3-MANIFOLDS WITH ABELIAN EMBEDDINGS IN S^4

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Abstract. We consider embeddings of 3-manifolds in S^4 such that each of the two complementary regions has an abelian fundamental group. In particular, we show that an homology handle M has such an embedding if and only if $\pi_1(M)'$ is perfect, and that the embedding is then essentially unique.

Every integral homology 3-sphere embeds as a topologically locally flat hypersurface in S^4, and has an essentially unique "simplest" such embedding, with contractible complementary regions. For other 3-manifolds which embed in S^4, the complementary regions cannot both be simply-connected, and it not clear whether they always have canonical "simplest" embeddings. If M is a closed hypersurface in $S^4 = X \cup_M Y$ then $H_1(M;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \oplus H_1(Y;\mathbb{Z})$, and so embeddings such that each of the complementary regions X and Y has an abelian fundamental group might be considered simplest. We shall say that such an embedding is abelian. Although most 3-manifolds that embed in S^4 do not have such embeddings, this class is of particular interest as the possible groups are known, and topological surgery in dimension 4 is available for abelian fundamental groups.

Homology 3-spheres have essentially unique abelian embeddings (although they may have other embeddings). This is also known for $S^2 \times S^1$ and $S^3/Q(8)$, by results of Aitchison (published in [21]) and Lawson [16], respectively. In Theorems 10 and 11 below we show that if M is an orientable homology handle (i.e., such that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$) then it has an abelian embedding if and only if $\pi_1(M)$ has perfect commutator subgroup, and then the abelian embedding is essentially unique. (There are homology handles which do not embed in S^4 at all!) The 3-manifolds obtained by 0-framed surgery on 2-component links with unknotted components always have abelian embeddings, and the complementary regions for such embeddings are homotopy equivalent to standard 2-complexes. These shall be our main source of examples. In particular, we shall give an example in which $X \simeq Y \simeq S^1 \vee S^2$, but the pairs (X, M) and (Y, M) are not homotopy equivalent. We do not yet have examples of a 3-manifold with several inequivalent abelian embeddings.

The first two sections fix our notation, recall some earlier work, and give some results on homotopy equivalences. In §3 we define the notions of abelian and nilpotent embeddings. Sections 4–6 consider abelian embeddings of 3-manifolds M with torsion free homology (i.e., $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^\beta$, where $\beta \leq 4$ or $\beta = 6$). In §7 we consider briefly some embeddings of rational homology spheres. In particular, we shall show that ten 3-manifolds with elementary amenable fundamental group have abelian embeddings. The question remains open for one further such 3-manifold. In §8 we give some simple observations on the possible homotopy types

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of the complementary regions for the final class of abelian embeddings, for which $\pi_1(X) \cong \pi_1(Y) \cong \mathbb{Z} \oplus (\mathbb{Z}/k\mathbb{Z})$, for some $k > 1$. While such examples do exist, much less is known in this case.

All 3-manifolds considered here shall be closed, connected and orientable. An embedding j is *smoothable* if it is smooth with respect to some smooth structure on S^4, equivalently, if each complementary region is a handlebody. Although the embeddings that we shall construct are usually smooth embeddings in the standard 4-sphere, we wish to apply surgery arguments, and so "embedding" shall mean "topologically locally flat embedding", unless otherwise qualified. Embeddings j and $\tilde{j}$ are *equivalent* if there are self-homeomorphisms ϕ of M and ψ of S^4 such that $\psi j = \tilde{j}\phi$. If all abelian embeddings are equivalent to j, we shall say that j is *essentially unique*.

1. NOTATION AND BACKGROUND

Let $j : M \to S^4$ be an embedding of a closed connected 3-manifold, and let X and Y be the closures of the components of $S^4 \setminus M$. The Mayer-Vietoris sequence for $S^4 = X \cup_M Y$ and Poincaré-Lefshetz duality give isomorphisms $H_i(M;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \oplus H_i(Y;\mathbb{Z})$ for $i = 1$ and 2, $H_2(X;\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$ and $H_2(Y;\mathbb{Z}) \cong H^1(X;\mathbb{Z})$, while $H_i(X;\mathbb{Z}) = H_i(Y;\mathbb{Z}) = 0$ for $i > 2$. Since $\chi(X) + \chi(Y) = \chi(S^4) + \chi(M) = 2$, we may assume that $\chi(X) \le 1 \le \chi(Y)$. Let $\beta = \beta_1(M;\mathbb{Z})$, $\pi = \pi_1(M)$, $\pi_X = \pi_1(X)$ and $\pi_Y = \pi_1(Y)$, and let j_X and j_Y be the inclusions of M into X and Y, respectively.

Our commutator convention is that if G is a group and $g, h \in G$ then $[g,h] = ghg^{-1}h^{-1}$. The commutator subgroup is $G' = [G,G]$, and the second derived group is $G'' = [G',G']$. The lower central series is defined by $G_{[1]} = G$ and $G_{[n+1]} = [G, G_n]$ for all $n \ge 1$. Let $F(r)$ be the free group of rank r.

If V is a cell-complex we shall write $C_*(\widetilde{V}) = C_*(V;\mathbb{Z}[\pi_1(V)])$ for the cellular chain complex of the universal cover $\widetilde{V}$ with its natural structure as a $\mathbb{Z}[\pi_1(V)]$-module (and similarly for pairs of spaces).

Our examples may all be constructed using bipartedly slice links. Let $M(L)$ be the closed 3-manifold obtained by 0-framed surgery on the link L. We say that L is *bipartedly slice* (respectively, *trivial* or *ribbon*) if it has a partition $L = L_+ \cup L_-$ into two sublinks which are each slice links (respectively, trivial or ribbon links). The partition then determines an embedding $j_L : M \to S^4$, given by ambient surgery on an equatorial S^3 in $S^4 = D_+ \cup D_-$. We add 2-handles to these 4-balls along L_+ on one side and along L_- on the other. If L_+ and L_- are smoothly slice then j_L is smooth, and if they are trivial each complementary region may be obtained by adding 1- and 2-handles to the 4-ball. (The notation j_L is ambiguous, for if L has more than two components it may have several different partitions leading to distinct embeddings. Moreover we must choose a set of slice discs for each of L_+ and L_-.) If each complementary region for an embedding j may be obtained from the 4-ball by adding 1- and 2-handles, must $j = j_L$ for some 0-framed link L?

In [14] we said that an embedding j is *minimal* if the induced homomorphism $j_\Delta : \pi \to \pi_1(X) \times \pi_1(Y)$ is an epimorphism. In fact this is equivalent to each of j_X and j_Y inducing an epimorphism.

Lemma 1. *The homomorphisms $j_{X*} = \pi_1(j_X)$ and $j_{Y*} = \pi_1(j_Y)$ are both epimorphisms if and only if $j_\Delta = (j_{X*}, j_{Y*})$ is an epimorphism.*

Proof. Let $K_X = \mathrm{Ker}(j_{X*})$ and $K_Y = \mathrm{Ker}(j_{Y*})$. If j_{X*} and j_{Y*} are epimorphisms then they induce isomorphisms $\pi/K_X \to \pi_X$ and $\pi/K_Y \to \pi_Y$. Hence $\pi/K_X K_Y \cong \pi_X/j_{X*}(K_Y)$ and $\pi/K_X K_Y \cong \pi_Y/j_{Y*}(K_X)$. Since $\pi_1(X \cup_M Y) = 1$, these quotients must all be trivial. If $g \in K_X$ and $h \in K_Y$ then $j_\Delta(gh) = (j_{X*}(h), j_{Y*}(g))$. Hence j_Δ is an epimorphism.

Conversely, if j_Δ is an epimorphism then so are its components j_{X*} and j_{Y*}. $\square$

The term "minimal" is unsatisfactory for several reasons, and we shall henceforth say that an embedding satisfying the equivalent conditions of Lemma 1 is *bi-epic*. Embeddings obtained from other embeddings by nontrivial "2-knot surgery" [14] are never bi-epic. However, if $j = j_L$ for some bipartedly ribbon link L then j is bi-epic, since π, π_X and π_Y are generated by images of the meridians of L.

Example. *There are* 3*-manifolds with more than one bi-epic embedding.*

The link L obtained from the Borromean rings by replacing one component by its $(2,1)$-cable and another by its $(3,1)$-cable may be partitioned as the union of two trivial links in three ways. The resulting three embeddings of $M(L)$ in S^4 each have $Y \simeq S^1 \vee 2S^2$, but the groups π_X have presentations $\langle a, b | [a, b^2]^3 \rangle$, $\langle a, c | [a, c^3]^2 \rangle$, and $\langle b, c | [b^2, c^3] \rangle$, respectively, and so are distinct. In the first two cases π has torsion, while in the third case X is aspherical. (None of these groups is abelian.) This example can obviously be generalized in various ways. The homology sphere in Figure 3 of [14] is another example; the embedding determined by the link is bi-epic, but the 3-manifold also has an embedding with both complementary regions contractible. However the latter embedding may not derive from a 0-framed link representing the homology sphere.

The cases when j_Δ is an isomorphism are quite rare.

Lemma 2. *If j_Δ is an isomorphism then either $M \cong F \times S^1$ for some aspherical closed orientable surface F or $M \cong \#^r(S^2 \times S^1)$ for some $r \geq 0$.*

Proof. If $\pi \cong \pi_X \times \pi_Y$ with π_X infinite and $\pi_Y \neq 1$ then $M \cong F \times S^1$ for some aspherical closed orientable surface F [8]. If $\pi_Y = 1$ then j_{X*} is an isomorphism, and so π must be a free group [5]. Hence $M \cong \#^r(S^2 \times S^1)$ for some $r \geq 1$. Finally, if π_X and π_Y are both finite and have nontrivial abelianization then their orders have a common prime factor p, and so π has $(\mathbb{Z}/p\mathbb{Z})^2$ as a subgroup, which is not possible. We may also exclude $\pi_X \cong \pi_Y \cong I^*$, for a similar reason, and so there remains only the case $\pi = 1$, when $M = S^3 = \#^0(S^2 \times S^1)$. $\square$

These 3-manifolds do in fact have bi-epic embeddings with j_Δ an isomorphism.

2. HOMOTOPY EQUIVALENCES

In this section we shall give some lemmas on recognizing the homotopy types of certain spaces and pairs of spaces arising later. One simple but important observation is that the natural homomorphisms $H_2(X; \mathbb{Z}) \to H_2(X, M; \mathbb{Z})$ is 0, since it factors through $H_2(S^4; \mathbb{Z}) \to H_2(S_4, Y; \mathbb{Z})$, and similarly for $H_2(Y; \mathbb{Z}) \to H_2(Y, M; \mathbb{Z})$. Equivalently, the intersection pairings are trivial on $H_2(X; \mathbb{Z})$ and $H_2(Y; \mathbb{Z})$. (See Theorem 11 below for one use of this observation.)

Theorem 3. *Let U and V be connected finite cell complexes such that c.d.$U \leq 2$ and c.d.$V \leq 2$. If $f : U \to V$ is a 2-connected map then $\chi(U) \geq \chi(V)$, with equality if and only if f is a homotopy equivalence.*

Proof. Up to homotopy, we may assume that f is a cellular inclusion, and that V has dimension ≤ 3. Let $\pi = \pi_1(U)$ and let $C_* = C_*(\widetilde{V}, \widetilde{U})$. Then $H_q(C_*) = 0$ if $q \leq 2$, since f is 2-connected, and $H_q(C_*) = 0$ if $q > 3$, since $c.d.U$ and $c.d.V \leq 2$. Hence $H_3(C_*) \oplus C_2 \oplus C_0 \cong C_3 \oplus C_1$, by Schanuel's Lemma, and so $H_3(C_*)$ is a stably free $\mathbb{Z}[\pi]$-module of rank $-\chi(C_*) = \chi(U) - \chi(V)$. Hence $\chi(U) \geq \chi(V)$, with equality if and only if $H_3(C_*) = 0$, since group rings are weakly finite, by a theorem of Kaplansky. (See [19] for a proof.) The result follows from the long exact sequence of the pair $(\widetilde{Y}, \widetilde{X})$ and the theorems of Hurewicz and Whitehead. □

If $c.d.X \leq 2$ then $C_*(\widetilde{X})$ is chain homotopy equivalent to a finite projective complex of length 2, which is a partial resolution of the augmentation module $\mathbb{Z}$. Chain homotopy classes of such partial resolutions are classified by $Ext^3_{\mathbb{Z}[\pi]}(\mathbb{Z}, \Pi) = H^3(\pi; \Pi)$, where Π is the module of 2-cycles.

Corollary 4. *If U is a connected finite complex such that $c.d.U \leq 2$ and $\pi_1(U) \cong \mathbb{Z}$ then $U \simeq S^1 \vee \bigvee^{\chi(U)} S^2$.*

Proof. Since $c.d.U \leq 2$ and projective $\mathbb{Z}[\pi_1(U)]$-modules are free, $C_*(\widetilde{U})$ is chain homotopy equivalent to a finite free $\mathbb{Z}[\pi_1(U)]$-complex P_* of length ≤ 2, and $\chi(U) = \Sigma(-1)^i rank(P_i)$. Since $\pi_2(U) \cong H_2(U; \mathbb{Z}[\pi_1(U)])$ is the module of 2-cycles in $C_*(\widetilde{U})$, it is free of rank $\chi(U)$. Let $f : S^1 \vee \bigvee^{\chi(U)} S^2 \to U$ be the map determined by a generator for $\pi_1(U)$ and representatives of a basis for $\pi_2(U)$. Then f is a homotopy equivalence, by the theorem. □

Theorem 3.2 of [11] gives an analogue of Theorem 3 for maps between closed 4-manifolds. The argument extends to the following relative version.

Lemma 5. *Let $f : (X_1, A_1) \to (X_2, A_2)$ be a map of orientable PD_4-pairs such that $f|_{A_1} : A_1 \to A_2$ is a homotopy equivalence. Then f is a homotopy equivalence of pairs if and only if $\pi_1(f)$ is an isomorphism and $\chi(X_1) = \chi(X_2)$.*

Proof. Since $f|_{A_1} : A_1 \to A_2$ is a homotopy equivalence, f has degree 1, and hence is 2-connected as a map from X_1 to X_2. The rest of the argument is as in Theorem 3.2 of [11]. □

In certain cases we can identify the homotopy type of a pair.

Lemma 6. *Let (X, A) and (X', A') be pairs such that the inclusions $\iota_A : A \to X$ and $\iota_{A'} : A' \to X'$ induce epimorphisms on fundamental groups. If X and X' are aspherical and $f : A \to A'$ is a homotopy equivalence such that $\pi_1(f)(\mathrm{Ker}(\pi_1(\iota_A))) = \mathrm{Ker}(\pi_1(\iota_{A'}))$ then f extends to a homotopy equivalence of pairs $(X, A) \simeq (X', A')$.*

Proof. The fundamental group conditions imply that $g = \iota_{A'} f$ extends to a map from the relative 2-skeleton $X^{[2]} \cup A$. The further obstructions to extending g to a map from X to X' lie in $H^{q+1}(X, A; \pi_q(X'))$, for $q \geq 2$. Since X' is aspherical these groups are 0. The other hypotheses imply that any extension $h : X \to X'$ induces an isomorphism on fundamental groups, and hence is a homotopy equivalence. □

We would like to have an analogue of Lemma 6 for the cases when $\pi_X \cong \mathbb{Z}$ and $\chi(X) = 1$. If $(X, \partial X)$ is a PD_4-pair such that $X \simeq S^1 \vee S^2$ then $\pi_2(X) \cong \mathbb{Z}[\pi_X]$ and $\pi_3(X) \cong \Gamma_W(\mathbb{Z}[\pi_X])$, where Γ_W is the quadratic functor of Whitehead. Let $(X, \partial X)$ and $(\widehat{X}, \partial \widehat{X})$ be two such PD_4-pairs, and let ι_X and $\iota_{\widehat{X}}$ be the inclusions of the boundaries. Then any homotopy equivalence $f : \partial X \to \partial \widehat{X}$ such that $f\iota_X \sim$

$\iota_{\widehat{X}}$ extends across the relative 3-skeleton $X^{[3]} \cup \partial X$, since $H^3(X, \partial X; f^*\pi_2(\widehat{X})) \cong H_1(X; \mathbb{Z}[\pi_X]) = 0$. The only obstruction to extending such an f to a map from X to $\widehat{X}$ lies in $H^4(X, \partial X; f^*\pi_3(\widehat{X})) \cong H_0(X; f^*\pi_3(\widehat{X})) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi_X]} \Gamma_W(\mathbb{Z}[\pi_X])$. (Any such extension would be a homotopy equivalence.) This obstruction is perhaps determined by the equivariant intersection pairings on $\pi_2(X)$ and $\pi_2(\widehat{X})$. Can we use the additional constraints that $(X, \partial X)$ and $(\widehat{X}, \partial\widehat{X})$ are codimension-0 submanifolds of S^4? (Note also that a further extension to the case when $\pi_1(Y) \cong \mathbb{Z}$ and $\chi(Y) > 1$ would imply the Unknotting Theorem for orientable surfaces in S^4.)

3. ABELIAN EMBEDDINGS

In so far as we hope to apply 4-dimensional topological surgery to the complementary regions, we need to assume that π_X and π_Y are "good" in the sense of [9]. At present, the class of groups known to be good is somewhat larger than the class of elementary amenable groups. The consequences of the Mayer-Vietoris sequence noted above together with the fact that the higher L^2 Betti numbers of amenable groups vanish give a simple but useful constraint.

Lemma 7. *If $\beta_1^{(2)}(\pi_X) = 0$ then either $\chi(X) = 0$ and $c.d.\pi_X \leq 2$ or $\chi(X) = 1$ and $def(\pi_X) \geq 0$.*

Proof. This follows from Theorem 2.5 of [11], since $def(\pi_X) \geq 1 - \chi(X)$. $\square$

In particular, if π_X is elementary amenable and $\chi(X) = 0$ then $\pi_X \cong \mathbb{Z}$ or $\mathbb{Z}*_m$ (with presentation $\langle a, t | tat^{-1} = a^m\rangle$), for some $m \neq 0$. (See Corollary 2.6.1 of [11].)

The first L^2-Betti number vanishes also for semidirect products $N \rtimes \mathbb{Z}$ with N finitely generated. (This observation is used in Theorem 12 below.)

Example. *If $M = M(-2; (1, 0))$ or $M(-2; (1, 4))$ and j is bi-epic then $X \simeq Kb$.*

In each case π is polycyclic and $\pi/\pi' \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. Hence $\chi(X) = 0$, and so $c.d.\pi_X \leq 2$. Since π_X is a quotient of π and $\pi_X/\pi_X' \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ we must have $\pi_X \cong \mathbb{Z}*_{-1} = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$. Since $c.d.X \leq 2$ and $\chi(X) = 0$ the classifying map $c_X : X \to Kb = K(\mathbb{Z} \rtimes_{-1} \mathbb{Z}, 1)$ is a homotopy equivalence.

The subclass of nilpotent groups is of particular interest. If π_X is nilpotent then j_{X*} is onto, since $H_1(j_X)$ is onto, and any subset of a nilpotent group G whose image generates the abelianization G/G' generates G. Moreover, since $def(\pi_X) \geq 0$ and π_X is nilpotent it is generated by at most 3 elements, by Theorem 2.7 of [17]. Since j_{X*} is onto, $c.d.X \leq 2$, by Theorem 5.1 of [14]. (Similarly, if π_Y is nilpotent then j_{Y*} is onto and $c.d.Y \leq 2$.) If π_X and π_Y are each nilpotent then j is bi-epic, by Lemma 1 above.

There are also purely algebraic reasons why nilpotent groups should be of particular interest. Firstly, there is the well-known connection between homology, lower central series and (Massey) products (as used in [14]). Secondly, if a group G is finite or solvable and every homomorphism $f : H \to G$ which induces an epimorphism on abelianization is an epimorphism then G must be nilpotent. (See pages 132 and 460 of [20].) However even the class of 2-generator nilpotent groups of deficiency 0 is not known. It seems to be quite large; for instance, there is a $\mathbb{N}il^4$-group with presentation $\langle x, y \mid x[x, [x, y]] = [x, [x, y]]x,\ y[x, y] = [x, y]y\rangle$. (We expect that nilpotent groups of large Hirsch length should have negative deficiency, and so should not arise in this context.)

If we restrict further to the abelian case the possible groups are known. If π_X is abelian and $\chi(X)=0$ then either $\pi_X\cong\mathbb{Z}$ and $H_2(X;\mathbb{Z})=0$ or $\pi_X\cong\mathbb{Z}^2$ and $H_2(X;\mathbb{Z})\cong\mathbb{Z}$. If π_X is abelian and $\chi(X)=1$ then $\chi(Y)=1$ also, and $\beta_1(\pi_X)-\beta_2(\pi_X)\geq def(\pi_X)=0$. In the latter case it follows easily that $\pi_X\cong\mathbb{Z}/k\mathbb{Z}$, $\mathbb{Z}\oplus\mathbb{Z}/k\mathbb{Z}$, $\mathbb{Z}^2$ or $\mathbb{Z}^3$. Hence either $\beta=0$ and $\pi_X\cong\mathbb{Z}/k\mathbb{Z}$ or $\beta=2$ and $\pi_X\cong\mathbb{Z}\oplus\mathbb{Z}/k\mathbb{Z}$, for some $k\geq 1$, or $\beta=1,3,4$ or 6 and $\pi_X\cong\mathbb{Z}^{\lfloor\frac{\beta+1}{2}\rfloor}$. (See also Theorem 7.1 of [14].)

Lemma 8. *If π_X is abelian of rank at most 1 then X is homotopy equivalent to a finite 2-complex. If moreover π_X is cyclic then $X\simeq S^1$ or $S^1\vee_\ell S^2$, for some $\ell\in\mathbb{Z}$. or $P_\ell=S^1\vee_\ell e^2$, with $\ell\neq 0$.*

Proof. The first assertion follows from the facts that $c.d.X\leq 2$, as just observed, and that the $\mathcal{D}(2)$ property holds for cyclic groups (see page 235 of [15]) and for the groups $\mathbb{Z}\oplus\mathbb{Z}/\ell\mathbb{Z}$ [7]. If $\pi_X\cong\mathbb{Z}/\ell\mathbb{Z}$ is cyclic then $X\simeq S^1$ or $S^1\vee S^2$, if $\ell=0$, or $P_\ell=S^1\vee_\ell e^2$, if $\ell\neq 0$ [6]. □

We shall show later that a similar result holds when $\pi_X\cong\mathbb{Z}^2$.

We shall say that an embedding j is *abelian* or *nilpotent* if π_X and π_Y are each abelian or nilpotent, respectively. Ten of the thirteen 3-manifolds with elementary amenable fundamental groups and which embed in S^4 (see [4]) have abelian embeddings. (Apart from the Poincaré homology 3-sphere S^3/I^*, which does not embed smoothly, these derive from the empty link $\emptyset$, the unknot U, the 2-component links 4^2_1, 5^2_1, 6^2_1, 8^2_2, 9^2_{53}, 9^2_{61} and the Borromean rings 6^3_2.) In at least four cases (S^3, $S^3/Q(8)$, S^3/I^* and $S^2\times S^1$) the abelian embedding is essentially unique. The "half-turn" flat 3-manifold $G_2=M(-2;(1,0))$ and the $\mathbb{N}il^3$-manifold $M(-2;(1,4))$ bound regular neighbourhoods of embeddings of the Klein bottle Kb in S^4, but have no abelian embeddings. The status of one $\mathbb{S}ol^3$-manifold is not yet known.

When $\beta\leq 1$ we must have $\chi(X)=1-\beta$. If L is a 2-component slice link with unknotted components (such as the trivial 2-component link, or the Milnor boundary link) and $M=M(L)$ then $\beta=2$ and M has an abelian embedding (with $\chi(X)=\chi(Y)=1$), and also an embedding with $\chi(X)=1-\beta=-1$ and $\pi_Y=1$. However it shall follow from the next lemma that if $\beta>2$ then M cannot have both an abelian embedding and also one with $\chi(X)=1-\beta$.

Lemma 9. *Let $j:M\to S^4$ be an embedding j such that $H_1(Y;\mathbb{Z})=0$, and let $S\subset\Lambda_\beta=\mathbb{Z}[\pi/\pi']$ be the multiplicative system consisting of all elements s with augmentation $\varepsilon(s)=1$. If the augmentation homomorphism $\varepsilon:\Lambda_{\beta S}\to\mathbb{Z}$ factors through an integral domain $R\neq\mathbb{Z}$ then $H_1(M;R)$ has rank $\beta-1$ as an R-module.*

Proof. Let $*$ be a basepoint for M and $A(\pi)=H_1(M,*;\Lambda_\beta)$ be the Alexander module of π. (See Chapter 4 of [12].) Since $H_2(X;\mathbb{Z})=0$, the inclusion of representatives for a basis of $H_1(X;\mathbb{Z})\cong\mathbb{Z}^\beta$ induces isomorphisms $F(\beta)/F(\beta)_{[n]}\cong\pi/\pi_{[n]}$, for all $n\geq 1$, by a theorem of Stallings. (See Lemma 3.1 of [14].) Hence $A(\pi)_S\cong(\Lambda_{\beta S})^\beta$, by Lemma 4.9 of [12]. Since ε factors through R, the exact sequence of the pair $(M,*)$ with coefficients R gives an exact sequence

$$0\to H_1(M;R)\to R\otimes_{\Lambda_\beta}A(\pi)\cong R^\beta\to R\to R\otimes_{\Lambda_\beta}\mathbb{Z}=\mathbb{Z}\to 0,$$

from which the lemma follows. □

4. HOMOLOGY SPHERES AND HANDLES

If M is an integral homology 3-sphere then it bounds a contractible 4-manifold, and so has an abelian embedding with X and Y each contractible. They are determined up to homeomorphism by their boundaries [9], and so the abelian embedding is unique. Moreover, the complementary regions are homeomorphic. When $M = S^3$, the result goes back to the Brown-Mazur-Schoenflies Theorem. (In this special case the embedding is essentially unique!)

It is not clear whether non-simply connected homology spheres must have embeddings with one or both of π_X and π_Y nontrivial. Figure 3 of [14] gives an example with $\pi_X \cong \pi_Y \cong I^*$, the binary icosahedral group. In this case the homology sphere is the result of surgery on a complicated 4-component bipartedly trivial link, and probably has no simpler description. The Poincaré homology 3-sphere S^3/I^* is not the result of 0-framed surgery on any bipartedly slice link, since it does not embed smoothly.

If instead M is an orientable homology handle, i.e., if $\pi/\pi' = H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, so M has the homology of $S^2 \times S^1$, then π'/π'' is a finitely generated torsion module over $\mathbb{Z}[\pi/\pi'] \cong \Lambda = \mathbb{Z}[t, t^{-1}]$. Equivariant Poincaré duality and the universal coefficient theorem together define a nonsingular hermitean pairing b on π'/π'', with values in $\mathbb{Q}(t)/\Lambda$, called the Blanchfield pairing. The pairing is *neutral* if π'/π'' has a submodule N which is its own annihilator with respect to b, i.e., such that $N = \{m \in \pi'/\pi'' \mid b(m, n) = 0 \ \forall n \in N\}$.

Theorem 10. *Let M be an orientable homology handle. If M embeds in S^4 then the Blanchfield pairing on $\pi'/\pi'' = H_1(M;\mathbb{Z}[\pi/\pi'])$ is neutral. There is an abelian embedding $j : M \to S^4$ if and only if π' is perfect, and then $X \simeq S^1$ and $Y \simeq S^2$.*

Proof. The first assertion follows on applying equivariant Poincaré-Lefshetz duality to the infinite cyclic cover of the pair (X, M). (See the proof of Theorem 2.4 of [12].)

If j is abelian then $\pi_X \cong \mathbb{Z}$ and $\pi_Y = 1$, while $H_2(X;\mathbb{Z}) = 0$ and $H_2(Y;\mathbb{Z}) \cong \mathbb{Z}$. Since $c.d.X \leq 2$ and $\pi_X \cong \mathbb{Z}$, it follows that $\pi_2(X) = H_2(X;\mathbb{Z}[\pi_X])$ is a free $\mathbb{Z}[\pi_X]$-module of rank $\chi(X) = 0$. Hence $\pi_2(X) = 0$, and so maps $f : S^1 \to X$ and $g : S^2 \to Y$ representing generators for π_X and $\pi_2(Y)$ are homotopy equivalences. Since $H_2(X, M;\mathbb{Z}[\pi_X]) \cong \overline{H^2(X;\mathbb{Z}[\pi_X])} = 0$, by equivariant Poincaré-Lefshetz duality, $\pi'/\pi'' = H_1(M;\mathbb{Z}[\pi_X]) = 0$, by the homology exact sequence for the infinite cyclic cover of the pair (X, M). Hence π' is perfect.

Suppose, conversely, that π' is perfect. Then M embeds in S^4, by the main result of [13], and examination of the proof shows that the embedding constructed in the theorem is abelian. □

The product $M = S^2 \times S^1$ may be obtained by 0-framed surgery on the unknot, and so has a standard abelian embedding with $X \cong S^1 \times D^3$ and $Y \cong D^2 \times S^2$. (In fact Y must be of this form whenever $M = S^2 \times S^1$, by a result of Aitchison, published in [21].)

If K is an Alexander polynomial 1 knot then $M(K)$ has an abelian embedding, and if K is a knot such that $M(K)$ embeds in S^4 then K is algebraically slice, by Theorem 10. However if K is a slice knot with nontrivial Alexander polynomial then $M(K)$ embeds in S^4 but no embedding is abelian.

Part of the argument for the next theorem was prompted by my reading of Section 2 of [16].

Theorem 11. *Let M be an orientable homology handle. Then M has an essentially unique abelian embedding.*

Proof. We may suppose that j_1 and j_2 are abelian embeddings of M. There is a homotopy equivalence of pairs $(X_1, M) \simeq (X_2, M)$ which extends id_M, by Lemma 6. This is homotopic *rel* M to a homeomorphism F, since $L_5(\mathbb{Z})$ acts trivially on the structure set $\mathcal{S}_{TOP}(X_2, \partial X_2)$. (This follows from the Wall-Shaneson theorem and the existence of the E_8-manifold. See also Theorem 6.7 of [11].)

We may assume the homotopy equivalences $Y_1 \simeq S^2$ and $Y_2 \simeq S^2$ are so chosen that the corresponding maps f_1 and f_2 from M to S^2 induce the same class in $H^2(M;\mathbb{Z})$. We may also assume that f_1 and f_2 agree on the 2-skeleton of M, by Theorem 8.4.11 of [22]. Let $p : M \to M \vee S^3$ be a pinch map, and $\eta : S^3 \to S^2$ be the Hopf fibration. Let d_t be a self map of S^3 of degree t, and let $q_t = id_M \vee d_t$. Then $f_2 \sim (f_1 \vee \eta)q_t p$, for some $t \in \mathbb{Z}$. Let Z_i be the mapping cylinder of f_i, for $i = 1, 2$. Then Y_i is homotopy equivalent to Z_i *rel* M, for $i = 1, 2$.

Let $P = MCyl(\eta) = \overline{\mathbb{CP}^2 \setminus D^4}$ and $W = MCyl(f_1 \vee \eta)$. The inclusions of S^3 and M into $M \vee S^3$ and $q_t p$ induce maps θ, ψ and ξ from (P, S^3), (Z_1, M) and (Z_2, M), respectively, to $(W, M \vee S^3)$. These induce isomorphisms of $H^2(P;\mathbb{Z})$, $H^2(Z_1;\mathbb{Z})$ and $H^2(Z_2;\mathbb{Z})$ with $H^2(W;\mathbb{Z}) \cong \mathbb{Z}$. Let σ generate $H^2(W;\mathbb{Z})$. The groups $H_4(P, S^3;\mathbb{Z})$, $H_4(Z_1, M;\mathbb{Z})$ and $H_4(Z_2, M;\mathbb{Z})$ are also infinite cyclic, with generators $[P, S^3]$, $[Z_1, M]$ and $[Z_2, M]$, respectively, but $H_4(W, M \vee S^3;\mathbb{Z}) \cong H_3(S^3;\mathbb{Z}) \oplus H_3(M;\mathbb{Z})$, and $\xi_*[Z_2, M] = t.\theta_*[P, S^3] + \psi_*[Z_1, M]$. Hence

$$\xi^*\sigma^2 \cap [Z_2, M] = \sigma^2 \cap \xi_*[Z_2, M] = t\sigma^2 \cap \theta_*[P, S^3] + \sigma^2 \cap \psi_*[Z_1, M].$$

The inclusion of (P, S^3) into $(\mathbb{CP}^2, D^4)$ induces isomomorphisms on (relative) cohomology, and so $\sigma^2 \cap \theta_*[P, S^3] = \theta^*\sigma^2 \cap [P, S^3] \neq 0$. Since the middle dimensional intersection pairing is trivial in each of (Z_1, M) and (Z_2, M), $t = 0$, and so $f_1 \sim f_2$. Hence there is a homotopy equivalence of pairs $(Y_1, M) \to (Y_2, M)$ which extends id_M. This is homotopic *rel* M to a homeomorphism G, by simply-connected surgery. The map $h = F \cup G$ is a homeomorphism of S^4 such that $hj_1 = j_2$. □

Example. *The manifold $M = M(11_{n42})$ has an essentially unique abelian embedding, although $M = M(K)$ for infinitely many distinct knots K.*

The knot 11_{n42} is the Kinoshita-Terasaka knot, which is the simplest non-trivial knot with Alexander polynomial 1. This bounds a smoothly embedded disc D in D^4, such that $\pi_1(D^4 \setminus D) \cong \mathbb{Z}$, obtained by desingularizing a ribbon disc. (See Figure 1.4 of [12].) Hence M has a smooth abelian embedding. Since 11_{n42} has unkotting number 1, it has an annulus presentation, and so there are infinitely many knots K_n such that $M(K_n) \cong M$ [1]. These knots must all have Alexander polynomial 1, and so each determines an abelian embedding. Are all of these embeddings smooth, and are they smoothly equivalent?

The connected sum of the homology 3-sphere represented by Figure 3 of [14] with $S^2 \times S^1$ has an embedding with both complementary regions having nontrivial fundamental group. Does every nontrivial homology handle have an embedding for which $\pi_Y \neq 1$?

A quite different extension of Aitchison's result follows from the next theorem.

Theorem 12. *If X fibres over S^1 then $\chi(X) = 0$, M is a mapping torus, the projection $p : M \to S^1$ extends to a map from X to S^1 and $\pi_1(j_X)$ is surjective.*

Conversely, if these conditions hold then M has an embedding $\widehat{j}$ such that $\widehat{X}$ fibres over S^1 and $(\widehat{X}, M)$ is s-cobordant rel M to (X, M).

Proof. If X fibres over S^1, with fibre F, then $M = \partial X$ is the mapping torus of a self-homeomorphism of ∂F and the projection $p : M \to S^1$ extends to a map from X to S^1. Moreover, $\chi(X) = 0$ and π_X is an extension of $\mathbb{Z}$ by the finitely presentable normal subgroup $\pi_1(F)$. Hence $\beta_1^{(2)}(\pi_X) = 0$, by Theorem 7.2.6 of [18], and so $c.d.\pi_X \leq 2$, by Lemma 7. Hence $\pi_1(F)$ is free, by Corollary 8.6 of [2], and so $F \cong \#^r(S^1 \times D^2)$, for some $r \geq 0$. Moreover, $\pi_1(j_X)$ is surjective.

If M is a mapping torus, the projection $p : M \to S^1$ extends to a map from X to S^1 and $\pi_1(j_X)$ is surjective then π_X is an extension of $\mathbb{Z}$ by a finitely presentable normal subgroup. Since $\chi(X) = 0$, the space X is aspherical, and so $\pi_X \cong F(r) \rtimes \mathbb{Z}$, for some $r \geq 0$. Let X^∞ be the covering space associated to the subgroup $F(r)$, and let j_{X^∞} be the inclusion of $M^\infty = \partial X^\infty$ into X^∞. Let τ be a generator of the covering group $\mathbb{Z}$. Fix a homotopy equivalence $h : X^\infty \to N = \#^r(S^1 \times D^2)$. Then there is a self-homeomorphism t_N of N such that $t_N h \sim h\tau$. Let $\theta : \partial N \to N$ be the inclusion, and let $\widehat{X} = M(t_N)$ be the mapping torus of t_N. Then there is a homotopy equivalence $\alpha : M^\infty \to \partial N$ such that $\theta\alpha \sim hj_{X^\infty}$, by a result of Stallings and Zieschang. (See Theorem 2 of [10].) We may modify h on a collar neighbourhood of ∂X^∞ so that $h|_{\partial X^\infty} = \alpha$. Hence h determines a homotopy equivalence of pairs $(X, M) \simeq (\widehat{X}, \partial\widehat{X})$. Since M and $\partial\widehat{X}$ are orientable (Haken) manifolds we may further arrange that $h|_M : M \to \partial\widehat{X}$ is a homeomorphism. Hence X and $\widehat{X}$ are s-cobordant *rel* ∂, since $L_5(F(r))$ acts trivially on the s-cobordism structure set $\mathcal{S}^s_{TOP}(\widehat{X}, \partial\widehat{X})$. (See Theorem 6.7 of [11].)

The union $\Sigma = \widehat{X} \cup_M Y$ is an homotopy 4-sphere, and so is homeomorphic to S^4. Then the final assertion is satisfied by the composite $\widehat{j} : M \subset \widehat{X} \subset \Sigma \cong S^4$. □

In particular, if $\beta = 1$ then $\chi(X) = 0$ and M is a rational homology handle.

Corollary 13. *Let K be a fibred 1-knot. Then $M = M(K)$ has a bi-epic embedding if and only if K is a homotopy ribbon knot.*

Proof. If j is a bi-epic embedding then $\pi_1(j_X)$ is surjective. The restriction from $H^1(X; \mathbb{Z})$ to $H^1(M; \mathbb{Z})$ is an isomorphism and $\chi(X) = 0$, since $\beta = 1$, and so the projection $p : M \to S^1$ extends to a map from X to S^1. Thus the hypotheses of the theorem are satisfied, and so M has an embedding $\widehat{j}$ such that $\widehat{X}$ fibres over S^1. Hence K is a homotopy ribbon knot [3].

If K is a fibred homotopy ribbon knot then the monodromy for the fibration extends over a handlebody [3]. Hence M bounds a mapping torus X such that the inclusion $M \subset X$ induces an epimorphism from π to $\pi_1(X)$ and an isomorphism on the abelianizations. Let Y be the 4-manifold obtained by adjoining a 2-handle to D^4 along K. Then $\Sigma = X \cup_M Y$ is a homotopy 4-sphere, and the inclusion of M into Σ is bi-epic. □

For example, if k is a fibred 1-knot with exterior $E(k)$ and genus g, then $K = k\#{-k}$ is a fibred ribbon knot, and $M(K)$ bounds a thickening X of $E(k) \subset S^3 \subset S^4$, which fibres over S^1, with fibre $\natural^g(S^1 \times D^2)$.

Any 1-knot K such that $M(K)$ embeds in S^4 must be algebraically slice, by Theorem 10. However, there are obstructions beyond neutrality of the Blanchfield pairing to slicing a knot, which probably also obstruct embeddings of homology handles.

5. $\pi/\pi' \cong \mathbb{Z}^2$

When $\pi/\pi' \cong \mathbb{Z}^2$ there is again a simple necessary condition for M to have an abelian embedding.

Lemma 14. *Let M be a 3-manifold with fundamental group π such that $\pi/\pi' \cong \mathbb{Z}^2$. If $j : M \to S^4$ is an abelian embedding then $X \simeq Y \simeq S^1 \vee S^2$, and $H_1(M;\mathbb{Z}[\pi_X])$ and $H_1(M;\mathbb{Z}[\pi_Y])$ are cyclic $\mathbb{Z}[\pi_X]$- and $\mathbb{Z}[\pi_Y]$-modules (respectively), of projective dimension ≤ 1.*

Proof. Since j is abelian $\pi_X \cong \pi_Y \cong \mathbb{Z}$ and $\chi(X) = \chi(Y) = 1$. Moreover, since j_{X*} and j_{Y*} are epimorphisms $c.d.X \leq 2$ and $c.d.Y \leq 2$, by Theorem 5.1 of [14]. Hence $X \simeq Y \simeq S^1 \vee S^2$, by Corollary 4.

As in Theorem 10 we consider the homology exact sequences of the infinite cyclic covers of the pairs (X, M) and (Y, M), in conjunction with equivariant Poincaré-Lefshetz duality. Since $H_i(X;\mathbb{Z}[\pi_X]) = 0$ for $i \neq 0$ or 2 and $H_2(X;\mathbb{Z}[\pi_X]) \cong \mathbb{Z}[\pi_X]$, we have $H_2(X, M;\mathbb{Z}[\pi_X]) \cong \overline{H^2(X;\mathbb{Z}[\pi_X])} \cong \mathbb{Z}[\pi_X]$ also. Hence there is an exact sequence

$$0 \to H_2(M;\mathbb{Z}[\pi_X]) \to \mathbb{Z}[\pi_X] \to \mathbb{Z}[\pi_X] \to H_1(M;\mathbb{Z}[\pi_X]) \to 0.$$

Therefore either $H_1(M;\mathbb{Z}[\pi_X]) \cong H_2(M;\mathbb{Z}[\pi_X]) \cong \mathbb{Z}[\pi_X]$ or $H_2(M;\mathbb{Z}[\pi_X])$ is a cyclic torsion module with a short free resolution, and $H_2(M;\mathbb{Z}[\pi_X]) = 0$. In either case $H_1(M;\mathbb{Z}[\pi_X])$ is a cyclic module of projective dimension ≤ 1.

A similar argument applies for the pair (Y, M). □

To use Lemma 14 to show that some M has no abelian embedding we must consider all possible bases for $Hom(\pi, \mathbb{Z})$, or, equivalently, for π/π'.

Example. *Let L be the link obtained from the Whitehead link $Wh = 5^2_1$ by tying a reef knot ($3_1\#{-3_1}$) in one component. Then no embedding of $M(L)$ is abelian.*

The link group πL has the presentation

$$\langle a, b, c, r, s, t, u, v, w \mid as^{-1}vsa^{-1} = w = brb^{-1},\ cac^{-1} = b,\ rcr^{-1} = a,\ wcw^{-1} = b,$$
$$rvr^{-1} = tut^{-1},\ sts^{-1} = u,\ usu^{-1} = t,\ vsv^{-1} = r\rangle,$$

and $\pi_1(M(L)) \cong \pi L/\langle\langle\lambda_a, \lambda_r\rangle\rangle$, where $\lambda_a = c^{-1}wr^{-1}a$ and $\lambda_r = vu^{-1}s^{-1}t^{-1}rsa^{-1}b$ are the longitudes of L. Let $b = \beta a$, $c = \gamma a$ and $t = r\tau$. Then $w = \gamma r$ in $\pi = \pi_1(M(L))$, and so π has the presentation

$$\langle a, \beta, \gamma, r, s, \tau, v \mid [r, a] = \gamma^{-1}\beta r\beta^{-1}r^{-1} = r\gamma^{-1}r^{-1},\ \gamma a\gamma^{-1}a^{-1} = \beta,\ sr\tau s = r\tau sr\tau,$$
$$as^{-1}vsa^{-1} = \gamma r,\ vs = rv,\ v = \tau sr\tau s^{-1}\tau^{-1} = \beta^{-1}s^{-1}\tau s^2r\tau s^{-1}\rangle.$$

Now let $s = \sigma r$ and $v = \xi r$. Then π/π'' has the metabelian presentation

$$\langle a, \beta, \gamma, r, \sigma, \tau, \xi \mid [r, a] = \gamma^{-1}\beta.r\beta^{-1}r^{-1} = r\gamma^{-1}r^{-1},\ \gamma.a\gamma^{-1}a^{-1} = \beta,$$
$$r^{-1}\sigma r.r\tau\sigma r^{-1} = \tau\sigma.r^2\tau r^{-2},\ ar^{-1}\sigma^{-1}\xi ra^{-1}.a\sigma a^{-1} = \gamma.r\gamma^{-1}r^{-1},\ \xi = r\xi\sigma^{-1}r^{-1},$$
$$\xi = \tau\sigma.r^2\tau r^{-2}.r\sigma^{-1}\tau^{-1}r^{-1} = \beta^{-1}.r^{-1}\tau r.\sigma.r^2\tau r^{-2}.r\sigma^{-1}r^{-1},\ [[\,,\,],[\,,\,]] = 1\rangle,$$

in which $\beta, \gamma, \sigma, \tau$ and ξ represent elements of π', which is the normal closure of the images of these generators. The first relation expresses the commutator $[r, a]$ as a product of conjugates of these generators. Using the third relation to eliminate β, we see that π'/π'' is generated as a module over $\mathbb{Z}[\pi/\pi'] = \mathbb{Z}[a^{\pm}, r^{\pm}]$ by the images of γ, σ, τ and ξ, with the relations

$$(1 - r)[\gamma] = 0,$$

$$(r^2 - r + 1)[\sigma] = r(r^2 - r + 1)[\tau] = 0,$$
$$[\xi] = (1 - r)[\sigma],$$

and

$$2[\sigma] + 2[\tau] = (a - 1)[\gamma].$$

If we extend coefficients to the rationals to simplify the analysis, we see that $P = H_1(M; \mathbb{Q}[\pi/\pi']) = \mathbb{Q} \otimes \pi'/\pi''$ is generated by $[\gamma]$ and $[\tau]$, with the relations

$$(1 - r)[\gamma] = (r^2 - r + 1)[\tau] = 0.$$

Let $\{x, y\}$ be a basis for π/π'. Then $x = a^m r^n$ and $y = a^p r^q$, where $|mq - np| = 1$. Let $\{x^*, y^*\}$ be the Kronecker dual basis for $Hom(\pi, \mathbb{Z})$, and let M_x and M_y be the infinite cyclic covering spaces corresponding to $\mathrm{Ker}(x^*)$ and $\mathrm{Ker}(y^*)$, respectively. Then $H_1(M_x; \mathbb{Q}) \cong (P/(y - 1)P \oplus \langle y \rangle)/(x.y = y + [x, y])$. If this module is cyclic as a module over the PID $\mathbb{Q}[x, x^{-1}]$ then so is the submodule

$$P/(y - 1)P \cong \mathbb{Q}[\pi/\pi']/(r^2 - r + 1, y - 1) \oplus \mathbb{Q}[\pi/\pi']/(r - 1, y - 1).$$

On substituting $y = a^p r^q$ we find that this is so if and only if $p = 0$ and $q = \pm 1$. But then $x = a^{\pm 1}$, and a similar calculation show that $H_1(M_y; \mathbb{Q})$ is not cyclic as a $\mathbb{Q}[y, y^{-1}]$-module. Thus no basis for π/π' satisfies the criterion of Lemma 14, and M has no abelian embedding.

We shall assume henceforth that $M = M(L)$, where L is a 2-component link with components slice knots and linking number $\ell = 0$. Let x and y be the images of the meridians of L in π, and let D_x and D_y be slice discs for the components of L, embedded on opposite sides of the equator $S^3 \subset S^4$. Then the complementary regions for the embedding j_L determined by L are $X_L = (D^4 \setminus N(D_x)) \cup D_y \times D^2$ and $Y_L = (D^4 \setminus N(D_y)) \cup D_x \times D^2$. The kernels of the natural homomorphisms from π to π_{X_L} and π_{Y_L} are the normal closures of y and x, respectively. If one of the components of L is unknotted then the corresponding complementary region is a handlebody of the form $S^1 \times D^3 \cup h^2$. Inverting the handle structure gives a handlebody structure $M \times [0, 1] \cup h^2 \cup h^3 \cup h^4$.

If the components of L are unknotted then j_L is abelian, and $\pi_X \cong \pi_Y \cong \mathbb{Z}$.

If L is interchangeable there is a self-homeomorphism of $M(L)$ which swaps the meridians. Hence X_L is homeomorphic to Y_L, and S^4 is a twisted double.

The next result has fairly strong hypotheses, but we shall give an example after the theorem showing that some such hypotheses are necessary.

Theorem 15. *Let M be a 3-manifold with fundamental group π such that $\pi/\pi' \cong \mathbb{Z}^2$, and suppose that j_1 and j_2 are abelian embeddings of M in S^4. If $(X_1, M) \simeq (X_2, M)$ and $(Y_1, M) \simeq (Y_2, M)$ then j_1 and j_2 are equivalent.*

Proof. As in Theorem 11, since $L_5(\mathbb{Z})$ acts trivially on the structure sets there are homeomorphisms $F : X_1 \to X_2$ and $G : Y_1 \to Y_2$ which agree on M. The union $h = F \cup G$ is a homeomorphism such that $hj_1 = j_2$. □

To find examples where the complementary regions are *not* homeomorphic we should start with a link L which is not interchangeable. The simplest condition that ensures that a link with unknotted components is not interchangeable is asymmetry of the Alexander polynomial, and the smallest such link with linking number 0 is 8^2_{13}. Since $\pi = \pi_1(M)$ is a quotient of πL, there remains something to be checked.

Example. *The complementary regions of the embedding of $M(8^2_{13})$ determined by the link $L = 8^2_{13}$ are not homeomorphic (although they are homotopy equivalent).*

Let $M = M(L)$. The link group $\pi L = \pi 8^2_{13}$ has the presentation

$$\langle s,t,u,v,w,x,y,z \mid yv = wy,\ zx = wz,\ ty = zt,\ uy = zu,\ sv = us,\ vs = xv,$$
$$wu = tw,\ xs = tx\rangle$$

and the longitudes are $u^{-1}t$ and $x^2z^{-1}ys^{-1}w^{-1}xv^{-1}$. Hence $\pi = \pi_1(M)$ has the presentation

$$\langle s,t,v,w,x,y \mid yv = wy,\ tyt^{-1}x = wtyt^{-1},\ x^2ty^{-1}t^{-1}ys^{-1}w^{-1}xv^{-1} = 1,$$
$$sv = ts,\ vs = xv,\ wt = tw,\ xs = tx\rangle.$$

Setting $s = x\alpha$, $t = x\beta$, $v = x\gamma$ and $w = x\delta$, we obtain the presentation

$$\langle \alpha,\beta,\gamma,\delta,x,y \mid [x,y] = xy\gamma(xy)^{-1}.x\delta x^{-1},\ \beta.y\beta y^{-1} = \delta.x\beta x^{-1}.xy\beta^{-1}(xy)^{-1}.[x,y],$$
$$x^2\beta x^{-2}.x^2y^{-1}\beta^{-1}yx^{-2} = \gamma\delta.x\alpha x^{-1}.xy^{-1}[x,y]^{-1}yx^{-1}$$
$$\delta x\beta = \beta x\delta,\ \alpha x\gamma = \beta x\alpha,\ \gamma x\alpha = x\gamma,\ x\alpha = \beta x\rangle$$

in which α,β,γ and δ represent elements of π', which is the normal closure of the images of these generators. The subquotient π'/π'' is generated as a module over $\mathbb{Z}[\pi/\pi'] \cong \Lambda_2 = \mathbb{Z}[x^{\pm}, y^{\pm}]$. by the images of γ, and δ, with the relations

$$(x+1)(y-1)(x-1)[\gamma] = xy[\gamma] - x[\delta],$$
$$(x-1)^2[\gamma] = (x-1)[\delta],$$

and

$$(x^2 - x + 1)[\gamma] = 0,$$

since $[\alpha] = x^{-1}(x-1)[\gamma]$ and $[\beta] = (x-1)[\gamma]$. Adding the first two relations and rearranging gives

$$[\delta] = -((x^2 - x + 1)y + 2 - 2x)[\gamma] = 2(x-1)[\gamma].$$

Hence $\pi'/\pi'' \cong \Lambda_2/(x^2 - x + 1, 3(x-1)^2) = \Lambda_2/(x^2 - x + 1, 3)$. As a module over the subring $\mathbb{Z}[x, x^{-1}]$, this is infinitely generated, but as a module over $\mathbb{Z}[y, y^{-1}]$ it has two generators. Therefore there is no automorphism of π which induces an isomorphism $\mathrm{Ker}(\pi_1(j_X)) = \pi' \rtimes \langle x\rangle \cong \mathrm{Ker}(\pi_1(j_Y)) = \pi' \rtimes \langle y\rangle$. Hence (X, M) and (Y, M) are not homotopy equivalent as pairs, although $X \simeq Y$.

Does M have any other abelian embeddings with neither complementary component homeomorphic to X, perhaps corresponding to distinct link presentations? Is this 3-manifold homeomorphic to a 3-manifold $M(\tilde{L})$ via a homeomorphism which does not preserve the meridians?

There is just one 3-manifold with π elementary amenable and $\beta = 2$ which embeds in S^4 [4]. This is the $\mathbb{N}il^3$-manifold $M = M(1;(1,1))$, and $\pi = \pi_1(M)$ is the free nilpotent group of class 2 on 2-generators: $\pi \cong F(2)/F(2)_{[3]}$. This manifold may be obtained by 0-framed surgery on the Whitehead link $Wh = 5^2_1$, and the corresponding embedding is abelian.

All epimorphisms from $F(2)/F(2)_{[3]}$ to $\mathbb{Z}$ are equivalent under composition with automorphisms, and each automorphism of $F(2)/F(2)_{[3]}$ is induced by a self-diffeomorphism of M. If j is an abelian embedding such that (X, M) and (Y, M) are homotopy equivalent (*rel* M) to (X_{Wh}, M), then j is equivalent to j_{Wh}, and the two complementary regions are homeomorphic. However, since X and Y are not aspherical, Lemma 6 does not apply to provide a homotopy equivalence of pairs. Is j_{Wh} essentially unique?

6. THE HIGHER RANK CASES

Theorem 10 and Lemma 14 have analogues when $\beta = 3$ (and, in part, when $\beta = 4$ or 6).

Lemma 16. *Let M be a 3-manifold with fundamental group π such that $\pi/\pi' \cong \mathbb{Z}^3$. If $j : M \to S^4$ is an abelian embedding then $X \simeq T$ and $Y \simeq S^1 \vee 2S^2$, while $H_1(M; \mathbb{Z}[\pi_X]) \cong \mathbb{Z}$ and $H_1(M; \mathbb{Z}[\pi_Y])$ is a torsion $\mathbb{Z}[\pi_Y]$-module of projective dimension 1 and which can be generated by two elements, or is 0. The component X is determined up to homeomorphism by its boundary M.*

Proof. The classifying map $c_X : X \to K(\pi_X, 1) \simeq T$ is a homotopy equivalence, by Theorem 3, since $c.d.X = c.d.T = 2$ and $\chi(X) = \chi(T) = 0$. The equivalence $Y \simeq S^1 \vee 2S^2$ follows from Corollary 4, since $\pi_Y \cong \mathbb{Z}$ and $\chi(Y) = 2$.

Since $H_2(X; \mathbb{Z}[\pi_X]) = 0$, the exact sequence of homology for the pair (X, M) with coefficients $\mathbb{Z}[\pi_X]$ reduces to an isomorphism $H_1(M; \mathbb{Z}[\pi_X]) \cong \overline{H^2(X; \mathbb{Z}[\pi_X])} \cong \mathbb{Z}$.

Similarly, there is an exact sequence

$$0 \to \mathbb{Z} \to H_2(M; \mathbb{Z}[\pi_Y]) \to \mathbb{Z}[\pi_Y]^2 \to \mathbb{Z}[\pi_Y]^2 \to H_1(M; \mathbb{Z}[\pi_Y]) \to 0,$$

since $H_2(Y; \mathbb{Z}[\pi_Y]) \cong \mathbb{Z}[\pi_Y]^2$ and $H_2(Y, M; \mathbb{Z}[\pi_Y]) \cong \overline{H^2(Y; \mathbb{Z}[\pi_Y])}$. Let $A = \pi'/\pi''$, considered as a $\mathbb{Z}[\pi/\pi']$-module. Then A is finitely generated as a module, since $\mathbb{Z}[\pi/\pi']$ is a noetherian ring. Let $\{x, y, z\}$ be a basis for π/π' such that $j_{X*}(y) = 0$ and $j_{Y*}(x) = j_{Y*}(z) = 0$. Then $H_1(M; \mathbb{Z}[\pi_X]) \cong (A/(y-1)A) \oplus \mathbb{Z} \cong \mathbb{Z}$, so $A = (y-1)A$. Similarly, $H_1(M; \mathbb{Z}[\pi_Y])$ is an extension of $\mathbb{Z}^2$ by $A/(x-1, z-1)A$. Together these observations imply that $H_1(M; \mathbb{Z}[\pi_Y])$ is a torsion $\mathbb{Z}[\pi_Y]$-module, and so the fourth homomorphism in the above sequence is a monomorphism. Thus $H_1(M; \mathbb{Z}[\pi_Y])$ is a torsion $\mathbb{Z}[\pi_Y]$-module with projective dimension ≤ 1, and is clearly generated by two elements. (Note also that a torsion $\mathbb{Z}[\pi_Y]$-module of projective dimension 0 is 0.)

The final assertion holds since Lemma 6 applies, and $L_5(\mathbb{Z}^2)$ acts trivially on the structure set $\mathcal{S}^{TOP}(X, \partial X)$, by Theorem 6.7 of [11]. $\square$

The link $L = 9^3_{21}$ has an unique partition as a bipartedly slice link, and for the corresponding embedding $\pi_{X_L} \cong F(2)$ (the free group of rank 2) and $\pi_{Y_L} \cong \mathbb{Z}$. Then $M = M(9^3_{21}) \cong (S^2 \times S^1) \# M(5^2_1)$, so $\pi \cong \mathbb{Z} * F(2)/F(2)_{[3]}$, with presentation $\langle x, y, z \mid [x, y] \leftrightharpoons x, y \rangle$. It is not hard to show that the kernel of any epimorphism $\phi : \pi \to \langle t \rangle \cong \mathbb{Z}$ has rank ≥ 1 as a $\mathbb{Z}[t, t^{-1}]$-module. Hence M has no abelian embedding, by Lemma 16.

The 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ has an abelian embedding, as the boundary of a regular neighbourhood of an unknotted embedding of T in S^4. This manifold may be obtained by 0-framed surgery on the Borromean rings $Bo = 6^3_2$, and also on 9^3_{18}. The three bipartite partitions of Bo lead to equivalent embeddings. (However these are clearly not isotopic!) The link 9^3_{18} has two bipartedly slice partitions (both bipartedly trivial). Any such embedding of T^3 has $X \cong T \times D^2$ and $Y \simeq S^1 \vee 2S^2$. Does T^3 have an essentially unique abelian embedding?

Suppose that $\beta = 3$ and M has an embedding j such that $H_1(Y; \mathbb{Z}) = 0$. If $f : \pi \to \mathbb{Z}^2$ is an epimorphism with kernel κ and $R = \mathbb{Z}[\pi/\kappa]_{f(S)}$ then $H_1(M; R)$ has rank 2, by Lemma 9, and so the condition of Lemma 16 does not hold. Therefore no such 3-manifold can also have an abelian embedding.

The argument of Lemma 14 extends almost verbatim to show that if $\pi \cong \mathbb{Z}^4$ and j is an abelian embedding then $X \simeq Y \simeq T \vee S^2$. (Generators for $\pi_X \cong \mathbb{Z}^2$

and $\pi_2(X) \cong \mathbb{Z}[\pi_X]$ determine a map from $T^{[1]} \vee S^2$ to X. This extends to a 2-connected map from $T \vee S^2$ to X, which is a homotopy equivalence by Theorem 3.) Lemmas 9 and Lemma 16 again imply that when $\beta = 4$ no 3-manifold which has an embedding j such that $H_1(Y;\mathbb{Z}) = 0$ can also have an abelian embedding. However, if L is the 4-component link obtained from Bo by adjoining a parallel of one component, then $M(L)$ has an abelian embedding with $X \cong Y$ and $\chi(X) = 1$, and also has an embedding with $\chi(X) = -1$.

The first part of the argument of Lemma 14 can also be applied when $\beta = 6$, to show that if j is an abelian embedding then $\pi_2(X)$ is again isomorphic to $\mathbb{Z}[\pi_X]$. In this case there is a map g from the 2-skeleton $T^{3[2]}$ to X such that $\pi_1(g)$ is an isomorphism. If g is 2-connected then it is a homotopy equivalence, by Theorem 3. Lemmas 9 and Lemma 16 again imply that such an M has no embedding with $H_1(Y;\mathbb{Z}) = 0$. We shall not give more details, as there are no natural examples demanding attention in these cases.

7. 2-component links with $\ell \neq 0$

If M is a rational homology sphere with an abelian embedding then $\pi/\pi' \cong (\mathbb{Z}/\ell\mathbb{Z})^2$ and $\pi_X \cong \pi_Y \cong \mathbb{Z}/\ell\mathbb{Z}$, for some $\ell \neq 0$. In particular, if L is a 2-component link with linking number $\ell \neq 0$ then $M(L)$ is a rational homology sphere, and if the components of L are unknotted then j_L is abelian. Six of the eight rational homology 3-spheres with elementary amenable groups and which embed in S^4 have such link presentations, with $\ell \leq 4$. (In particular, $S^3 = M(Ho)$, where $Ho = 2^2_1$ is the Hopf link!) Since M is an integral homology 3-sphere if $\ell = 1$, we may assume that $\ell > 1$.

There is again a necessary condition for the existence of such an embedding.

Lemma 17. *Let M be a 3-manifold with fundamental group π such that $\pi/\pi' \cong (\mathbb{Z}/\ell\mathbb{Z})^2$, for some $\ell \neq 0$. If $j : M \to S^4$ is an abelian embedding then $X \simeq Y \simeq P_\ell$, and $H_1(M;\mathbb{Z}[\pi_X])$ and $H_1(M;\mathbb{Z}[\pi_Y])$ are quotients of $\mathbb{Z}^{\ell-1}$, as abelian groups.*

Proof. The first assertion holds by Lemma 8. The second part then follows from the exact sequences of homology for the universal covering spaces of the pairs (X, M) and (Y, M), since $\widetilde{X} \simeq \widetilde{Y} \simeq \vee^{\ell-1} S^2$. $\square$

The first case to consider is $\ell = 2$. In this case $\pi_X = \pi_Y = \mathbb{Z}/2\mathbb{Z}$ and $X \simeq Y \simeq RP^2$. The quaternion manifold $M = S^3/Q(8)$ may be obtained by 0-framed surgery on the $(2,4)$-torus link 4^2_1. This manifold has an essentially unique abelian embedding, with X and Y homeomorphic to the total space N of the disc bundle over RP^2 with Euler number 2 [16]. Does Lawson's argument of Lawson for constructing an exotic self homotopy equivalence extend to all X with $\pi_X = Z/2Z$?

The links 9^2_{38}, 9^2_{57} and 9^2_{58} each have unknotted components, asymmetric Alexander polynomial and linking number 2. They are candidates for examples with X and Y not homeomorphic.

Let L be the link obtained from the $(2,4)$-torus link by tying a slice knot with non-trivial Alexander polynomial (such as the stevedore's knot 6_1) in one component. Then $M(L)$ embeds in S^4, but does not satisfy Lemma 17, and so has no abelian embedding.

The other two cases of most interest are when $\ell = 3$ or 4. Let $L = 6^2_1$. Then $M(L) = M(0; (3,1), (3,1), (3,-1))$, which is a $\mathbb{N}il^3$-manifold with Seifert base the flat orbifold $S(3,3,3)$. This link is interchangeable, and so $X_L \cong Y_L$. If j is an abelian embedding for M then $X \simeq Y \simeq P_3$. When ℓ is odd $L_5^s(\mathbb{Z}/\ell\mathbb{Z}) = 0$, and $Wh(\mathbb{Z}/3\mathbb{Z}) = 0$, so the homeomorphism type of the pair (X, M) is determined by its homotopy type *rel* ∂.

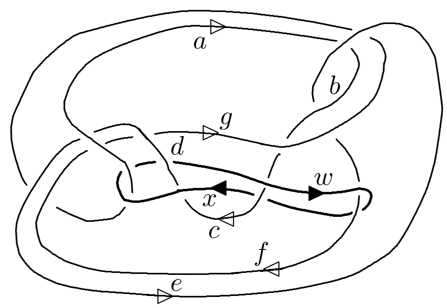
The simplest link with $\ell = 4$ is $L = 8_2^2$. In this case $M(L)$ is the $\mathbb{N}il^3$-manifold $M(-1; (2,1), (2,3))$. This link is interchangeable, and so $X_L \cong Y_L$. Each of the links 9_{53}^2 and 9_{61}^2 gives a $\mathbb{S}ol^3$-manifold with an abelian embedding. Are these links interchangeable?

There remains one more $\mathbb{S}ol^3$-manifold which embeds in S^4. This manifold arises from surgery on the link $L = (U, 8_{20})$ of the Figure below, with components the unknot U and the slice knot 8_{20}, and with $\ell = 4$. (This is the link of Figure 1 of [4], but the diagram has been changed so that 8_{20} is visibly a ribbon knot. Warning! The diagram for 8_{20} on the right in Figure 1 of [4] is *not* in obvious ribbon form!)

The Wirtinger presentation associated to this diagram gives rise to the following presentation for $\pi = \pi_1(M(U, 8_{20}))$:

$$\langle a, b, c, d, e, f, g; w, x \mid be = ea,\ adg = fad,\ bgw = gwc,\ beb^{-1}gw = gwf,\ ea = ad,$$
$$cx = xd,\ adw = xad,\ gfx = fxa,\ e^{-1}gwxa^{-1}b^{-1}gwadfxa^{-2} = 1,\ adfc = 1\rangle.$$

Here the last two relators correspond to the longitudes which commute with the meridians a and x.



$L = (U, 8_{20})$

The knot 8_{20} bounds a slice disc $D \subset D^4$ obtained by desingularizing the ribbon disc of the Figure. The associated "ribbon group" $\pi_1(D^4 \setminus D)$ has a presentation obtained by adjoining the relations $b^{-1}g = 1$ and $e^{-1}f = 1$ to the Wirtinger presentation for $\pi 8_{20}$. (See Theorem 1.15 of [12].) Let X_L be the region obtained from D^4 by deleting a regular neighbourhood of this slice disc D and adding a 2-handle along the unknotted component U, and let Y_L be the complementary region. Then π_{X_L} has a presentation $\langle a, b, e \mid be = ea = ab,\ ababa^{-1}b = 1\rangle$, since $b = c = d = g$ and $e = f = aba^{-1}$ in $\pi_1(D^4 \setminus D)$. (Here $ababa^{-1}b$ is the image of the longitude for U.) This simplifies to a presentation $\langle a, b \mid aba^{-1} = b^{-1}ab,\ a^2b^2 = 1\rangle$ for the non-abelian group $\mathbb{Z}/3\mathbb{Z} \rtimes_{-1} \mathbb{Z}/4\mathbb{Z}$. On the other hand, $\pi_{Y_L} \cong \mathbb{Z}/4\mathbb{Z}$.

Every subgroup of finite index in π is the group of a $\mathbb{S}ol^3$-manifold, and so can be generated by at most 3 elements. Therefore Lemma 17 is of no avail in deciding whether $M(U, 8_{20})$ has an abelian embedding. Is there a 2-component link with unknotted components which gives this 3-manifold?

8. Some remarks on the mixed cases

If M is a 3-manifold with an abelian embedding such that $\pi_X \cong G_k = \mathbb{Z}\oplus(\mathbb{Z}/k\mathbb{Z})$, for some $k > 1$, then $\pi_Y \cong G_k$ also, and so $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^2\oplus(\mathbb{Z}/k\mathbb{Z})^2$, which requires four generators. The simplest examples may be constructed from 4-component links obtained by replacing one component of the Borromean rings Bo by its $(2k, 2)$ cable.

In this case even the determination of the homotopy types of the complements is not clear. The group G_k has minimal presentations

$$\mathcal{P}_{k,n} = \langle a, t \mid a^k,\ ta^n = a^n t\rangle,$$

where $0 < n < k$ and $(n, k) = 1$. The associated 2-complexes $S_{k,n} = S^1 \vee P_k \cup_{[t,a^n]} e^2$ have Euler characteristic 1, and it is easy to see that there are maps between them which induce isomorphisms on fundamental groups. We may identify $S_{k,n}$ with $T \cup MC \cup P_k$, where MC is the mapping cylinder of the degree-n map $z \mapsto z^n$ from $\{1\} \times S^1 \subset T$ to the 1-skeleton $S^1 \subset P_k$. In particular, $S_k = S_{k,1} = T \cup_{a^k} e^2$ is the 2-skeleton of $S^1 \times P_k$. From these descriptions it is easy to see that (1) automorphisms of G_k which fix the torsion subgroup $A = \langle a\rangle$ may be realized by self homeomorphisms of $S_{k,n}$ which act by reflections and Dehn twists on T, and fix the second 2-cell; and (2) the automorphism which fixes t and inverts a is induced by an involution of $S_{k,n}$.

Let $C(k, n)_*$ be the cellular chain complex of the universal cover of $S_{k,n}$. A choice of basepoint for $S_{n,k}$ determines lifts of the cells of $S_{k,n}$, and hence isomorphisms $C(k, n)_0 \cong \Gamma$, $C(k, n)_1 \cong \Gamma^2$ and $C(k, n)_2 \cong \Gamma^2$. The differentials are given by $\partial_1 = (a - 1, t - 1)$ and $\partial_2^n = \begin{pmatrix} (t-1)\nu_n & \rho \\ 1-a & 0 \end{pmatrix}$, where $\nu_n = \Sigma_{0\le i<n} a^i$ and $\rho = \nu_k$. Let $\{e_1, e_2\}$ be the standard basis for $C(k, n)_2$. Then $\Pi_{k,n} = \pi_2(S_{k,n}) = \mathrm{Ker}(\partial_2^n)$ is generated by $g = \rho e_1 - n(t-1)e_2$ and $h = (a-1)e_2$, with relations $(a-1)g = n(t-1)h$ and $\rho h = 0$. It can be shown that $\Pi_{k,n} \cong \alpha^*\Pi_{k,m}$, where α is the automorphism of G_k such that $\alpha(t) = t$ and $\alpha(a) = a^r$, where $n \equiv rm \ mod\ k$. Is there a chain homotopy equivalence $C(k, n)_* \simeq \alpha^* C(k, m)_*$?

Is every finite 2-complex S with $\pi_1(S) \cong G_k$ and $\chi(S) = 1$ homotopy equivalent to $S_{k,n}$, for some n? The key invariants are the Γ-module $\pi_2(S)$ and the k-invariant in $H^3(G_k; \pi_2(S))$. Let $S_{\langle t\rangle}$ be the finite covering space with fundamental group $\langle t\rangle \cong \mathbb{Z}$. If M is a finitely generated submodule of a free Γ-module then $H^i(\langle t\rangle; M) = 0$ for $i \neq 1$, while $H^1(\langle t\rangle; M) = M_t = M/(t - 1)M$. Hence the spectral sequence

$$H^p(A; (H^q(\langle t\rangle; M)) \Rightarrow H^{p+q}(G_k; M)$$

collapses, to give $H^{p+1}(G_k; M) \cong H^p(A; M_t)$. If $M = \pi_2(S)$ then $M_t \cong H_2(S_{\langle t\rangle}; \mathbb{Z})$, as a $\mathbb{Z}[A]$-module. When $M = \Pi_{k,n}$ it is easy to see that $M_t \cong \mathbb{Z} \oplus I_A$, where I_A is the augmentation ideal of $\mathbb{Z}[A]$, and so $H^2(A; M_t) \cong H^2(A; \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$.

Let V and W be finite 2-complexes with $\pi_1(V) \cong \pi_1(W) \cong G_k$, and let $\Gamma = \mathbb{Z}[G_k]$. Then $\chi(V) \ge 1$ and $\chi(W) \ge 1$, and an application of Schanuel's Lemma to the chain complexes of the universal covers gives

$$\pi_2(V) \oplus \Gamma^{\chi(W)} \cong \pi_2(W) \oplus \Gamma^{\chi(V)}.$$

Taking $W = S_{k,1}$, we see that $H^3(G; \pi_2(V)) \cong \mathbb{Z}/k\mathbb{Z}$, for all such V.

Even if we can determine the homotopy types of the 2-complexes S with $\pi_1(S)$ and $\chi(S) = 1$, and the homotopy types of the pairs (X, M) for a given M, the surgery obstruction groups $L_5^s(G)$ are commensurable with $L_4(\mathbb{Z}/k\mathbb{Z})$, which has rank $\lfloor \frac{k+1}{2} \rfloor$, and so characterizing such abelian embeddings up to isotopy may be difficult.

The S^1-bundle spaces $M(-2;(1,0))$ (the half-turn flat 3-manifold G_2), and $M(-2;(1,4))$ (a $\mathbb{N}il^3$-manifold) do not have abelian embeddings, since $\beta = 1$ but π/π' has nontrivial torsion. Moreover, they cannot be obtained by surgery on a 2-component link, since in each case π requires 3 generators. However, they may be obtained by 0-framed surgery on the links 8^3_9 and 9^3_{19}, respectively. For the embeddings defined by these links $X \simeq Kb$ and $\pi_Y = \mathbb{Z}/2\mathbb{Z}$. As in Lemma 16, X is homeomorphic to the corresponding disc bundle space, since Lemma 6 applies, and $L_5(\mathbb{Z} \rtimes_{-1} \mathbb{Z})$ acts trivially on the structure set $\mathcal{S}_{TOP}(X, \partial X)$, by Theorem 6.7 of [11]. (See also Theorem 12 above.) As discussed in §1, Y is homotopy equivalent to a finite 2-complex, and hence $Y \simeq RP^2 \vee S^2$. Are the corresponding embeddings of Kb unknotted?

It is easy to find 3-component bipartedly trivial links L such that X_L is aspherical and π_{X_L} is a solvable Baumslag-Solitar group $\mathbb{Z}*_m$. Lemma 6 and surgery arguments again apply to show that X_L is determined up to homeomorphism by M. In this case Y is homotopy equivalent to a finite 2-complex, by Lemma 8, since $\pi_1(Y) \cong \mathbb{Z}/(m-1)\mathbb{Z}$, $\chi(Y) = 2$ and $c.d.Y \leq 2$. Hence $Y \simeq P_{m-1} \vee S^2$ [6].

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