they are a consequence of Proposition 3.2.9. The most difficult part of the proof of Theorem 3.6.2 is showing that the τ_d satisfy the braid relations. This is proved using the Khovanov-Lauda diagram calculus which was briefly mentioned in §2.2. Like Theorem 3.2.8 this result holds over an arbitrary ring. To prove that the graded module defined by the presentation in Theorem 3.6.2 has the correct rank the constructions of the graded Specht module $S^{\boldsymbol{\lambda}}$ over a field from Theorem 3.2.6, from [23, 49], are used.

One of the main points of Theorem 3.6.2 is that it makes it possible to do calculations in the graded Specht modules defined over an arbitrary ring. Prior to Theorem 3.6.2 the only way to compute inside the graded

Specht modules was, in effect, to use the isomorphism $\mathscr{R}_n^\Lambda \xrightarrow{\sim} \mathscr{H}_n^\Lambda$ of Theorem 3.1.1 to work in the ungraded setting then use the inverse isomorphism $\mathscr{H}_n^\Lambda \xrightarrow{\sim} \mathscr{R}_n^\Lambda$ to get back to the graded setting. This made it difficult to keep track of, and to exploit, the grading on $S^{\boldsymbol{\lambda}}$ — and it was only possible to work with Specht modules defined over a field.

Theorem 3.6.2 also gives the relations for $S^{\boldsymbol{\lambda}}$ as an $\mathscr{R}_n$-module. From this perspective Theorem 3.6.2 can be used to give another construction of the graded Specht modules. For $\alpha, \beta \in Q^+$ let $\mathscr{R}_{\alpha,\beta} = \mathscr{R}_\alpha \otimes \mathscr{R}_\beta$. Definition 2.2.1 implies that there is a non-unital embedding $\mathscr{R}_{\alpha,\beta} \hookrightarrow \mathscr{R}_{\alpha+\beta}$ which maps $e(\mathbf{i}) \otimes e(\mathbf{j})$ to $e(\mathbf{i} \vee \mathbf{j})$, where $\mathbf{i} \vee \mathbf{j}$ is the sequence obtained by concatenating $\mathbf{i}$ and $\mathbf{j}$. Under this embedding the identity element of $\mathscr{R}_{\alpha,\beta}$ maps to

$$e_{\alpha,\beta} = \sum_{\mathbf{i}\in I^\alpha, \mathbf{j}\in I^\beta} e(\mathbf{i} \vee \mathbf{j}).$$

Definition 2.2.1 implies that $\mathscr{R}_{\alpha+\beta}$ is free as an $\mathscr{R}_{\alpha,\beta}$-module, so the functor

$$\operatorname{Ind}_{\alpha,\beta}^{\alpha+\beta}(M \boxtimes N) = (M \boxtimes N)e_{\alpha,\beta} \otimes_{\mathscr{R}_{\alpha,\beta}} \mathscr{R}_{\alpha+\beta}$$

is a left adjoint to the natural restriction map. Iterating this construction, given $\beta_1, \dots, \beta_\ell \in Q^+$ and $\mathscr{R}_{\beta_k}$ modules M_k, for $1 \le k \le \ell$, define

$$M_1 \circ \cdots \circ M_\ell = \operatorname{Ind}_{\beta_1,\dots,\beta_\ell}^{\beta_1+\cdots+\beta_\ell}(M_1 \boxtimes \cdots \boxtimes M_\ell).$$

The definition of the graded Specht modules by generators and relations in Theorem 3.6.2 makes the following result almost obvious. This description of the Specht modules is part of the folklore of these algebras with several authors [21, 126] using it as the definition of Specht modules.

3.6.3. **Corollary** (Kleshchev, Mathas and Ram [75, Theorem 8.2]). *Suppose that $\lambda^{(k)} \in \mathcal{P}_{1,\beta_k}$, for $\beta_k \in Q^+$ and $1 \le k \le \ell$, so that $\boldsymbol{\lambda} \in \mathcal{P}_\beta$, where $\beta = \beta_1 + \cdots + \beta_\ell$. Then there is an isomorphism of graded $\mathscr{R}_n^\Lambda$-modules (and graded $\mathscr{R}_n$-modules),*

$$S^{\boldsymbol{\lambda}}\langle \deg \mathsf{t}^{\lambda^{(1)}} + \cdots + \deg \mathsf{t}^{\lambda^{(\ell)}} \rangle \cong (S^{\lambda^{(1)}} \circ \cdots \circ S^{\lambda^{(\ell)}})\langle \deg \mathsf{t}^{\boldsymbol{\lambda}} \rangle,$$

where on the right hand side $S^{\lambda^{(k)}}$ is considered as an $\mathscr{R}_{\beta_k}$-module, for $1 \le k \le \ell$.

A second application of Theorem 3.6.2 is a generalization of James' famous result [54, Theorem 8.15] for symmetric groups which describes what happens to the Specht modules when they are tensored with the sign representation. First some notation.

Following [75, §3.3], for $\mathbf{i} \in I^n$ let $-\mathbf{i} = (-i_1, \cdots - i_n) \in I^n$. Recalling the multicharge $\boldsymbol{\kappa}$ from §1.2, set $\boldsymbol{\kappa}' = (-\kappa_\ell, \dots, -\kappa_1)$ and let $\Lambda' = \Lambda(\boldsymbol{\kappa}') \in P^+$. Similarly, if $\beta = \sum_i a_i\alpha_i \in Q^+$ let $\beta' = \sum_{i\in I} a_i\alpha_{-i}$. Inspecting Definition 2.2.9, there is a unique isomorphism of graded algebras

$$\mathtt{sgn}\colon \mathscr{R}_\beta^\Lambda \longrightarrow \mathscr{R}_{\beta'}^{\Lambda'}; \quad e(\mathbf{i}) \mapsto e(-\mathbf{i}), \quad y_r \mapsto -y_r, \quad \text{and} \quad \psi_s \mapsto -\psi_s, \tag{3.6.4}$$

for all admissible r and s and $\mathbf{i} \in I^\beta$. The involution $\mathtt{sgn}$ induces an equivalence of categories $\operatorname{Rep}(\mathscr{R}_{\beta'}^{\Lambda'}) \longrightarrow \operatorname{Rep}(\mathscr{R}_\beta^\Lambda)$ which sends an $\mathscr{R}_{\beta'}^{\Lambda'}$-module M to the $\mathscr{R}_\beta^\Lambda$-module $M^{\mathtt{sgn}}$, where the $\mathscr{R}_\beta^\Lambda$-action is twisted by $\mathtt{sgn}$.

3.6.5. **Corollary** (Kleshchev, Mathas and Ram [75, Theorem 8.5]). *Suppose that $\boldsymbol{\mu} \in \mathcal{P}_\beta$, for $\beta \in Q^+$. Then $S^{\boldsymbol{\mu}} \cong (S_{\boldsymbol{\mu}'})^{\mathtt{sgn}}$ and $S_{\boldsymbol{\mu}} \cong (S^{\boldsymbol{\mu}'})^{\mathtt{sgn}}$ as $\mathscr{R}_\beta^\Lambda$-modules.*

In [75] this is proved by checking the relations in Theorem 3.6.2. As noted in [50, Proposition 3.26], this can be proved more transparently by noting that, up to sign, the involution $\mathtt{sgn}$ maps the ψ-basis of $\mathscr{R}_n^\Lambda$ to the ψ'-basis of $\mathscr{R}_{\beta'}^{\Lambda'}$. Some care must be taken with the notation here. For example, if $\boldsymbol{\mu} \in \mathcal{P}_\beta$ then $\boldsymbol{\mu}' \in \mathcal{P}_{\beta'}$. See [50, §3.7] for more details.

We give an application of these results to the graded decomposition numbers. First, by Corollary 3.5.12 if $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$ there exists $\mathbf{i} \in I^n$ and a sequence of multipartitions $\boldsymbol{\mu}_0 = \underline{\mathbf{0}}, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_n = \boldsymbol{\mu}$ in $\mathcal{K}^\Lambda$ such that $\boldsymbol{\mu}_{k+1}$ is obtained from $\boldsymbol{\mu}_k$ by adding a good i_k-node, for $0 \le k < n$. It follows from the modular branching rules [20, Theorem 4.12], and properties of crystal graphs, that there exists a unique sequence of multipartitions $\mathbf{m}(\boldsymbol{\mu}_0) = \underline{\mathbf{0}}, \mathbf{m}(\boldsymbol{\mu}_1), \dots, \mathbf{m}(\boldsymbol{\mu}_n) = \boldsymbol{\mu}$ such that $\mathbf{m}(\boldsymbol{\mu}_{k+1})$ is obtained from $\mathbf{m}(\boldsymbol{\mu}_k)$ by adding a good $-i_k$-node and $\mathbf{m}(\boldsymbol{\mu}_{k+1}) \in \mathcal{K}_{k+1}^{\Lambda'}$, for $1 \le k \le n$. The **Mullineux conjugate** of $\boldsymbol{\mu}$ is the multipartition $\mathbf{m}(\boldsymbol{\mu})$. Thus, $D^{\mathbf{m}(\boldsymbol{\mu})}$ is a non-zero irreducible $\mathscr{R}_{\beta'}^{\Lambda'}$-module. We emphasize that the $\mathscr{R}_{\beta'}^{\Lambda'}$-module $D^{\mathbf{m}(\boldsymbol{\mu})}$ is defined using the ψ-basis of $\mathscr{R}_{\beta'}^{\Lambda'}$ and hence the crystal theory used in §3.5, with respect to the multicharge $\boldsymbol{\kappa}'$.

3.6.6. **Theorem.** *Suppose that $\boldsymbol{\mu} \in \mathcal{K}_\beta^\Lambda$, for $\beta \in Q^+$. Then $(D^{\mathbf{m}(\boldsymbol{\mu})})^{\mathtt{sgn}} \cong D^{\boldsymbol{\mu}}$ as $\mathscr{R}_\beta^\Lambda$-modules.*

Proof. As $\mathtt{sgn}$ is an equivalence of categories, $(D^{\mathbf{m}(\boldsymbol{\mu})})^{\mathtt{sgn}} \cong D^{\boldsymbol{\nu}}\langle d \rangle$ for some $\boldsymbol{\nu} \in \mathcal{K}_\beta^\Lambda$ and $d \in \mathbb{Z}$ by Corollary 3.2.7. Since $\mathtt{sgn}$ is homogeneous, by Theorem 2.1.4(a),

$$\mathit{dim}_q (D^{\mathbf{m}(\boldsymbol{\mu})})^{\mathtt{sgn}} = \mathit{dim}_q D^{\mathbf{m}(\boldsymbol{\mu})} = \overline{\dim D^{\mathbf{m}(\boldsymbol{\mu})}} = \overline{\dim (D^{\mathbf{m}(\boldsymbol{\mu})})^{\mathtt{sgn}}},$$

so that $d=0$ and $(D^{\mathbf{m}(\boldsymbol{\mu})})^{\mathtt{sgn}} \cong D^{\boldsymbol{\nu}}$. To show that $\boldsymbol{\nu}=\boldsymbol{\mu}$ it is now enough to work in the ungraded setting. Therefore, we can either use the modular branching rules of [5, 46], or their graded counterparts from [20, Theorem 4.12], together with what is by now an almost standard argument due to Kleshchev [70, Theorem 4.7], to show that $\boldsymbol{\nu}=\boldsymbol{\mu}$. □

As we have defined it, $\mathtt{sgn}$ induces an equivalence $\operatorname{Rep}(\mathscr{R}^{\Lambda'}_{\beta'}) \longrightarrow \operatorname{Rep}(\mathscr{R}^{\Lambda}_{\beta})$. As $\mathtt{sgn}$ is an involution, we also write $\mathtt{sgn}\colon \operatorname{Rep}(\mathscr{R}^{\Lambda}_{\beta}) \longrightarrow \operatorname{Rep}(\mathscr{R}^{\Lambda'}_{\beta'})$ for the inverse equivalence. With this small abuse of language, the last two results can be written as $(S^{\boldsymbol{\lambda}})^{\mathtt{sgn}} \cong S_{\boldsymbol{\lambda}'}$ and $(D^{\boldsymbol{\mu}})^{\mathtt{sgn}} \cong D^{\mathbf{m}(\boldsymbol{\mu})}$ as $\mathscr{R}^{\Lambda'}_{\beta'}$-modules, for $\boldsymbol{\lambda}\in\mathcal{P}_\beta$ and $\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}$.

3.6.7. **Corollary.** *Suppose that F is a field and that $\boldsymbol{\lambda}\in\mathcal{P}_\beta$ and $\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{n}$. Then $d_{\boldsymbol{\mu}\boldsymbol{\mu}}(q)=1$, $d_{\mathbf{m}(\boldsymbol{\mu})'\boldsymbol{\mu}}(q)=q^{\operatorname{def}\boldsymbol{\mu}}$ and $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\ne 0$ only if $\mathbf{m}(\boldsymbol{\mu}) \trianglerighteq \boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$. Moreover, if $F=\mathbb{C}$ then $0<\deg d^{\mathbb{C}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)<\operatorname{def}\boldsymbol{\mu}$ whenever $\mathbf{m}(\boldsymbol{\mu}) \triangleright \boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$.*

Proof. Suppose that $\boldsymbol{\lambda}\in\mathcal{P}_\beta$ and $\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{\beta}$. Then

$$\begin{aligned}
[S^{\boldsymbol{\lambda}}:D^{\boldsymbol{\mu}}]_q &= [(S^{\boldsymbol{\lambda}})^{\mathtt{sgn}}:(D^{\boldsymbol{\mu}})^{\mathtt{sgn}}]_q \\
&= [S_{\boldsymbol{\lambda}'}:D^{m(\boldsymbol{\mu})}]_q, && \text{by Corollary 3.6.5 and Theorem 3.6.6,}\\
&= q^{\operatorname{def}\boldsymbol{\mu}}[(S^{\boldsymbol{\lambda}'})^{\circledast}:D^{m(\boldsymbol{\mu})}]_q, && \text{by Corollary 3.3.5,}\\
&= q^{\operatorname{def}\boldsymbol{\mu}}\overline{[S^{\boldsymbol{\lambda}'}:D^{m(\boldsymbol{\mu})}]_q}, && \text{by Theorem 2.1.4(a) and §3.5.}
\end{aligned}$$

By Theorem 2.1.4(c), if $\boldsymbol{\tau}\in\mathcal{K}^{\Lambda}_{n}$ and $\boldsymbol{\sigma}\in\mathcal{P}_n$ then $d_{\boldsymbol{\tau}\boldsymbol{\tau}}(1)=1$ and $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\ne0$ only if $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$. Therefore, $d_{\mathbf{m}(\boldsymbol{\mu})'\boldsymbol{\mu}}(q)=q^{\operatorname{def}\boldsymbol{\mu}}\overline{d_{\mathbf{m}(\boldsymbol{\mu})\mathbf{m}(\boldsymbol{\mu})}}=q^{\operatorname{def}\boldsymbol{\mu}}$ and $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\ne0$ only if $\mathbf{m}(\boldsymbol{\mu})'\trianglerighteq\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$. The argument so far is valid over any field. Now suppose that $F=\mathbb{C}$. Then $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\in\delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}+q[\mathbb{N}]$, by Corollary 3.5.11, so the remaining statement about the degrees of the graded decomposition numbers follows. □

Corollary 3.6.7 was conjectured by Fayers [38]. He was interested in this property of the graded decomposition numbers in characteristic zero because it leads to a more efficient algorithm for computing the graded decomposition numbers $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$, for $\boldsymbol{\lambda}\in\mathcal{P}_n$ and $\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_{n}$. (When $e>n$ a very fast algorithm is given in [50, §5].)

3.7. **Graded adjustment matrices.** All of the results in this section have their origin in the work of James [55] and Geck [40] on *adjustment matrices*. Brundan and Kleshchev have given two different approaches to graded decomposition matrices in [19, §6] and [20, §5.6]. In this section we give third cellular algebra approach. Even though our definitions and proofs are different, it is easy to see that everything in this section is equivalent to definitions or theorems of Brundan and Kleshchev — or to graded analogues of results of James and Geck.

Before we introduce the adjustment matrices, let $\mathcal{A}[I^n]$ be the free $\mathcal{A}$-module generated by I^n. The **q-character** of a finite dimensional $\mathscr{R}_n$-module M is

$$\operatorname{Ch}_q M=\sum_{\mathbf{i}\in I^n}\mathit{dim}_q M_{\mathbf{i}}\cdot\mathbf{i}\in\mathcal{A}[I^n],$$

where $M_{\mathbf{i}}=Me(\mathbf{i})$, for $\mathbf{i}\in I^n$. For example, $\operatorname{Ch}_q S^{\boldsymbol{\lambda}}=\sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})}q^{\deg(\mathfrak{t})}\cdot\mathbf{i}^{\mathfrak{t}}$.

3.7.1. **Theorem** ([67, Theorem 3.17])**.** *Suppose that $\mathcal{Z}$ is a field. Then the map*

$$\operatorname{Ch}_q:[\operatorname{Rep}(\mathscr{R}_n)]\longrightarrow\mathcal{A}[I^n];[M]\mapsto\operatorname{Ch}_q M$$

is injective.

As every $\mathscr{R}^{\Lambda}_n$-module can be considered as an $\mathscr{R}_n$-module by inflation, it follows that the restriction of Ch_q to $[\operatorname{Rep}(\mathscr{R}^{\Lambda}_n)]$ is still injective. Extend the map $\circledast$ to $\mathcal{A}[I^n]$ by defining $\big(\sum_{\mathbf{i}}f_{\mathbf{i}}(q)\cdot\mathbf{i}\big)^{\circledast}=\sum_{\mathbf{i}}\overline{f_{\mathbf{i}}(q)}\cdot\mathbf{i}$. Then $(\operatorname{Ch}_q[M])^{\circledast}=\operatorname{Ch}_q[M^{\circledast}]$, for all $M\in\operatorname{Rep}(\mathscr{R}^{\Lambda}_n)$.

In this section we compare representations of cyclotomic KLR algebras over different fields. Write $S^{\boldsymbol{\lambda}}_{\mathcal{Z}}$ for the graded Specht module of the algebra $\mathscr{R}^{\Lambda}_n(\mathcal{Z})$ defined over the ring $\mathcal{Z}$, for $\boldsymbol{\lambda}\in\mathcal{P}_n$. Similarly, if F is a field and $\boldsymbol{\mu}\in\mathcal{K}^{\Lambda}_n$ let $D^{\boldsymbol{\mu}}_F$ be the corresponding graded irreducible $\mathscr{R}^{\Lambda}_n(F)$-module. If K is an extension of F then $D^{\boldsymbol{\mu}}_K\cong D^{\boldsymbol{\mu}}_F\otimes_F K$ since $D^{\boldsymbol{\mu}}_F$ is absolutely irreducible by Theorem 2.1.4.

Suppose that $\boldsymbol{\mu}\in\mathcal{P}_n$. By Theorem 3.2.8, or by Theorem 3.6.2, the graded Specht module $S^{\boldsymbol{\mu}}_{\mathbb{Z}}$ is defined over $\mathbb{Z}$ and $S^{\boldsymbol{\mu}}_{\mathcal{Z}}\cong S^{\boldsymbol{\mu}}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathcal{Z}$ for any commutative ring $\mathcal{Z}$. The graded Specht module $S^{\boldsymbol{\mu}}_{\mathbb{Z}}$ has basis $\{\,\psi_{\mathfrak{t}}\mid\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\mu})\,\}$ and it comes equipped with a $\mathbb{Z}$-valued bilinear form $\langle\ ,\ \rangle$ which is determined by

$$\langle\psi_{\mathfrak{s}},\psi_{\mathfrak{t}}\rangle\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}}=\psi_{\mathfrak{s}}\psi_{\mathfrak{t}\mathfrak{t}^{\boldsymbol{\lambda}}}=\psi_{\mathfrak{s}}\psi^{\star}_{d(\mathfrak{t})}y^{\boldsymbol{\mu}}e(\mathbf{i}^{\boldsymbol{\lambda}}),\tag{3.7.2}$$

where $y^{\boldsymbol{\mu}}\in\mathscr{R}^{\Lambda}_n$ is given in Definition 3.2.2. Following (1.3.3), define the radical of $S^{\boldsymbol{\mu}}_{\mathbb{Z}}$ to be

$$\operatorname{rad}S^{\boldsymbol{\mu}}_{\mathbb{Z}}=\{\,x\in S^{\boldsymbol{\mu}}_{\mathbb{Z}}\mid\langle x,y\rangle=0\text{ for all }y\in S^{\boldsymbol{\mu}}_{\mathbb{Z}}\,\}.$$

In fact, by (3.7.2), $\operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Z} = \{\, x \in S^{\boldsymbol\mu}_{\mathbb Z} \mid xa = 0 \text{ for all } a \in (\mathscr R^\Lambda_n)^{\trianglerighteq\boldsymbol\mu} \,\}$.

3.7.3. **Definition.** *Suppose that $\boldsymbol\mu \in \mathcal P_n$. Let $D^{\boldsymbol\mu}_{\mathbb Z} = S^{\boldsymbol\mu}_{\mathbb Z} / \operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Z}$.*

By definition, $\operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Z}$ is a graded submodule of $S^{\boldsymbol\mu}_{\mathbb Z}$, so $D^{\boldsymbol\mu}_{\mathbb Z}$ is a graded $\mathscr R^\Lambda_n(\mathbb Z)$-module. Hence, $D^{\boldsymbol\mu}_{\mathbb Z} \otimes_{\mathbb Z} \mathcal Z$ is a graded $\mathscr R^\Lambda_n(\mathcal Z)$-module for any ring $\mathcal Z$.

The following result should be compared with [19, Theorem 6.5].

3.7.4. **Theorem.** *Suppose that $\boldsymbol\mu \in \mathcal P_n$. Then $\operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Z}$ is a $\mathbb Z$-lattice in $\operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Q}$ and $D^{\boldsymbol\mu}_{\mathbb Z}$ is a $\mathbb Z$-lattice in $D^{\boldsymbol\mu}_{\mathbb Q}$. Consequently, $D^{\boldsymbol\mu}_{\mathbb Q} = D^{\boldsymbol\mu}_{\mathbb Z} \otimes_{\mathbb Z} \mathbb Q$ and $\operatorname{Ch}_q D^{\boldsymbol\mu}_{\mathbb Z} = \operatorname{Ch}_q D^{\boldsymbol\mu}_{\mathbb Q}$.*

Proof. Fix an ordering $\operatorname{Std}(\boldsymbol\mu) = \{\mathsf t_1, \dots, t_z\}$ of $\operatorname{Std}(\boldsymbol\mu)$ and let $G^{\boldsymbol\mu}_{\mathbb Z} = (\langle\psi_{\mathsf s}, \psi_{\mathsf t}\rangle)$ be the Gram matrix of $S^{\boldsymbol\mu}_{\mathbb Z}$. As $\mathbb Z$ is a principal ideal domain, by the Smith normal form there exists a pair of bases $\{a_r\}$ and $\{b_s\}$ of $S^{\boldsymbol\mu}_{\mathbb Z}$ such that $(\langle a_r, b_s\rangle) = \operatorname{diag}(d_1.d_2, \dots, d_z)$ for some non-negative integers such that $d_1|d_2|\dots|d_z$, where $d_r = 0$ only if $d_s = 0$ for all $s \ge r$. That is, $d_1, \dots, d_z$ are the elementary divisors of the Gram Matrix $G^{\boldsymbol\mu}_{\mathbb Z}$. As the form is homogeneous, we may assume that the bases $\{a_r\}$ and $\{b_s\}$ are homogeneous with $\deg a_r = \deg \mathsf t_r = -\deg b_r$. Moreover, in view of Proposition 3.2.9(a), we can also assume that $a_r e(\mathbf i) = \delta_{\mathbf i^{\mathsf t_r},\mathbf i} a_r$ and $b_s e(\mathbf i) = \delta_{\mathbf i^{\mathsf t_s},\mathbf i} b_s$, for $1 \le r, s \le z$ and $\mathbf i \in I^n$. Comparing with the definitions above, it follows that $\{\, a_r \mid d_r = 0 \,\}$ is a basis of $\operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Z}$ and that $\{\, a_r + \operatorname{rad} S^{\boldsymbol\mu}_{\mathbb Z} \mid d_r = 0 \,\}$ is a basis of $D^{\boldsymbol\mu}_{\mathbb Z}$. All of our claims now follow. □

For an arbitrary field F, it is usually not the case that $D^{\boldsymbol\mu}_F$ is isomorphic to $D^{\boldsymbol\mu}_{\mathbb Z} \otimes_{\mathbb Z} F$ as an $\mathscr R^\Lambda_n(F)$-module. Indeed, if F is a field of characteristic $p > 0$ then the argument of Theorem 3.7.4 shows that

$$\dim_F D^{\boldsymbol\mu}_F = \{\, 1 \le r \le z \mid d_r \not\equiv 0 \pmod p \,\} \le \operatorname{rank}_{\mathbb Z} D^{\boldsymbol\mu}_{\mathbb Z} = \dim_{\mathbb Q} D^{\boldsymbol\mu}_{\mathbb Q},$$

with equality if and only if all of the non-zero elementary divisors of $G^{\boldsymbol\mu}_{\mathbb Z}$ are coprime to p.

3.7.5. **Definition** (cf. Brundan and Kleshchev [20, §5.6])**.** *Suppose that F is a field. For $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal K^\Lambda_n$ define Laurent polynomials $a^F_{\boldsymbol\lambda\boldsymbol\mu}(q) \in \mathbb N[q, q^{-1}]$ by*

$$a^F_{\boldsymbol\lambda\boldsymbol\mu}(q) = \sum_{d\in\mathbb Z} [D^{\boldsymbol\lambda}_{\mathbb Z} \otimes_{\mathbb Z} F : D^{\boldsymbol\mu}_F\langle d\rangle]\, q^d.$$

*The matrix $\mathbf a^F_q = \big(a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)\big)$ is the **graded adjustment matrix** of $\mathscr R^\Lambda_n(F)$.*

Recall that $d_{\boldsymbol\lambda\boldsymbol\mu}(q)$ is a graded decomposition number of $\mathscr R^\Lambda_n$. When want to emphasize the base field F then we write $d^F_{\boldsymbol\lambda\boldsymbol\mu}(q) = [S^{\boldsymbol\lambda}_F : D^{\boldsymbol\mu}_F]_q$ and $\mathbf d^F_q = \big(d^F_{\boldsymbol\lambda\boldsymbol\mu}(q)\big)$. Note that e is always fixed.

3.7.6. **Theorem** (cf. Brundan and Kleshchev [20, Corollary 5.11, Theorem 5.17])**.** *Suppose that F is a field. Then:*

a) *If $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal K^\Lambda_n$ then $a^F_{\boldsymbol\lambda\boldsymbol\lambda}(1) = 1$ and $a^F_{\boldsymbol\lambda\boldsymbol\mu}(q) \ne 0$ only if $\boldsymbol\lambda \trianglerighteq \boldsymbol\mu$. Moreover, $\overline{a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)} = a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)$.*
b) *We have, $\mathbf d^F_q = \mathbf d^{\mathbb Q}_q \circ \mathbf a^F_q$. That is, if $\boldsymbol\lambda \in \mathcal P_n$ and $\boldsymbol\mu \in \mathcal K^\Lambda_n$ then*

$$[S^{\boldsymbol\lambda}_F : D^{\boldsymbol\mu}_F]_q = d^F_{\boldsymbol\lambda\boldsymbol\mu}(q) = \sum_{\boldsymbol\nu \in \mathcal K^\Lambda_n} d^{\mathbb Q}_{\boldsymbol\lambda\boldsymbol\nu}(q) a^F_{\boldsymbol\nu\boldsymbol\mu}(q).$$

Proof. By construction, every composition factor of $D^{\boldsymbol\lambda}_{\mathbb Z} \otimes F$ is a composition factor of $S^{\boldsymbol\lambda}_F$, so the first two properties of the Laurent polynomials $a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)$ follow from Theorem 2.1.4. By Theorem 3.7.4, the adjustment matrix induces a well-defined map of Grothendieck groups $\mathbf a^F_q : [\operatorname{Rep}(\mathscr R^\Lambda_n(\mathbb Q))] \longrightarrow [\operatorname{Rep}(\mathscr R^\Lambda_n(F))]$ given by

$$\mathbf a^F_q\big([D^{\boldsymbol\lambda}_{\mathbb Q}]\big) = [D^{\boldsymbol\lambda}_{\mathbb Z} \otimes F] = \sum_{\boldsymbol\mu \in \mathcal K^\Lambda_n} a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)[D^{\boldsymbol\mu}_F].$$

Taking q-characters, $\operatorname{Ch}_q D^{\boldsymbol\lambda}_{\mathbb Q} = \sum_{\boldsymbol\mu} a^F_{\boldsymbol\lambda\boldsymbol\mu}(q) \operatorname{Ch}_q D^{\boldsymbol\mu}_F$. Applying $\circledast$ to both sides gives $\operatorname{Ch}_q D^{\boldsymbol\lambda}_{\mathbb Q} = \sum_{\boldsymbol\mu} \overline{a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)} \operatorname{Ch}_q D^{\boldsymbol\mu}_F$. Therefore, $\overline{a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)} = a^F_{\boldsymbol\lambda\boldsymbol\mu}(q)$ by Theorem 3.7.1, completing the proof of part (a). For (b), since $S^{\boldsymbol\lambda}_F \cong S^{\boldsymbol\lambda}_{\mathbb Z} \otimes_{\mathbb Z} F$,

$$[S^{\boldsymbol\lambda}_F] = \mathbf a^F_q\big([S^{\boldsymbol\lambda}_{\mathbb Q}]\big) = \mathbf a^F_q\Big(\sum_{\boldsymbol\nu \in \mathcal K^\Lambda_n} d^{\mathbb Q}_{\boldsymbol\lambda\boldsymbol\nu}(q)[D^{\boldsymbol\nu}_{\mathbb Q}]\Big) = \sum_{\boldsymbol\nu \in \mathcal K^\Lambda_n} \sum_{\boldsymbol\mu \in \mathcal K^\Lambda_n} d^{\mathbb Q}_{\boldsymbol\lambda\boldsymbol\nu}(q) a^F_{\boldsymbol\nu\boldsymbol\mu}(q)[D^{\boldsymbol\mu}_F].$$

Comparing the coefficient of $[D^{\boldsymbol\mu}_F]$ on both sides completes the proof. □

Corollary 3.5.11 determines the graded decomposition numbers of the cyclotomic Hecke algebras in characteristic zero. There are several different algorithms for computing the graded decomposition numbers in characteristic zero [38, 43, 50, 76, 82, 124]. To determine the graded decomposition numbers in positive characteristic it is enough to compute the adjustment matrices of Theorem 3.7.6. The simplest case will be when $a^F_{\boldsymbol\lambda\boldsymbol\mu}(q) = \delta_{\boldsymbol\lambda\boldsymbol\mu}$, for all $\boldsymbol\lambda, \boldsymbol\mu \in \mathcal K^\Lambda_n$. Unfortunately, we currently have no idea when this happens. Two failed conjectures for when $\mathbf a^F_q$ is the identity matrix are discussed in Example 3.8.4 and Example 3.8.5 below.

3.8. **Gram determinants and graded adjustment matrix examples.** The graded cellular basis of $\mathscr{R}_n^\Lambda = \mathscr{R}_n^\Lambda(\mathbb{Z})$ given by Theorem 3.2.8 (or Theorem 3.6.2), defines a $\mathbb{Z}$-valued homogeneous symmetric bilinear form on the graded Specht modules $S^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n$. Using Theorem 3.6.2 it is possible to calculate this form. In general, the homogeneous bilinear form is difficult to compute, however, it gives a lot of information about the Specht modules and the simple modules of $\mathscr{R}_n^\Lambda$.

By (1.3.2), if $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then the inner product $\langle \psi_\mathsf{s}, \psi_\mathsf{t} \rangle$ can be computed inside the Specht module $S^{\boldsymbol{\lambda}}$ using (3.7.2). This section computes the Gram matrices $G_\mathbb{Z}^{\boldsymbol{\lambda}} = \big(\langle \psi_\mathsf{s}, \psi_\mathsf{t} \rangle\big)$ in several examples.

3.8.1. **Example** (Semisimple algebras) Suppose that $(\Lambda, \alpha_{i,n}) \le 1$, for all $i \in I$. Let $\boldsymbol{\lambda} \in \mathcal{P}_n$ and $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then $\langle \psi_\mathsf{s}, \psi_\mathsf{t} \rangle = \delta_{\mathsf{st}}$ because $\mathbf{i}^\mathsf{s} = \mathbf{i}^\mathsf{t}$ if and only if $\mathsf{s} = \mathsf{t}$ by Lemma 2.4.1. Hence, $G_\mathbb{Z}^{\boldsymbol{\lambda}}$ is the identity matrix for all $\boldsymbol{\lambda} \in \mathcal{P}_n$. $\Diamond$

3.8.2. **Example** (Nil-Hecke algebras) Suppose that $\Lambda = n\Lambda_i$ and $\beta = n\alpha_i$, for some $i \in I$. Let $\boldsymbol{\lambda} = (1|1|\dots|1) \in \mathcal{P}_n$, as in §2.5, and suppose $\mathsf{s}, \mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\langle \psi_\mathsf{s}, \psi_\mathsf{t} \rangle \psi_{\mathsf{t}^{\boldsymbol{\lambda}}} = \psi_\mathsf{s} \psi_\mathsf{t}^\star y_1^{n-1} y_2^{n-2} \dots y_{n-1}$, by (3.7.2) and Example 3.2.4. By Proposition 2.5.2, $\psi_\mathsf{s} \psi_\mathsf{t}^\star = \psi_\mathsf{u}$, where $\mathsf{u} = \mathsf{s} d(\mathsf{t})^{-1}$, if $\ell(d(\mathsf{u})) = \ell(d(\mathsf{s})) + \ell(d(\mathsf{t}))$ and otherwise $\psi_\mathsf{s} \psi_\mathsf{t}^\star = 0$. On the other hand, by the last paragraph of the proof of Proposition 2.5.2, or simply by counting degrees, $\psi_\mathsf{u} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = 0$ if $\mathsf{u} \ne \mathsf{t}_{\boldsymbol{\lambda}}$ and $\psi_{\mathsf{t}_{\boldsymbol{\lambda}}} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = (-1)^{n(n-2)/2} \psi_\mathsf{t}^{\boldsymbol{\lambda}}$. Hence, $\langle \psi_\mathsf{s}, \psi_\mathsf{t} \rangle = \delta_{\mathsf{st}'}$, where $\mathsf{t}' = \mathsf{t}_{\boldsymbol{\lambda}} d'(\mathsf{t})$ is the tableau that is conjugate to t. Hence, $G_\mathbb{Z}^{\boldsymbol{\lambda}}$ is $(-1)^{n(n-2)/2}$ times the anti-diagonal identity matrix. $\Diamond$

3.8.3. **Example** Suppose $e = 2$, $\Lambda = \Lambda_0$ and $\lambda = (2,2,1)$. Then $\operatorname{Std}(\lambda)$ contains the five tableaux:

	$\mathsf{t}_1 = \mathsf{t}^\lambda$	t_2	t_3	t_4	t_5
t	$\begin{smallmatrix}1&2\\3&4\\5&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&5\\4&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&4\\5&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\3&5\\4&\end{smallmatrix}$	$\begin{smallmatrix}1&4\\2&5\\3&\end{smallmatrix}$
$d(\mathsf{t})$	1	$s_2 s_4$	s_2	s_4	$s_2 s_4 s_3$
$\deg \mathsf{t}$	2	-2	0	0	0
$\mathbf{i}^\mathsf{t}$	01100	01100	01100	01100	01010

We want to compute the Gram matrix $G_\mathbb{Z}^{\boldsymbol{\lambda}} = \big(\langle \psi_\mathsf{s}, \psi_\mathsf{t} \rangle\big)$ of $S_\mathbb{Z}^\lambda$. Now $\langle \psi_\mathsf{t}, \psi_\mathsf{t} \rangle \ne 0$ only if $\mathbf{i}^\mathsf{s} = \mathbf{i}^\mathsf{t}$, by Proposition 3.2.9(a), and if $\deg \mathsf{s} + \deg \mathsf{t} = 0$ since the bilinear form is homogeneous of degree zero. Hence, the only possible non-zero inner products are

$$\langle \psi_{\mathsf{t}_1}, \psi_{\mathsf{t}_2} \rangle = \langle \psi_{\mathsf{t}^\lambda}, \psi_{\mathsf{t}^\lambda} \psi_2 \psi_4 \rangle = \langle \psi_{\mathsf{t}^\lambda} \psi_4, \psi_{\mathsf{t}^\lambda} \psi_2 \rangle = \langle \psi_{\mathsf{t}_4}, \psi_{\mathsf{t}_2} \rangle$$

together with $\langle \psi_{\mathsf{t}_2}, \psi_{\mathsf{t}_2} \rangle$, $\langle \psi_{\mathsf{t}_4}, \psi_{\mathsf{t}_4} \rangle$ and $\langle \psi_{\mathsf{t}_5}, \psi_{\mathsf{t}_5} \rangle$. By (2.2.3), if $a \in \{2,4\}$ then

$$\langle \psi_{\mathsf{t}^\lambda} \psi_a, \psi_{\mathsf{t}^\lambda} \psi_a \rangle = \langle \psi_{\mathsf{t}^\lambda} \psi_a^2, \psi_{\mathsf{t}^\lambda} \rangle = \pm \langle \psi_{\mathsf{t}^\lambda} (y_a - y_{a+1}), \psi_{\mathsf{t}^\lambda} \rangle = 0,$$

since $\psi_{\mathsf{t}^\lambda} y_r = 0$, for $1 \le r \le 5$. To compute the remaining inner products we have to go back to the definition of the bilinear form (3.7.2). By Definition 3.2.2, $y^\lambda = y_2 y_4$ so

$$\langle \psi_{\mathsf{t}_1}, \psi_{\mathsf{t}_2} \rangle \psi_{\mathsf{t}^\lambda} = \psi_{\mathsf{t}^\lambda} \psi_2 \psi_4 y_2 y_4 = \psi_{\mathsf{t}^\lambda} \psi_2 y_2 \psi_4 y_4 = \psi_{\mathsf{t}^\lambda} (y_3 \psi_2 + 1)(y_5 \psi_4 + 1) = \psi_{\mathsf{t}^\lambda},$$

by Proposition 3.2.9(c). Hence, $\langle \psi_{\mathsf{t}_1}, \psi_{\mathsf{t}_2} \rangle = 1 = \langle \psi_{\mathsf{t}_3}, \psi_{\mathsf{t}_4} \rangle$. Finally, using (2.2.3),

$$\langle \psi_{\mathsf{t}_5}, \psi_{\mathsf{t}_5} \rangle \psi_{\mathsf{t}^{\boldsymbol{\lambda}}} = \psi_{\mathsf{t}^{\boldsymbol{\lambda}}} \psi_2 \psi_4 \psi_3^2 \psi_2 \psi_4 y_2 y_4 = \psi_{\mathsf{t}^{\boldsymbol{\lambda}}} \psi_2 \psi_4 (2 y_3 y_4 - y_3^2 - y_4^2) \psi_2 \psi_4 y_2 y_4.$$

Now $v_{\mathsf{t}^{\boldsymbol{\lambda}}} \psi_2 y_3 = v_{\mathsf{t}^{\boldsymbol{\lambda}}} (y_2 \psi_1 + 1) = v_{\mathsf{t}^{\boldsymbol{\lambda}}}$ and, similarly, $v_{\mathsf{t}^{\boldsymbol{\lambda}}} y_4 \psi_4 = -v_{\mathsf{t}^{\boldsymbol{\lambda}}}$. Consequently $v_{\mathsf{t}^{\boldsymbol{\lambda}}} \psi_2 \psi_4 y_a^2 = 0$, for $a = 3, 4$, so that $\psi_{\mathsf{t}^{\boldsymbol{\lambda}}} \psi_2 \psi_4 \psi_3^2 = -2 \psi_{\mathsf{t}^{\boldsymbol{\lambda}}}$. Similarly $\langle \psi_{\mathsf{t}_5}, \psi_{\mathsf{t}_5} \rangle = -2$. Therefore, the Gram matrix of $S^{(2,2,1)}$ is

$$G_\mathbb{Z}^\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Consequently, the elementary divisors of $G_\mathbb{Z}^\lambda$ are $1,1,1,1,2$. Therefore, if $v = -1$ and $\mathcal{Z} = \mathbb{Q}$ then $S_\mathbb{Q}^\lambda = D_\mathbb{Q}^\lambda$ is irreducible, as is easily checked using Corollary 1.7.6. Now suppose that $v = 1$ and $\mathcal{Z} = \mathbb{F}_2$, so that $\mathscr{H}_n^\Lambda \cong \mathbb{F}_2 \mathfrak{S}_5$. Then the calculation of $G_\mathbb{Z}^\lambda$ shows that the Specht module S^λ is reducible with $\dim_{\mathbb{F}_2} D_{\mathbb{F}_2}^\lambda = 4 < 5 = \dim_\mathbb{Q} D_\mathbb{Q}^\lambda$. It follows that if $e = p = 2$ then $D^{(1^5)}$ is also a composition factor of $S^{\boldsymbol{\lambda}}$, so $a_{(2,2,1),(1^5)}^{\mathbb{F}_2} = 1$. $\Diamond$

3.8.4. **Example** Kleshchev and Ram [78, Conjecture 7.3] made a conjecture which, in type A, is equivalent to saying that the adjustment matrices $\mathbf{a}_q^F$ of the (cyclotomic) KLR algebras are trivial when $e = \infty$. Williamson [128] has given an example which shows that, in general, this is not true. Williamson's example comes from geometry [65], however, when it is translated into the language that we are using here it corresponds to a statement about the simple module $D^{\boldsymbol{\mu}}$, for $\boldsymbol{\mu} = (2|2|1|1|3|3|2|2)$, for the cyclotomic quiver Hecke algebra $\mathscr{R}_{16}^\Lambda$ with $e = \infty$ and $\Lambda = 2\Lambda_1 + 2\Lambda_2 + 2\Lambda_3 + 2\Lambda_4$. Fix the multicharge $\boldsymbol{\kappa} = (4,4,3,3,2,2,1,1)$ and set

$\mathbf{i} = (4,5,3,4,2,3,4,5,2,3,1,2,3,4,1,2)$. So $y^{\boldsymbol{\mu}} = y_1 y_9 y_{15} y_{19}$. There are 5 standard $\boldsymbol{\mu}$-tableaux of degree zero with residue sequence $\mathbf{i}$, namely:

t	$\ell(d(\mathsf{t}))$
$\left(\,\boxed{4\,\vert\,8} \mid \boxed{1\,\vert\,2} \mid \boxed{13} \mid \boxed{3} \mid \boxed{5\,\vert\,6\,\vert\,7} \mid \boxed{9\,\vert\,10\,\vert\,14} \mid \boxed{15\,\vert\,16} \mid \boxed{11\,\vert\,12}\,\right)$	23
$\left(\,\boxed{1\,\vert\,2} \mid \boxed{4\,\vert\,8} \mid \boxed{13} \mid \boxed{3} \mid \boxed{9\,\vert\,10\,\vert\,14} \mid \boxed{5\,\vert\,6\,\vert\,7} \mid \boxed{15\,\vert\,16} \mid \boxed{11\,\vert\,12}\,\right)$	28
$\left(\,\boxed{4\,\vert\,8} \mid \boxed{1\,\vert\,2} \mid \boxed{13} \mid \boxed{3} \mid \boxed{9\,\vert\,10\,\vert\,14} \mid \boxed{5\,\vert\,6\,\vert\,7} \mid \boxed{11\,\vert\,12} \mid \boxed{15\,\vert\,16}\,\right)$	28
$\left(\,\boxed{4\,\vert\,8} \mid \boxed{1\,\vert\,2} \mid \boxed{10} \mid \boxed{3} \mid \boxed{9\,\vert\,13\,\vert\,14} \mid \boxed{5\,\vert\,6\,\vert\,7} \mid \boxed{15\,\vert\,16} \mid \boxed{11\,\vert\,12}\,\right)$	31
$\left(\,\boxed{4\,\vert\,8} \mid \boxed{1\,\vert\,2} \mid \boxed{3} \mid \boxed{13} \mid \boxed{9\,\vert\,10\,\vert\,14} \mid \boxed{5\,\vert\,6\,\vert\,7} \mid \boxed{15\,\vert\,16} \mid \boxed{11\,\vert\,12}\,\right)$	31

The Gram matrix for this component of the Specht module $S^{\boldsymbol{\mu}}$ is

$$\begin{pmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

Calculating this matrix is non-trivial because the lengths of the permutations $d(\mathsf{t})$ are reasonably large. This matrix was computed using the authors' implementation of the graded Specht modules in Sage [122]. Brundan, Kleshchev and McNamara [22, Example 2.16] obtain exactly the same matrix, up to a permutation of the rows and columns, as part of the Gram matrix for the homogeneous bilinear form of the corresponding proper standard module for $\mathscr{R}_n$.

The elementary divisors of this matrix are $1,1,2,0,0$, so the dimension of $D^{\boldsymbol{\mu}} e(\mathbf{i})$ is 2 in characteristic 2 and 3 in all other characteristics. Consequently, the dimension of $D^{\boldsymbol{\mu}}$, and hence the adjustment matrix $\mathbf{a}_q^F$ for $\mathscr{R}_{16}^{\Lambda}(F)$, depends on the characteristic of F — as was first proved by Williamson geometrically. ♢

3.8.5. **Example** Consider the case when $\Lambda = \Lambda_0$, so that $\mathscr{H}_n^{\Lambda}$ is the Iwahori-Hecke algebra of the symmetric group. The *James conjecture* [55, §4] says that if F is a field of characteristic $p > 0$ and $\lambda, \mu \in \mathcal{P}_n$ then $a_{\lambda\mu}(q) = \delta_{\lambda\mu}$ if $ep > n$. A natural strengthening of this conjecture is that the adjustment matrix of $\mathscr{R}_\beta^{\Lambda}$ is trivial whenever $\operatorname{def}\beta < p$. For the symmetric groups, the condition $\operatorname{def}\beta < p$ exactly corresponds to the case when the defect group of the block $\mathscr{R}_\beta^{\Lambda}$ is abelian.

The James conjecture is known to be true for blocks of weight at most 4 [36, 37, 55, 112]. Moreover, for every defect $w \geq 0$ there exists a *Rouquier block* of defect w for which the James conjecture holds [56]. Starting from the Rouquier blocks, there was some hope that the derived equivalences of Chuang and Rouquier [26] could be used to prove the James conjecture for all blocks.

Notwithstanding all of the evidence in favour of the James conjecture, it turns out that the conjecture is wrong! Again, Williamson [129, §6] has cruelly (or kindly, depending on your perspective) produced counterexamples to the James conjecture. At the same time he also found counterexamples to the *Lusztig conjecture* [86] for SL_n. These examples rely upon Williamson's recent work with Elias which gives generators and relations for the category of Soergel bimodules [32]. As of writing, the smallest known counterexample to the James conjecture occurs in a block of defect 561 in $\mathbb{F}_{839}\mathfrak{S}_{467874}$. It is unlikely that Williamson's counterexample can be verified using the techniques that we are describing here. ♢

Brundan and Kleshchev [20, §5.6] remarked that $a^F_{\boldsymbol{\lambda\mu}}(q) \in \mathbb{N}$ in all of the examples that they had computed. They asked whether this might always be the case. The next examples show that, in general, $a^F_{\lambda\mu}(q) \notin \mathbb{N}$.

3.8.6. **Example** (Evseev [33, Corollary 5]) Suppose that $e = 2$, $\Lambda = \Lambda_0$ and let $\lambda = (3, 2^2, 1^2)$ and $\mu = (1^9)$. Take $F = \mathbb{F}_2$ to be a field of characteristic 2 and let $\mathbf{a}_q^F = (a_{\lambda\boldsymbol{\mu}}(q))$ be the adjustment matrix.

As part of a general argument Evseev shows that $a_{\lambda\mu}(q) = q + q^{-1}$. In fact, this is not hard to see directly. Comparing the decomposition matrix for $\mathbb{F}_2\mathfrak{S}_9$ given by James [54] with the graded decomposition matrices when $e = 2$ given in [97], shows that $d^{\mathbb{Q}}_{\lambda\mu} = 0$, $d^{\mathbb{F}_2}_{\lambda\mu} = 2$, and that $a_{\lambda\mu}(1) = 2$. Now $D^{\mu}_{\mathbb{F}_2} = D^{\mu}_{\mathbb{F}_2} e(\mathbf{i}^{\mu})$ is one dimensional, so any composition factor of $S^{\lambda}_{\mathbb{F}_2}$ that is isomorphic to $D^{\mu}_{\mathbb{F}_2}\langle d\rangle$, for some $d \in \mathbb{Z}$, must be contained in $S^{\lambda}_{\mathbb{F}_2} e(\mathbf{i}^{\mu})$. There are exactly six standard λ-tableau with residue sequence $\mathbf{i}^{\mu}$, namely:

$$\begin{array}{ccccccc}
\deg \mathsf{t} & 1 & 1 & 1 & 1 & 1 & -1 \\
\mathsf{t} &
\begin{array}{|c|c|c|}\hline 1&6&9\\\hline 2&7\\\cline{1-2} 3&8\\\cline{1-2} 4\\\cline{1-1} 5\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&4&9\\\hline 2&5\\\cline{1-2} 3&8\\\cline{1-2} 6\\\cline{1-1} 7\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&7\\\cline{1-2} 3&8\\\cline{1-2} 6\\\cline{1-1} 9\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&7\\\cline{1-2} 5&8\\\cline{1-2} 6\\\cline{1-1} 9\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&4&7\\\hline 2&5\\\cline{1-2} 3&6\\\cline{1-2} 8\\\cline{1-1} 9\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&4&7\\\hline 2&5\\\cline{1-2} 3&8\\\cline{1-2} 6\\\cline{1-1} 9\\\cline{1-1}\end{array}
\end{array}$$

As D^μ is one dimensional, and concentrated in degree zero, it follows that $a^{\mathbb{F}_2}_{\lambda\mu} = d^{\mathbb{F}_2}_{\lambda\mu}(q) = q + q^{-1}$. We can see a shadow of the adjustment matrix entry in the Gram matrix of $S^{\boldsymbol{\lambda}}_{\mathbb{Z}} e(\mathbf{i}^\mu)$ which is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & -2 & 2 & 0 \end{pmatrix}$$

The elementary divisors of this matrix are $2, 2, 0, 0, 0, 0$, with the 2's in degrees ± 1. Therefore, the graded dimension of $D^{\lambda}_{\mathbb{F}_2} e(\mathbf{i}^\mu)$ decreases by $q + q^{-1}$ in characteristic 2. ♢

3.8.7. **Example** Motivated by the runner removable theorems of [25, 58] and Example 3.8.6, take $e = 3$, $F = \mathbb{F}_2$, $\lambda = (3, 2^4, 1^3)$ and $\mu = (1^{14})$. (The partitions λ and μ are obtained from the corresponding partitions in Example 3.8.6 by conjugating, adding an empty runner, and then conjugating again.) Again, we work over $\mathbb{F}_2$ and consider the corresponding adjustment matrices.

Calculating with Specht [95] we find that $d^{\mathbb{Q}}_{\lambda\mu} = 0$ and that $d^{\mathbb{F}_2}_{\lambda\mu} = 2$. Once again, it turns out that there are exactly six λ-tableaux with 3-residue sequence $\mathbf{i}^\mu$, with five of these having degree 1 and one having degree -1. (Moreover, the Gram matrix of $S^\lambda e(\mathbf{i}^\mu)$ exactly matches the Gram matrix given in Example 3.8.6.) Hence, exactly as in Example 3.8.6, $a^{\mathbb{F}_2}_{\lambda\mu}(q) = q + q^{-1} = d^{\mathbb{F}_2}_{\lambda\mu}(q)$.

As the runner removable theorems compare blocks for different e over the same field we cannot expect to find an example of a non-polynomial adjustment matrix entry in odd characteristic in this way. Nonetheless, it seems fairly certain that non-polynomial adjustment matrix entries exist for all e and all $p > 0$.

Evseev [33, Corollary 5] gives three other examples of adjustment matrix entries which are equal to $q + q^{-1}$ when $e = p = 2$. All of them have similar analogues when $e = 3$ and $p = 2$. Finally, if we try adding further empty runners to the partitions λ and μ, so that $e \geq 4$, then the corresponding adjustment matrix entry is zero. Interestingly, all of these partitions have weight 4. ♢

4. Seminormal bases and the KLR grading

In this final section we link the KLR grading on $\mathscr{R}^\Lambda_n$ with the semisimple representation theory of $\mathscr{H}^\Lambda_n$ using the seminormal bases. We start by showing that by combining information from all of the KLR gradings for different cyclic quivers leads to an integral formula for the Gram determinants of the ungraded Specht modules.

4.1. **Gram determinants and graded dimensions.** In Theorem 1.7.3 we gave a "rational" formula for the Gram determinant of the ungraded Specht modules $\underline{S}^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n$. We now give an integral formula for these determinants and give both a combinatorial and a representation theoretic interpretation of this formula.

Suppose that the Hecke parameter v from Definition 1.1.1 is an indeterminate over $\mathbb{Q}$ and consider an integral cyclotomic Hecke algebra $\mathscr{H}^\Lambda_n$ over the field $\mathcal{Z} = \mathbb{Q}(v)$ where $\Lambda \in P^+$ such that $(\Lambda, \alpha_{i,n}) \leq 1$, for all $i \in I$. Then $\mathscr{H}^\Lambda_n$ is semisimple by Corollary 1.6.11.

4.1.1. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n$. For $e \geq 2$ and $\mathbf{i} \in I^n_e$ define*

$$\deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) = \sum_{\mathfrak{t} \in \operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda})} \deg_e \mathfrak{t},$$

where $\operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda}) = \{\, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \mid \mathbf{i}^{\mathfrak{t}} = \mathbf{i} \,\}$. *Set* $\deg_e(\boldsymbol{\lambda}) = \sum_{\mathbf{i} \in I^n_e} \deg_{e,\mathbf{i}}(\boldsymbol{\lambda})$. *For p a positive prime set* $\operatorname{Deg}_p(\boldsymbol{\lambda}) = \sum_{k \geq 1} \deg_{p^k}(\boldsymbol{\lambda})$.

By definition, $\deg_e(\boldsymbol{\lambda}), \operatorname{Deg}_p(\boldsymbol{\lambda}) \in \mathbb{Z}$. For $e > 0$ let $\Phi_e(x)$ be the eth cyclotomic polynomial in the indeterminate x.

4.1.2. **Theorem** (Hu-Mathas [52, Theorem 3]). *Suppose that $\Lambda \in P^+$ and $(\Lambda, \alpha_{i,n}) \leq 1$, for all $i \in I$. Let $\boldsymbol{\lambda} \in \mathcal{P}_n$. Then*

$$\det \underline{G}^{\boldsymbol{\lambda}} = \prod_{e > 1} \Phi_e(v^2)^{\deg_e(\boldsymbol{\lambda})}.$$

Consequently, if $v = 1$ then $\det \underline{G}^{\boldsymbol{\lambda}} = \prod_{p \text{ prime}} p^{\operatorname{Deg}_p(\boldsymbol{\lambda})}$.

Proving this result is not hard: it amounts to interpreting Definition 1.6.6 in light of the KLR degree functions on $\operatorname{Std}(\boldsymbol{\lambda})$. There is a power of v in the statement of this result in [52]. This is not needed here because we have renormalised the quadratic relations in the Hecke algebra given in Definition 1.1.1.

The Murphy basis is defined over $\mathbb{Z}[v, v^{-1}]$. Therefore, $\det \underline{G}^{\boldsymbol{\lambda}} \in \mathbb{Z}[v, v^{-1}]$ and Theorem 4.1.2 implies that $\deg_e(\boldsymbol{\lambda}) \geq 0$ for all $\boldsymbol{\lambda} \in \mathcal{P}_n$ and $e \geq 2$. In fact, [52, Theorem 3.24] gives an analogue of Theorem 4.1.2 for the determinant of the Gram matrix restricted to $\underline{S}^{\boldsymbol{\lambda}} e(\mathbf{i})$, suitably interpreted, and the following is true:

4.1.3. **Corollary** ([52, Corollary 3.25]).
Suppose that $e \geq 2$, $\boldsymbol{\lambda} \in \mathcal{P}_n$ and $\mathbf{i} \in I_e^n$. Then $\deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) \geq 0$.

The definition of the integers $\deg_{e,\mathbf{i}}(\boldsymbol{\lambda})$ is purely combinatorial, so it should be possible to give a combinatorial proof of this result. It may be possible to do this using Theorem 3.4.3, however, as we now explain, we think that this is difficult.

Fix an integer $e \geq 2$ and a dominant weight $\Lambda \in P^+$ and consider the Hecke algebra $\mathscr{H}_n^\Lambda$ over a field F. If $\boldsymbol{\lambda} \in \mathcal{P}_n$ then, by definition,

$$\operatorname{Ch}_q S^{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \operatorname{Ch}_q D^{\boldsymbol{\mu}} \in \mathcal{A}[I^n].$$

Let $\partial : \mathcal{A}[I^n] \longrightarrow \mathbb{Z}[I^n]$ be the linear map given by $\partial(f(q) \cdot \mathbf{i}) = f'(1)\mathbf{i}$, where $f'(1)$ is the derivative of $f(q) \in \mathcal{A}$ evaluated at $q = 1$. Then $\partial \operatorname{Ch}_q S^{\boldsymbol{\lambda}} = \sum_{\mathbf{i}} \deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) \cdot \mathbf{i}$. The KLR idempotents are orthogonal, so $\mathit{dim}_q D^{\boldsymbol{\mu}}_{\mathbf{i}} = \overline{\mathit{dim}_q D^{\boldsymbol{\mu}}_{\mathbf{i}}}$ since $(D^{\boldsymbol{\mu}})^{\circledast} \cong D^{\boldsymbol{\mu}}$. Therefore, $\partial \operatorname{Ch}_q D^{\boldsymbol{\mu}} = 0$. Hence, applying ∂ to the formula for $\operatorname{Ch}_q S^{\boldsymbol{\lambda}}$ shows that

$$\sum_{\mathbf{i} \in I^n} \deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) \cdot \mathbf{i} = \partial \operatorname{Ch}_q S^{\boldsymbol{\lambda}} = \sum_{\mathbf{i} \in I^n} \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d'_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1) \dim D^{\boldsymbol{\mu}}_{\mathbf{i}} \cdot \mathbf{i}. \tag{4.1.4}$$

Consequently, $\deg_{e,\mathbf{i}} = \sum_{\boldsymbol{\mu}} d'_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1) \dim D^{\boldsymbol{\mu}}_{\mathbf{i}}$. So far we have worked over an arbitrary field. If $F = \mathbb{C}$ then $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in N[q]$, by Proposition 3.5.8, so that $d'_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1) \geq 0$. Therefore, $\deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) \geq 0$ as claimed. (In fact, using Theorem 3.7.6 it is easy to see that the righthand side of (4.1.4) is independent of F, as it must be.)

Theorem 1.7.4 shows that taking the $\mathfrak{p}$-adic valuation of the Gram determinant of $\underline{S}^{\boldsymbol{\lambda}}$ leads to the Jantzen sum formula for $\underline{S}^{\boldsymbol{\lambda}}$. Therefore, (4.1.4) suggests that

$$\sum_{k>0} [J_k(\underline{S}^{\boldsymbol{\lambda}}_F)] = \sum_{\boldsymbol{\mu} \rhd \boldsymbol{\lambda}} d'_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1) [\underline{D}^{\boldsymbol{\mu}}_F], \tag{4.1.5}$$

where we use the notation of Theorem 1.7.4. That is, Theorem 4.1.2 corresponds to writing the Jantzen sum formula as a non-negative linear combination of simple modules. In fact, what we have done is not enough to prove (4.1.5) — to do this it would be enough to prove analogous statements for the Gram determinants of the Weyl modules of the cyclotomic Schur algebras [29]. Nonetheless, (4.1.5) is true, being proved by Ryom-Hansen [117, Theorem 1] in level one and by Yvonne [131, Theorem 2.11] in general.

A better interpretation of (4.1.5) is given by the *grading filtrations* of the graded Specht modules [13, §2.4]. Let $\dot{\mathscr{R}}_n^\Lambda = \mathscr{H}om_{\mathscr{R}_n^\Lambda}(Y, Y)$, where $Y = \bigoplus_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} Y^{\boldsymbol{\mu}}$. Then $\dot{\mathscr{R}}_n^\Lambda$ is a graded basic algebra and the functor

$$\mathsf{F}_n : \operatorname{Rep}(\mathscr{R}_n^\Lambda) \longrightarrow \operatorname{Rep}(\dot{\mathscr{R}}_n^\Lambda); M \mapsto \mathscr{H}om_{\mathscr{R}_n^\Lambda}(Y, M), \qquad \text{for } M \in \operatorname{Rep}(\mathscr{R}_n^\Lambda),$$

is a graded Morita equivalence; see, for example, [50, §2.3-2.4]. Recall that $\mathbf{c}_q = \big(c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\big) = \mathbf{d}_q^T \circ \mathbf{d}_q$ is the Cartan matrix of $\mathscr{R}_n^\Lambda$. By Corollary 2.1.5, $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \mathit{dim}_q \mathscr{H}om_{\mathscr{R}_n^\Lambda}(Y^{\boldsymbol{\lambda}}, Y^{\boldsymbol{\mu}})$ so that

$$\mathit{dim}_q \dot{\mathscr{R}}_n^\Lambda = \sum_{\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{N}[q, q^{-1}].$$

Until further notice assume that $F = \mathbb{C}$. Then $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{N}[q]$ by Corollary 3.5.11. Therefore, $\mathit{dim}_q \dot{\mathscr{R}}_n^\Lambda \in \mathbb{N}[q]$ so that $\dot{\mathscr{R}}_n^\Lambda$ is a positively graded algebra. Let $\dot{M} = \bigoplus_d \dot{M}_d$ be a graded $\dot{\mathscr{R}}_n^\Lambda$-module. The **grading filtration** of $\dot{M}$ is the filtration $\dot{M} = G_a(\dot{M}) \supseteq G_{a+1}(\dot{M}) \supseteq \cdots \supseteq G_z(\dot{M}) \supset 0$, where

$$G_d(\dot{M}) = \bigoplus_{k \geq d} \dot{M}_k,$$

$a \leq z$, and $\mathit{dim}_q \dot{M} = m_a q^a + \cdots + m_z q^z$ for positive integers m_a and m_z. By definition, $G_r(\dot{M})$ is graded and it is an $\dot{\mathscr{R}}_n^\Lambda$-module precisely because $\dot{\mathscr{R}}_n^\Lambda$ is *positively* graded. The grading filtration of an $\mathscr{R}_n$-module M is the filtration given by $G_r(M) = \mathsf{F}_n^{-1}(G_r(\mathsf{F}_n(M)))$, for $r \in \mathbb{Z}$. As $[S^{\boldsymbol{\lambda}}] = \sum_{\boldsymbol{\mu}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)[D^{\boldsymbol{\mu}}]$, and $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}} + q\mathbb{N}[q]$, it follows that $S^{\boldsymbol{\lambda}} = G_0(S^{\boldsymbol{\lambda}})$ and that $G_r(S^{\boldsymbol{\lambda}}) = 0$ for $r > \operatorname{def} \boldsymbol{\lambda}$ by Corollary 3.6.7.

For $\boldsymbol{\lambda} \in \mathcal{P}_n$ and $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$ write $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \sum_{r \geq 0} d^{(r)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} q^r$, for $d^{(r)}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \in \mathbb{N}$.

4.1.6. **Lemma.** *Suppose that $F = \mathbb{C}$ and that $\boldsymbol{\lambda} \in \mathcal{P}_n$. If $0 \leq r \leq \operatorname{def} \boldsymbol{\lambda}$ then*

$$G_r(S^{\boldsymbol{\lambda}})/G_{r+1}(S^{\boldsymbol{\lambda}}) \cong \bigoplus_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} \big(D^{\boldsymbol{\mu}}\langle r \rangle\big)^{\oplus d^{(r)}_{\boldsymbol{\lambda}\boldsymbol{\mu}}}.$$

Proof. This is an immediate consequence of the definition of the grading filtration and Corollary 3.5.11. □

Comparing this with (4.1.5) suggests that $J_r(\underline{S}^{\boldsymbol\lambda}) = G_r(S^{\boldsymbol\lambda})$, for $r \geq 0$. Of course, there is no reason to expect that $J_r(\underline{S}^{\boldsymbol\lambda})$ is a graded submodule of $S^{\boldsymbol\lambda}$. Nonetheless, establishing a conjecture of Rouquier [82, (16)], Shan has proved the following when Λ is a weight of level 1.

4.1.7. **Theorem** (Shan [119, Theorem 0.1])**.** *Suppose that F is a field of characteristic zero, $\Lambda = \Lambda_0$, and that $\boldsymbol\lambda \in \mathcal{P}_n$. Then $J_r(\underline{S}^{\boldsymbol\lambda}) = G_r(S^{\boldsymbol\lambda})$ is a graded submodule of $S^{\boldsymbol\lambda}$ and $[J_r(\underline{S}^{\boldsymbol\lambda})/J_{r+1}(\underline{S}^{\boldsymbol\lambda}) : D^{\mu}\langle s\rangle] = \delta_{rs} d^{(r)}_{\boldsymbol\lambda\boldsymbol\mu}$, for all $\boldsymbol\mu \in \mathcal{K}^{\Lambda}_n$ and $r \geq 0$.*

Shan actually proves that the Jantzen, radical and grading filtrations of graded Weyl modules coincide for the Dipper-James v-Schur algebras [28]. This implies the result above because the Schur functor maps Jantzen filtrations of Weyl modules to Jantzen filtrations of Specht modules. There is a catch, however, because Shan remarks that it is unclear how her geometrically defined grading relates to the grading on the v-Schur algebra given by Ariki [6] and hence to the KLR grading on $\mathscr{R}^{\Lambda}_n$. As we now sketch, Theorem 4.1.7 can be deduced from Shan's result using recent work.

Since Shan's paper cyclotomic quiver Schur algebras have been introduced for arbitrary dominant weights [6, 50, 121], thus giving a grading on all of the cyclotomic Schur algebras introduced by Dipper, James and the author [29]. The key point, which is non-trivial, is that the module categories of the cyclotomic quiver Schur algebras are Koszul. When $e = \infty$ this is proved in [50] by using Corollary 3.5.11 and [18] to reduce parabolic category $\mathcal{O}$ for the general linear groups, which is known to be Koszul by [12, 13]. Maksimau [92] follows the recipe in [50] to prove Koszulity of Stroppel and Webster's cyclotomic quiver Schur algebras for arbitrary e by using [116] to reduce to affine parabolic category $\mathcal{O}$. Maksimau has to work much harder, however, because he first has to explicitly identify the parabolic Kazhdan-Lusztig polynomials that give the graded decomposition numbers of $\mathscr{R}^{\Lambda}_n$.

As the module categories of the cyclotomic quiver Schur are Koszul, an elementary argument [13, Proposition 2.4.1] shows that the radical and grading filtrations of the graded Weyl modules of these algebras coincide. By definition, the analogue of Lemma 4.1.6 describes the graded composition factors of the grading (=radical) filtrations of the graded Weyl modules — compare with [50, Corollary 7.24] when $e = \infty$ and [92, Theorem 1.1] in general. The graded Schur functors of [50, 92] sends graded Weyl modules to graded Specht modules, graded simple modules to graded simple $\mathscr{R}^{\Lambda}_n$-modules (or zero), grading filtrations to grading filtrations and Jantzen filtrations to Jantzen filtrations. Combining these facts with Shan's work [119] implies Theorem 4.1.7 when $\Lambda = \Lambda_0$. We note that the v-Schur algebras were first shown to be Koszul by Shan, Varagnolo and Vasserot [120]. It is also possible to match up Shan's grading on the v-Schur algebras with the gradings of [6, 121] using the uniqueness of Koszul gradings [13, Proposition 2.5.1]. As these papers use different conventions, it is necessary to work with the graded Ringel dual.

The obstacle to extending Theorem 4.1.7 to dominant weights $\Lambda \in P^+$ of higher level is in showing that the Jantzen and radical (=grading) filtrations of the graded Weyl modules of the cyclotomic quiver Schur algebras coincide. As the cyclotomic quiver Schur algebras are Koszul it is possible that this is straightforward. It seems to the author, however, that it is necessary to generalize Shan's arguments [119] to realize the Jantzen filtration geometrically using the language of [116].

4.2. **A deformation of the KLR grading.** Following [52], especially the appendix, we now sketch how to use the seminormal basis to prove that $\mathscr{R}^{\Lambda}_n \cong \mathscr{H}^{\Lambda}_n$ over a field (Theorem 3.1.1). The aim in doing this is not so much to give a new proof of the graded isomorphism theorem. Rather, we want to build a bridge between the KLR algebras and the well-understood semisimple representation theory of the cyclotomic Hecke algebras. In §4.3 we cross this bridge to construct a new graded cellular basis $\{B_{\mathsf{st}}\}$ of $\mathscr{H}^{\Lambda}_n$ which is independent of the choices of reduced expressions that are necessary in Theorem 3.2.6.

Throughout this section we consider a cyclotomic Hecke algebra $\mathscr{H}^{\Lambda}_n$ defined over a field F which has Hecke parameter $v \in F^{\times}$ of quantum characteristic $e \geq 2$. As in §1.2, the dominant weight $\Lambda \in P^+$ is determined by a multicharge $\boldsymbol\kappa \in \mathbb{Z}^{\ell}$. We set up a modular system for studying $\mathscr{H}^{\Lambda}_n = \mathscr{H}^{\Lambda}_n(F)$.

Let x be an indeterminate over F and let $\mathcal{O} = F[x]_{(x)}$ be the localization of $F[x]$ at the principal ideal generated by x. Let $K = F(x)$ be the field of fractions of $\mathcal{O}$. Let $\mathscr{H}^{\mathcal{O}}_n$ be the cyclotomic Hecke algebra with Hecke parameter $t = x + v$, a unit in $\mathcal{O}$, and cyclotomic parameters $Q_l = x^l + [\kappa_l]_t$, for $1 \leq l \leq \ell$. Then $\mathscr{H}^{K}_n = \mathscr{H}^{\mathcal{O}}_n \otimes_{\mathcal{O}} K$ is a split semisimple algebra by Theorem 2.4.8. Moreover, by definition, $\mathscr{H}^{\Lambda}_n = \mathscr{H}^{\Lambda}_n(F) \cong \mathscr{H}^{\mathcal{O}}_n \otimes_{\mathcal{O}} F$, where we consider F as an $\mathcal{O}$-module by letting x act on F as multiplication by zero.

As the algebra $\mathscr{H}^{K}_n$ is semisimple it has a seminormal basis $\{f_{\mathsf{st}}\}$ in the sense of Definition 1.6.4. With our choice of parameters, the content functions from (1.6.1) become $c^{\mathcal{Z}}_r(\mathsf{s}) = t^{2(c-b)}x^l + [\kappa_l + c - b]_t = t^{2(c-b)}x^l + [\mathsf{c}^{\mathbb{Z}}_r(\mathsf{s})]$ if $\mathsf{s}(l, b, c) = r$, for $1 \leq k \leq n$. Then, $L_r f_{\mathsf{st}} = c^{\mathcal{Z}}_r(\mathsf{s}) f_{\mathsf{st}}$, for $(\mathsf{s}, \mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$. By Corollary 1.6.9, the basis $\{f_{\mathsf{st}}\}$ determines a seminormal coefficient system $\boldsymbol\alpha = \{\, \alpha_r(\mathsf{t}) \mid \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n) \text{ and } 1 \leq r < n \,\}$ and a set of scalars $\{\, \gamma_{\mathsf{t}} \mid \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n) \,\}$.

For $\mathbf{i}\in I^n$ let $\operatorname{Std}(\mathbf{i})=\{\,\mathsf{s}\in\operatorname{Std}(\mathcal{P}_n)\mid \mathbf{i}^{\mathsf{s}}=\mathbf{i}\,\}$ be the set of standard tableaux with residue sequence $\mathbf{i}$. Define

$$f_{\mathbf{i}}^{\mathcal{O}}=\sum_{\mathsf{t}\in\operatorname{Std}(\mathbf{i})}F_{\mathsf{t}}. \tag{4.2.1}$$

By definition, $f_{\mathbf{i}}^{\mathcal{O}}\in\mathscr{H}_n^K$ but, in fact, $f_{\mathbf{i}}^{\mathcal{O}}\in\mathscr{H}_n^{\mathcal{O}}$. This idempotent lifting result dates back to Murphy [105] for the symmetric groups. For higher level it was first proved in [101]. In [52] it is proved for a more general rings $\mathcal{O}$.

4.2.2. **Lemma** ([52, Lemma 4.4]). *Suppose that $\mathbf{i}\in I^n$. Then $f_{\mathbf{i}}^{\mathcal{O}}\in\mathscr{H}_n^{\mathcal{O}}$.*

We will see that $f_{\mathbf{i}}^{\mathcal{O}}\otimes_{\mathcal{O}}1_F$ is the KLR idempotent $e(\mathbf{i})$, for $\mathbf{i}\in I^n$. Notice that $1=\sum_{\mathbf{i}}f_{\mathbf{i}}^{\mathcal{O}}$ and, further, that $f_{\mathbf{i}}^{\mathcal{O}}f_{\mathbf{j}}^{\mathcal{O}}=\delta_{\mathbf{ij}}f_{\mathbf{i}}^{\mathcal{O}}$, for $\mathbf{i},\mathbf{j}\in I^n$, by Theorem 1.6.7.

As detailed after Theorem 3.1.1, Brundan and Kleshchev construct their isomorphisms $\mathscr{R}_n^\Lambda\overset{\sim}{\longrightarrow}\mathscr{H}_n^\Lambda$ using certain rational functions $P_r(\mathbf{i})$ and $Q_r(\mathbf{i})$ in $F[y_1,\dots,y_n]$. The advantage of working with seminormal forms is that, at least intuitively, these rational functions "converge" and can be replaced with "nicer" polynomials. The main tool for doing this is the following result which generalizes Lemma 4.2.2.

Let $M_r=1-t^{-1}L_r+tL_{r+1}$, for $1\le r<n$. Then $M_rf_{\mathsf{st}}=M_r^{\mathcal{Z}}(\mathsf{s})f_{\mathsf{st}}$, where $M_r^{\mathcal{Z}}(\mathsf{s})=1-t^{-1}c_r^{\mathcal{Z}}(\mathsf{s})+tc_{r+1}^{\mathcal{Z}}(\mathsf{s})$. The constant term of $M_r^{\mathcal{Z}}(\mathsf{s})$ is equal to $v^{2c_r^{\mathbb{Z}}(\mathsf{s})-1}[1-c_r^{\mathbb{Z}}(\mathsf{s})+c_{r+1}^{\mathbb{Z}}(\mathsf{s})]_v\ne 0$. Consequently, M_r acts invertibly on f_{st} whenever $\mathsf{s}\in\operatorname{Std}(\mathbf{i})$ and $1-i_r+i_{r+1}\ne0$ in $I=\mathbb{Z}/e\mathbb{Z}$. This observation is part of the proof of part (a) of the next result. Similarly, set $\rho_r^{\mathcal{Z}}(\mathsf{s})=c_r^{\mathcal{Z}}(\mathsf{s})-c_{r+1}^{\mathcal{Z}}(\mathsf{s})$. Then $\rho_r^{\mathcal{Z}}(\mathsf{s})$ is invertible in $\mathcal{O}$ if $i_r\ne i_{r+1}$.

4.2.3. **Corollary** (Hu-Mathas [52, Corollary 4.6]). *Suppose that $1\le r<n$ and $\mathbf{i}\in I^n$.*

a) *If $i_r\ne i_{r+1}+1$ then* $\dfrac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}=\displaystyle\sum_{\mathsf{s}\in\operatorname{Std}(\mathbf{i})}\frac{1}{M_r^{\mathcal{Z}}(\mathsf{s})}F_{\mathsf{s}}\in\mathscr{H}_n^{\mathcal{O}}$.

b) *If $i_r\ne i_{r+1}$ then* $\dfrac{1}{L_r-L_{r+1}}f_{\mathbf{i}}^{\mathcal{O}}=\displaystyle\sum_{\mathsf{s}\in\operatorname{Std}(\mathbf{i})}\frac{1}{\rho_r^{\mathcal{Z}}(\mathsf{s})}F_{\mathsf{s}}\in\mathscr{H}_n^{\mathcal{O}}$.

The invertibility of $M_rf_{\mathbf{i}}^{\mathcal{O}}$, when $i_r\ne i_{r+1}+1$, allows us to define analogues of the KLR generators of $\mathscr{R}_n^\Lambda$ in $\mathscr{H}_n^{\mathcal{O}}$. The invertibility of $(L_r-L_{r+1})f_{\mathbf{i}}^{\mathcal{O}}$ is needed to show that these new elements generate $\mathscr{H}_n^{\mathcal{O}}$.

Define an embedding $I\hookrightarrow\mathbb{Z}; i\mapsto\hat{\imath}$ by letting $\hat{\imath}$ be the smallest non-negative integer such that $i=\hat{\imath}+e\mathbb{Z}$, for $i\in I$.

4.2.4. **Definition.** *Suppose that $1\le r<n$. Define elements $\psi_r^{\mathcal{O}}=\sum_{\mathbf{i}\in I^n}\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}$ in $\mathscr{H}_n^{\mathcal{O}}$ by*

$$\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}=\begin{cases}(T_r+t^{-1})\frac{t^{2\hat{\imath}_r}}{M_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+1},\\ (T_rL_r-L_rT_r)t^{-2\hat{\imath}_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+1}+1,\\ (T_rL_r-L_rT_r)\frac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.}\end{cases}$$

If $1\le r\le n$ then define $y_r^{\mathcal{O}}=\sum_{\mathbf{i}\in I^n}t^{-2\hat{\imath}_r-1}(L_r-[\hat{\imath}_r])f_{\mathbf{i}}^{\mathcal{O}}$.

We now describe an $\mathcal{O}$-deformation of cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$. This is a special case of one of the main results of [52] which allows greater flexibility in the choice of the ring $\mathcal{O}$.

4.2.5. **Theorem** (Hu-Mathas [52, Theorem A]). *As an $\mathcal{O}$-algebra, the algebra $\mathscr{H}_n^{\mathcal{O}}$ is generated by the elements*

$$\{\,f_{\mathbf{i}}^{\mathcal{O}}\mid\mathbf{i}\in I^n\,\}\cup\{\,\psi_r^{\mathcal{O}}\mid 1\le r<n\,\}\cup\{\,y_r^{\mathcal{O}}\mid 1\le r\le n\,\}$$

subject only to the following relations:

$$\prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1\ (\mathrm{mod}\ e)}}(y_1^{\mathcal{O}}-x^l-[\kappa_l-i_1])f_{\mathbf{i}}^{\mathcal{O}}=0,$$

$$f_{\mathbf{i}}^{\mathcal{O}}f_{\mathbf{j}}^{\mathcal{O}}=\delta_{\mathbf{ij}}f_{\mathbf{i}}^{\mathcal{O}},\qquad \textstyle\sum_{\mathbf{i}\in I^n}f_{\mathbf{i}}^{\mathcal{O}}=1,\qquad y_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}=f_{\mathbf{i}}^{\mathcal{O}}y_r^{\mathcal{O}},$$

$$\begin{aligned}\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}&=f_{s_r\cdot\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}}, & y_r^{\mathcal{O}}y_s^{\mathcal{O}}&=y_s^{\mathcal{O}}y_r^{\mathcal{O}},\\ \psi_r^{\mathcal{O}}y_{r+1}^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}&=(y_r^{\mathcal{O}}\psi_r^{\mathcal{O}}+\delta_{i_ri_{r+1}})f_{\mathbf{i}}^{\mathcal{O}}, & y_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}&=(\psi_r^{\mathcal{O}}y_r^{\mathcal{O}}+\delta_{i_ri_{r+1}})f_{\mathbf{i}}^{\mathcal{O}},\\ \psi_r^{\mathcal{O}}y_s^{\mathcal{O}}&=y_s^{\mathcal{O}}\psi_r^{\mathcal{O}}, & &\text{if } s\ne r,r+1,\\ \psi_r^{\mathcal{O}}\psi_s^{\mathcal{O}}&=\psi_s^{\mathcal{O}}\psi_r^{\mathcal{O}}, & &\text{if } |r-s|>1,\end{aligned}$$

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} \equiv \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftrightarrows i_{r+1},\\ (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \to i_{r+1},\\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1},\\ 0, & \text{if } i_r = i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

$$(\psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}} - \psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ -t^{1+\rho_r(\mathbf{i})} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \to i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\ 0, & \text{otherwise,} \end{cases}$$

where $\rho_r(\mathbf{i}) = \hat{\imath}_r - \hat{\imath}_{r+1}$ *and* $y_r^{\langle d\rangle} = t^{2d} y_r^{\mathcal{O}} + t^{-1}[d]$, *for* $d \in \mathbb{Z}$.

In fact, the statement of Theorem 4.2.5 is slightly different to [52, Theorem A]. This is because we are using a different choice of modular system $(K, \mathcal{O}, F)$ and because Definition 1.1.1 renormalizes the quadratic relations for the generators T_r of $\mathscr{H}_n^{\mathcal{O}}$, for $1 \le r < n$.

The strategy behind the proof of Theorem 4.2.5 is quite simple: we compute the action of the elements defined in Definition 4.2.4 on the seminormal basis use this to verify that they satisfy the relations in the theorem. To bound the rank of the algebra defined by the presentation in Theorem 4.2.5 we essentially count dimensions. By specializing $x = 0$, we obtain Theorem 3.1.1 as a corollary of Theorem 4.2.5.

To give a flavour of the type of calculations that were used to verify that the elements in Definition 4.2.4 satisfy the relations in Theorem 4.2.5, for $\mathsf{s} \in \operatorname{Std}(\mathbf{i})$ and $1 \le r < n$ define

$$\beta_r(\mathsf{s}) = \begin{cases} \dfrac{\alpha_r(\mathsf{s}) t^{2\hat{\imath}_r}}{M_r^{\mathcal{Z}}(\mathsf{s})}, & \text{if } i_r = i_{r+1},\\ \alpha_r(\mathsf{s})\rho_r^{\mathcal{Z}}(\mathsf{s}) t^{-2\hat{\imath}_r}, & \text{if } i_r = i_{r+1}+1,\\ \dfrac{\alpha_r(\mathsf{s})\rho_r^{\mathcal{Z}}(\mathsf{s})}{M_r^{\mathcal{Z}}(\mathsf{s})}, & \text{otherwise,} \end{cases} \tag{4.2.6}$$

Then Theorem 1.6.7 easily yields the following.

4.2.7. **Lemma.** *Suppose that* $1 \le r < n$ *and that* $(\mathsf{s}, \mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$. *Set* $\mathbf{i} = \mathbf{i}^{\mathsf{s}}$, $\mathbf{j} = \mathbf{i}^{\mathsf{t}}$, $\mathsf{u} = \mathsf{s}(r, r+1)$ *and* $\mathsf{v} = \mathsf{t}(r, r+1)$. *Then*

$$\psi_r^{\mathcal{O}} f_{\mathsf{st}} = \beta_r(\mathsf{s}) f_{\mathsf{ut}} - \delta_{i_r i_{r+1}} \frac{1}{\rho_r^{\mathcal{Z}}(\mathsf{s})} f_{\mathsf{st}}.$$

Moreover, if $\mathsf{s}(l, b, c) = r$ *then* $y_r^{\langle d\rangle} f_{\mathsf{st}} = t^{-1}\big(t^{2(c-b+d-i_r)} x^l + [c_k^{\mathbb{Z}}(\mathsf{s}) + d - \hat{\imath}_r]\big) f_{\mathsf{st}}$, *for* $1 \le r \le n$ *and* $d \in \mathbb{Z}$.

Armed with Lemma 4.2.7, and Definition 1.6.6, it is an easy exercise to verify that all of the relations in Theorem 4.2.5 hold in $\mathscr{H}_n^{\mathcal{O}}$. For the quadratic relations, Lemma 4.2.7 implies that $(\psi_r^{\mathcal{O}})^2 f_{\mathsf{st}} = 0$ if $\mathsf{s} \in \operatorname{Std}(\mathbf{i})$ and $i_r = i_{r+1}$ whereas if $i_r \ne i_{r+1}$ then $(\psi_r^{\mathcal{O}})^2 f_{\mathsf{st}} = \beta_r(\mathsf{s})\beta_r(\mathsf{u}) f_{\mathsf{st}}$, where $\mathsf{u} = \mathsf{s}(r, r+1)$. The quadratic relations in Theorem 4.2.5 now follow using (4.2.6) and Lemma 4.2.7. For example, suppose that $i_r \to i_{r+1}$ and $\mathsf{s} \in \operatorname{Std}(\mathbf{i})$. Pick nodes (l, b, c) and (l', b', c') such that $\mathsf{s}(l, b, c) = r$ and $\mathsf{s}(l', b', c') = r+1$. Then, using Lemma 4.2.7 and Definition 1.6.6,

$$\begin{aligned}(\psi_r^{\mathcal{O}})^2 f_{\mathsf{st}} &= t^{-2\hat{\imath}_{r+1}} \beta_r(\mathsf{s})\beta_r(\mathsf{u}) f_{\mathsf{st}} = t^{-2\hat{\imath}_{r+1}} M_r^{\mathcal{Z}}(\mathsf{u}) f_{\mathsf{st}}\\ &= t^{-2\hat{\imath}_{r+1}}\Big(1 + t^{2(c-b)+1} x^l - t^{2(c'-'b)-1} x^{l'} + t[c_r^{\mathbb{Z}}(\mathsf{s})] - t^{-1}[c_r^{\mathbb{Z}}(\mathsf{u})]\Big) f_{\mathsf{st}}\\ &= t^{-1-2\hat{\imath}_{r+1}}\Big(t^{2(c-b+1)} x^l - t^{2(c'-'b)} x^{l'} + t^{2c_{r+1}(\mathsf{s})}[1 + c_r(\mathsf{s}) - c_{r+1}(\mathsf{s})]\Big) f_{\mathsf{st}}.\end{aligned}$$

On the other hand, using Lemma 4.2.7 again,

$$\begin{aligned}(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathsf{st}} &= t^{-1}\Big(t^{2(c-b+1-\hat{\imath}_{r+1})} x^l - t^{2(c'-b'-\hat{\imath}_{r+1})} x^{l'} + [c_r^{\mathbb{Z}}(\mathsf{s}) + 1 - \hat{\imath}_{r+1}] - [c_{r+1}^{\mathbb{Z}}(\mathsf{s}) - \hat{\imath}_{r+1}]\Big) f_{\mathsf{st}}\\ &= t^{-1-2\hat{\imath}_{r+1}}\Big(t^{2(c-b+1)} x^l - t^{2(c'-b')} x^{l'} + t^{2c_{r+1}(\mathsf{s})}[1 + c_r^{\mathbb{Z}}(\mathsf{s}) - c_{r+1}^{\mathbb{Z}}(\mathsf{s})]\Big) f_{\mathsf{st}}\\ &= (\psi_r^{\mathcal{O}})^2 f_{\mathsf{st}}.\end{aligned}$$

Therefore, $(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}$ when $i_r \to i_{r+1}$. These calculations are not very pretty, but nor are they are hard — and they are very effective. The proof of the (deformed) braid relations is similar. As indicated by Remark 2.2.5, the quadratic relations play a role in the proof of the braid relations.

4.3. **A distinguished homogeneous basis.** One of the advantages of Theorem 4.2.5 is that it allows us to transplant questions about the KLR algebra $\mathscr{R}_n^\Lambda$ into the language of seminormal bases. In Definition 1.6.6 we defined a $*$-seminormal basis which provides a good framework for studying the semisimple cyclotomic Hecke algebras. The algebra $\mathscr{H}_n^\Lambda$ comes with two cellular algebra automorphisms, $*$ and $\star$, where $\star$ is the unique anti-isomorphism fixing the homogeneous generators and $*$ is the unique anti-isomorphism fixing the inhomogeneous generators.

4.3.1. **Definition** (Hu-Mathas [52, §5]). *A* $\star$-***seminormal coefficient system*** *is a collection of scalars*

$$\boldsymbol{\beta} = \{\, \beta_r(\mathsf{t}) \mid \mathsf{t} \in \operatorname{Std}(\mathcal{P}_n) \text{ and } 1 \le r \le n \,\}$$

such that $\beta_r(\mathsf{t}) = 0$ *if* $\mathsf{v} = \mathsf{t}(r, r+1)$ *is not standard, if* $\mathsf{v} \in \operatorname{Std}(\mathcal{P}_n)$ *then* $\beta_r(\mathsf{v})\beta_r(\mathsf{t})$ *is given by the product of the particular choice of coefficients in (4.2.6) and if* $1 \le r < n$ *then* $\beta_r(\mathsf{t})\beta_{r+1}(\mathsf{t}s_r)\beta_r(\mathsf{t}s_rs_{r+1}) = \beta_{r+1}(\mathsf{t})\beta_r(\mathsf{t}s_{r+1})\beta_{r+1}(\mathsf{t}s_{r+1}s_r)$.

Exactly as in Corollary 1.6.9, the $\star$-seminormal coefficient systems determine $\star$-seminormal bases $\{f_{\mathsf{st}}\}$ which, similar to Definition 1.6.4 consist of non-zero elements $f_{\mathsf{st}} \in H_{\mathsf{st}}$ such that $f_{\mathsf{st}}^\star = f_{\mathsf{ts}}$, for $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$. The left (and right) the action of $\psi_r^{\mathcal{O}}$ on f_{st} is exactly as in Lemma 4.2.7 but for a general $\star$-seminormal coefficient system $\boldsymbol{\beta}$.

Definition 4.3.1 gives us extra flexibility in choosing a $\star$-seminormal basis. By [52, (5.8)] there exists a $\star$-seminormal basis $\{f_{\mathsf{st}}\}$ such that the ψ-basis of Theorem 3.2.6 lifts to a $\psi^{\mathcal{O}}$-basis $\{\psi_{\mathsf{st}}^{\mathcal{O}}\}$ with the property that

$$\psi_{\mathsf{st}}^{\mathcal{O}} = f_{\mathsf{st}} + \sum_{(\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})} r_{\mathsf{uv}} f_{\mathsf{uv}}, \tag{4.3.2}$$

for some $r_{\mathsf{uv}} \in K$. In this way we recover Theorem 3.2.6 and with quicker proof than given initially in [49]. More importantly, by working with $\mathscr{H}_n^{\mathcal{O}}$ we can improve upon the ψ-basis.

4.3.3. **Theorem** (Hu-Mathas [52, Theorem 6.2, Corollary 6.3]). *Suppose that* $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$. *There exists a unique element* $B_{\mathsf{st}}^{\mathcal{O}} \in \mathscr{H}_n^{\mathcal{O}}$ *such that*

$$B_{\mathsf{st}}^{\mathcal{O}} = f_{\mathsf{st}} + \sum_{\substack{(\mathsf{u},\mathsf{v}) \in \operatorname{Std}^2(\mathcal{P}_n) \\ (\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})}} p_{\mathsf{uv}}^{\mathsf{st}}(x^{-1}) f_{\mathsf{uv}},$$

where $p_{\mathsf{uv}}^{\mathsf{st}}(x) \in xK[x]$. *Moreover,* $\{\, B_{\mathsf{st}}^{\mathcal{O}} \mid (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n) \,\}$ *is a cellular basis of* $\mathscr{H}_n^{\mathcal{O}}$.

The existence and uniqueness of this basis essentially come down to Gaussian elimination, although for technical reasons it is necessary to work over the $x\mathcal{O}$-adic completion of $\mathcal{O}$. Proving that $\{B_{\mathsf{st}}^{\mathcal{O}}\}$ is cellular is trickier.

As we will see, because we are using a $\star$-seminormal basis, the basis $\{B_{\mathsf{st}}^{\mathcal{O}}\}$ behaves well with respect to the KLR grading on $\mathscr{H}_n^\Lambda$. The main justification for using this seminormal basis as a proxy for choosing a "nice" basis for $\mathscr{H}_n^\Lambda$, a part from the fact that it works, is that Theorem 2.4.8 shows that the natural homogeneous basis of the semisimple cyclotomic quiver Hecke algebras is a $\star$-seminormal basis.

In characteristic zero the polynomials $p_{\mathsf{uv}}^{\mathsf{st}}(x)$ satisfy $0 < \deg p_{\mathsf{uv}}^{\mathsf{st}}(x) \le \frac12(\deg \mathsf{u} - \deg \mathsf{s} + \deg \mathsf{v} - \deg \mathsf{t})$, whenever $(\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})$ by [52, Proposition 6.4]. Moreover, if $\mathsf{s},\mathsf{t},\mathsf{u},\mathsf{v}$ are all standard tableaux of the same shape then $p_{\mathsf{uv}}^{\mathsf{st}}(x) = p_{\mathsf{u}}^{\mathsf{s}}(x)p_{\mathsf{v}}^{\mathsf{t}}(x)$, where $0 < \deg p_{\mathsf{u}}^{\mathsf{s}}(x) \le \frac12(\deg \mathsf{u} - \deg \mathsf{s})$ and $0 < \deg p_{\mathsf{v}}^{\mathsf{t}} \le \frac12(\deg \mathsf{v} - \deg \mathsf{t})$ whenever $\mathsf{u} \rhd \mathsf{s}$ and $\mathsf{v} \rhd \mathsf{t}$, respectively.

As the basis $\{B_{\mathsf{st}}^{\mathcal{O}}\}$ is defined over $\mathcal{O}$ we can reduce modulo the ideal $x\mathcal{O}$ to obtain a basis $\{B_{\mathsf{ts}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K\}$ of $\mathscr{H}_n^\Lambda = \mathscr{H}_n^\Lambda(K)$. This basis is hard to compute and we do not know whether it is homogeneous in general. Nonetheless, it is possible to construct a homogeneous basis $\{B_{\mathsf{st}}\}$ of $\mathscr{H}_n^\Lambda$ from $\{B_{\mathsf{st}}^{\mathcal{O}}\}$. By definition, if $\boldsymbol{\lambda} \in \mathcal{P}_n$ then $B_{\mathsf{t}^{\boldsymbol{\lambda}}\mathsf{t}^{\boldsymbol{\lambda}}}$ is the homogeneous component of $B_{\mathsf{t}^{\boldsymbol{\lambda}}\mathsf{t}^{\boldsymbol{\lambda}}}^{\mathcal{O}} \otimes 1_K$ of degree $2\deg \mathsf{t}^{\boldsymbol{\lambda}}$. In general, for $\mathsf{s},\mathsf{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ there exists homogeneous elements $D_{\mathsf{s}}, D_{\mathsf{t}} \in \mathscr{H}_n^\Lambda$ such that $B_{\mathsf{st}} = D_{\mathsf{s}}^\star B_{\mathsf{t}^{\boldsymbol{\lambda}}\mathsf{t}^{\boldsymbol{\lambda}}} D_{\mathsf{t}}$. In characteristic zero, B_{st} is the homogeneous component of $B_{\mathsf{st}}^{\mathcal{O}} \otimes 1_K$ of degree $\deg \mathsf{s} + \deg \mathsf{t}$, and all other components are of larger degree. In general, this appears to depend on the characteristic. For any field, by (4.3.2) and Theorem 4.3.3,

$$B_{\mathsf{ts}} = \psi_{\mathsf{st}} + \sum_{(\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})} a_{\mathsf{uv}} \psi_{\mathsf{uv}}, \tag{4.3.4}$$

for some $a_{\mathsf{uv}} \in K$ which are non-zero only if $\mathbf{i}^{\mathsf{u}} = \mathbf{i}^{\mathsf{s}}$, $\mathbf{i}^{\mathsf{v}} = \mathbf{i}^{\mathsf{t}}$ and $\deg \mathsf{u} + \deg \mathsf{v} = \deg \mathsf{s} + \deg \mathsf{t}$. Therefore, this basis resolves the ambiguities of Proposition 3.2.9(b). More importantly, we have the following.

4.3.5. **Theorem** (Hu-Mathas [52, Theorem 6.9]). *Suppose that* K *is a field. Then* $\{\, B_{\mathsf{st}} \mid (\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n) \,\}$ *is a graded cellular basis of* $\mathscr{R}_n^\Lambda$ *with weight poset* $(\mathcal{P}_n, \trianglerighteq)$, *cellular algebra automorphism* $\star$ *and with* $\deg B_{\mathsf{st}} = \deg \mathsf{s} + \deg \mathsf{t}$, *for* $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$. *Moreover, if* $(\mathsf{s},\mathsf{t}) \in \operatorname{Std}^2(\mathcal{P}_n)$ *then* $B_{\mathsf{st}} + \mathscr{H}_n^{\rhd\boldsymbol{\lambda}}$ *depends only on* s *and* t *and not on the choice of reduced expressions for the permutations* $d(\mathsf{s}), d(\mathsf{t}) \in \mathfrak{S}_n$.

By construction, the basis $\{B_{\mathsf{st}}$ depends on the field F. Moreover, if F is a field of positive characteristic then B_{st} depends upon the choice of the elements D_{s} and D_{t}, which are uniquely determined modulo $\mathscr{H}_n^{\rhd\boldsymbol{\lambda}}$. This is why $B_{\mathsf{st}}+\mathscr{H}_n^{\rhd\boldsymbol{\lambda}}$ is uniquely determined by s and t.

4.4. **A conjecture.** The construction of the basis $\{B^{\mathcal{O}}_{\mathsf{st}}\}$ of $\mathscr{H}_n^{\mathcal{O}}$ in Theorem 4.3.3, together with the degree constraints on the polynomials $p^{\mathsf{st}}_{\mathsf{uv}}(x)$ in characteristic zero, are reminiscent of the Kazhdan-Lusztig basis [66]. We do not have an analogue of the bar involution, however, a possible replacement for this is that the basis elements B_{st} are homogeneous. Motivated by this analogy with the Kazhdan-Lusztig basis we now define analogues of *cell representations* for the B-basis.

As the basis $\{B_{\mathsf{st}}\}$ of Theorem 4.3.5 is graded cellular we obtain a new homogeneous basis $\{\,B_{\mathsf{t}}\mid\mathsf{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$ of the graded Specht module $S^{\boldsymbol{\lambda}}$. Define the pre-order $\succeq_B$ on $\operatorname{Std}(\boldsymbol{\lambda})$ to be the transitive closure of the relation $\dot{\succeq}_B$ where $\mathsf{t}\dot{\succeq}_B\mathsf{v}$ if there exists $a\in\mathscr{R}^\Lambda_n$ such that $B_{\mathsf{t}}a=\sum_{\mathsf{s}}r_{\mathsf{s}}B_{\mathsf{s}}$ with $r_{\mathsf{v}}\neq0$. (So $\succeq_B$ is reflexive and transitive but not anti-symmetric.) Let $\sim_B$ be the equivalence relation on $\operatorname{Std}(\boldsymbol{\lambda})$ determined by $\succeq_B$ so that $\mathsf{t}\sim_B\mathsf{v}$ if and only if $\mathsf{t}\succeq_B\mathsf{v}\succeq_B\mathsf{t}$. For example, $\mathsf{t}^{\boldsymbol{\lambda}}\succeq_B\mathsf{t}\succeq_B\mathsf{t}_{\boldsymbol{\lambda}}$, for all $\mathsf{t}\in\operatorname{Std}(\boldsymbol{\lambda})$.

Let $\operatorname{Std}[\boldsymbol{\lambda}]$ be the set of $\sim_B$-equivalence classes in $\operatorname{Std}(\boldsymbol{\lambda})$. The set $\operatorname{Std}[\boldsymbol{\lambda}]$ is partially ordered by $\succeq_B$, where $\mathsf{T}\succeq_B\mathsf{V}$ if $\mathsf{t}\succeq_B\mathsf{v}$ for some $\mathsf{t}\in\mathsf{T}$ and $\mathsf{v}\in\mathsf{V}$. Write $\mathsf{T}\succeq_B\mathsf{v}$ if $\mathsf{t}\succeq_B\mathsf{v}$ for some $\mathsf{t}\in\mathsf{T}$ and $\mathsf{T}\succ_B\mathsf{v}$ if $\mathsf{T}\succeq_B\mathsf{v}$ and $\mathsf{v}\notin\mathsf{T}$. Define $S^{\boldsymbol{\lambda}}_{\mathsf{T}\succeq}$ to be the vector subspace of $S^{\boldsymbol{\lambda}}$ with basis $\{\,B_{\mathsf{v}}\mid\mathsf{T}\succeq_B\mathsf{v}\,\}$. Similarly, let $S^{\boldsymbol{\lambda}}_{\mathsf{T}\succ}$ be the vector space with basis $\{\,B_{\mathsf{v}}\mid\mathsf{T}\succ_B\mathsf{v}\,\}$. The definition of $\succeq_B$ ensures that $S^{\boldsymbol{\lambda}}_{\mathsf{T}\succeq}$ and $S^{\boldsymbol{\lambda}}_{\mathsf{T}\succ}$ are both graded $\mathscr{H}^\Lambda_n$-submodules of $S^{\boldsymbol{\lambda}}$ and that $S^{\boldsymbol{\lambda}}_{\mathsf{T}\succ}\subsetneq S^{\boldsymbol{\lambda}}_{\mathsf{T}\succeq}$. Therefore, $S^{\boldsymbol{\lambda}}_{\mathsf{T}}=S^{\boldsymbol{\lambda}}_{\mathsf{T}\succeq}/S^{\boldsymbol{\lambda}}_{\mathsf{T}\succ}$ is a graded $\mathscr{H}^\Lambda_n$-module. By choosing any total order of $\operatorname{Std}[\boldsymbol{\lambda}]$ which is compatible with $\succeq_B$ it is easy to see that $S^{\boldsymbol{\lambda}}$ has a filtration with subquotients being precisely the modules $S^{\boldsymbol{\lambda}}_{\mathsf{T}}$, for $\mathsf{T}\in\operatorname{Std}[\boldsymbol{\lambda}]$.

For $\boldsymbol{\lambda}\in\mathcal{P}_n$ let $\mathsf{T}^{\boldsymbol{\lambda}}=\{\,\mathsf{t}\in\operatorname{Std}(\boldsymbol{\lambda})\mid\mathsf{t}\sim_B\mathsf{t}^{\boldsymbol{\lambda}}\,\}$. In view of (3.7.2), if $\mathsf{s},\mathsf{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ and $\langle B_{\mathsf{s}},B_{\mathsf{t}}\rangle\neq0$ then $\mathsf{s}\sim_B\mathsf{t}^{\boldsymbol{\mu}}\sim_B\mathsf{t}$ so that $\mathsf{s},\mathsf{t}\in T^{\boldsymbol{\lambda}}$. Therefore, $\dim D^{\boldsymbol{\lambda}}\le|\mathsf{T}^{\boldsymbol{\lambda}}|$. Of course, if $\boldsymbol{\lambda}\notin\mathcal{K}^\Lambda_n$ then this bound is not sharp because $D^{\boldsymbol{\lambda}}=0$ whereas $|\mathsf{T}^{\boldsymbol{\lambda}}|\ge1$.

4.4.1. **Conjecture.** *Suppose that F is a field of characteristic zero and that $\boldsymbol{\lambda}\in\mathcal{P}_n$. Then $S^{\boldsymbol{\lambda}}_{\mathsf{T}}$ is an irreducible $\mathscr{H}^\Lambda_n$-module, for all $\mathsf{T}\in\operatorname{Std}[\boldsymbol{\lambda}]$.*

Conjecture 4.4.1 is not supported by a great deal of evidence. It is easy to check that the conjecture is true in the trivial cases considered in Example 3.8.1 and Example 3.8.2. With considerably more effort, using [24, Lemma 9.7] and results of [50, Appendix], it is possible to verify the conjecture when Λ is a weight of level 2 and $e>n$. In all of these cases, the conjecture can be checked because $B_{\mathsf{st}}=\psi_{\mathsf{st}}$, for all $(\mathsf{s},\mathsf{t})\in\operatorname{Std}^2(\mathcal{P}_n)$.

As discussed in [52, §3.3], and is implicit in (4.1.4), by fixing a composition series for $S^{\boldsymbol{\lambda}}$ and using a Gaussian elimination argument, it is possible to construct a basis $\{C_{\mathsf{t}}\}$ of $S^{\boldsymbol{\mu}}$ such that each module appearing in the composition series has a basis which is contained in $\{C_{\mathsf{t}}\}$ and such that if $\mathsf{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ then $C_{\mathsf{t}}=\psi_{\mathsf{t}}$ plus a linear combination of "higher terms" with respect to some total order on $\operatorname{Std}(\boldsymbol{\lambda})$. This defines a partition of $\operatorname{Std}(\boldsymbol{\lambda})=X_1\sqcup\dots\sqcup X_z$, where the tableaux in the set X_k are in bijection with a basis of the kth composition factor. That is, we have defined an equivalence relation on $\operatorname{Std}(\boldsymbol{\lambda})$, which is associated with a composition series, so that the analogue of Conjecture 4.4.1 holds for this equivalence relation. Our conjecture is an optimistic attempt to make this equivalence relation on $\operatorname{Std}(\boldsymbol{\lambda})$ explicit and canonical.

If $\mathcal{T}\subseteq\operatorname{Std}(\boldsymbol{\lambda})$ define $\operatorname{Ch}_q\mathcal{T}=\sum_{\mathsf{t}\in\mathcal{T}}q^{\deg\mathsf{t}}\cdot\mathbf{i}^{\mathsf{t}}\in\mathcal{A}[I^n]$. Then, by definition, $\operatorname{Ch}_q S^{\boldsymbol{\lambda}}_{\mathsf{T}}=\operatorname{Ch}_q\mathsf{T}$.

4.4.2. **Proposition.** *Let F be a field of characteristic zero and assume that Conjecture 4.4.1 holds.*

a) *Suppose that $\boldsymbol{\mu}\in\mathcal{K}^\Lambda_n$. Then $D^{\boldsymbol{\mu}}\cong S^{\boldsymbol{\lambda}}_{\mathsf{T}^{\boldsymbol{\mu}}}$. Consequently, $\operatorname{Ch}_q D^{\boldsymbol{\mu}}=\operatorname{Ch}_q\mathsf{T}^{\boldsymbol{\mu}}$.*

b) *For each $\mathsf{T}\in\operatorname{Std}[\boldsymbol{\lambda}]$, for $\boldsymbol{\lambda}\in\mathcal{P}_n$, there exists a unique $(\boldsymbol{\nu}_{\mathsf{T}},d_{\mathsf{T}})\in\mathcal{K}^\Lambda_n\times\mathbb{N}$ such that $\operatorname{Ch}_q\mathsf{T}=q^{d_{\mathsf{T}}}\operatorname{Ch}_q D^{\boldsymbol{\nu}_{\mathsf{T}}}$. Moreover,*

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)=\sum_{\substack{\mathsf{T}\in\operatorname{Std}[\boldsymbol{\lambda}]\\ \boldsymbol{\nu}_{\mathsf{T}}=\boldsymbol{\mu}}}q^{d_{\mathsf{T}}}.$$

Proof. By Corollary 3.2.7, $D^{\boldsymbol{\mu}}\neq0$ since $\boldsymbol{\mu}\in\mathcal{K}^\Lambda_n$. The irreducible module $D^{\boldsymbol{\mu}}$ is generated by $B_{\mathsf{t}^{\boldsymbol{\mu}}}+\operatorname{rad}S^{\boldsymbol{\mu}}=\psi_{\mathsf{t}^{\boldsymbol{\mu}}}+\operatorname{rad}S^{\boldsymbol{\mu}}$, so $D^{\boldsymbol{\mu}}\cong S^{\boldsymbol{\mu}}_{T_{\boldsymbol{\mu}}}$ since both modules are irreducible by Conjecture 4.4.1 For part (b), $S^{\boldsymbol{\lambda}}_{\mathsf{T}}\cong D^{\boldsymbol{\nu}}\langle d\rangle$, for some $\boldsymbol{\nu}\in\mathcal{K}^\Lambda_n$ and $d\in\mathbb{Z}$, because $S^{\boldsymbol{\lambda}}_{\mathsf{T}}$ is irreducible by Conjecture 4.4.1. Therefore, $\operatorname{Ch}_q S^{\boldsymbol{\lambda}}_{\mathsf{T}}=q^d\operatorname{Ch}_q D^{\boldsymbol{\nu}}$. The uniqueness of $(\boldsymbol{\nu}_{\mathsf{T}},d_{\mathsf{T}})=(\boldsymbol{\nu},d)\in\mathcal{K}^\Lambda_n\times\mathbb{Z}$ follows from Theorem 3.7.1 and Theorem 2.1.4. Moreover, $d\ge0$ by Corollary 3.5.11. As every composition factor of $S^{\boldsymbol{\lambda}}$ is isomorphic to $S^{\boldsymbol{\lambda}}_{\mathsf{T}}$ for some $\mathsf{T}\in\operatorname{Std}[\boldsymbol{\lambda}]$ the formula for $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$ is now immediate. □

Proposition 4.4.2 shows that Conjecture 4.4.1 encodes closed formulas for the characters and graded dimensions of the irreducible $\mathscr{H}^\Lambda_n$-modules and for the graded decomposition numbers of $\mathscr{H}^\Lambda_n$. For this result to really be useful we need to both verify Conjecture 4.4.1 and to explicitly determine the equivalence relation $\sim_B$. Our last result is a small step in this direction.

4.4.3. **Lemma.** *Suppose that* $\mathsf{s},\mathsf{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ *and that* $\mathsf{t}=\mathsf{s}(r,r+1)$ *such that* $\mathbf{i}^{\mathsf{s}}_{r+1}\ne\mathbf{i}^{\mathsf{s}}_r\pm1$*, where* $1\le r<n$ *and* $\boldsymbol{\lambda}\in\mathcal{P}_n$*. Then* $\mathsf{s}\sim_B\mathsf{t}$.

Proof. It follows from (4.3.4), and Theorem 3.6.2, $B_{\mathsf{s}}\psi_r=\psi_{\mathsf{t}}+\sum_{\mathsf{u}}a_{\mathsf{u}}\psi_{\mathsf{u}}=B_{\mathsf{t}}+\sum_{\mathsf{u}}b_{\mathsf{u}}B_{\mathsf{u}}$, where $a_{\mathsf{u}},b_{\mathsf{u}}\in F$ are non-zero only if $\ell(d(\mathsf{u}))<\ell(d(\mathsf{s}))$. Therefore, $\mathsf{s}\succ_B\mathsf{t}$. If $\mathbf{i}^{\mathsf{s}}_{r+1}\ne\mathbf{i}^{\mathsf{t}}_r$ then $e(\mathbf{i}^{\mathsf{s}})\psi_r^2=1$ by (2.2.3), so it follows that $\mathsf{s}\sim_B\mathsf{t}$.

Now consider the more interesting case when $\mathbf{i}^{\mathsf{s}}_{r+1}=\mathbf{i}^{\mathsf{s}}_r$ or, equivalently, $\mathbf{i}^{\mathsf{s}}_r=\mathbf{i}^{\mathsf{t}}_r$. Then, using (2.2.2),

$$B_{\mathsf{t}}y_{r+1}=\big(B_{\mathsf{s}}\psi_r-\sum_{\mathsf{u}}b_{\mathsf{u}}B_{\mathsf{u}}\big)y_{r+1}=B_{\mathsf{s}}(y_r\psi_r+1)-\sum_{\mathsf{u}}b_{\mathsf{u}}B_{\mathsf{u}}y_{r+1}.$$

In view of Proposition 3.2.9(c), B_{s} appears on the righthand side with coefficient 1. Hence, $\mathsf{t}\succ_B\mathsf{s}$ implying that $\mathsf{s}\sim_B\mathsf{t}$ as claimed. ☐

The B-basis, and hence Conjecture 4.4.1 and all of the results in this section (except that $d_{\mathsf{T}}\in\mathbb{Z}$ in Proposition 4.4.2), make sense over any field. We restrict Conjecture 4.4.1 to fields of characteristic zero because it would be foolhardy to venture into the realms of positive characteristic without some evidence. This said, whether or not Conjecture 4.4.1 is true in characteristic zero, we strongly believe that in *all characteristics* there exists a "canonical" graded cellular basis $\{C_{\mathsf{st}}\}$ of $\mathscr{R}^\Lambda_n$ such that the analogous version of Conjecture 4.4.1 holds for the $\sim_C$ equivalence classes.

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