Type A-admissible cells are Kazhdan–Lusztig

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ABSTRACT. Admissible W-graphs were defined and combinatorially characterised by Stembridge in [12]. The theory of admissible W-graphs was motivated by the need to construct W-graphs for Kazhdan–Lusztig cells, which play an important role in the representation theory of Hecke algebras, without computing Kazhdan–Lusztig polynomials. In this paper, we shall show that type A-admissible W-cells are Kazhdan–Lusztig as conjectured by Stembridge in his original paper.

1. INTRODUCTION

Let (W,S) be a Coxeter system and $\mathcal{H}(W)$ its Hecke algebra over $\mathbb{Z}[q,q^{-1}]$, the ring of Laurent polynomials in the indeterminate q. We are interested in representations of W and $\mathcal{H}(W)$ that can be described by combinatorial objects, namely W-graphs. In particular, we are interested in W-graphs corresponding to Kazhdan–Lusztig left cells.

In principle, when computing left cells one encounters the problem of having to compute a large number of Kazhdan–Lusztig polynomials before any explicit description of their W-graphs can be given. In [12], Stembridge introduced *admissible* W-graphs; these can be described combinatorially and can be constructed without calculating Kazhdan–Lusztig polynomials. Moreover, the W-graphs corresponding to Kazhdan–Lusztig left cells are admissible. Stembridge showed in [13] that for any given finite W there are only finitely many stongly connected admissible W-graphs. It was conjectured by Stembridge that in type A all strongly connected admissible W-graphs are isomorphic to Kazhdan–Lusztig left cells. In this paper we complete the proof of Stembridge's conjecture.

We shall work with *S-coloured graphs* (as defined in Section 3 below), of which W-graphs are examples. These graphs have both edges (bi-directional) and arcs (uni-directional). A *cell* in such a graph Γ is by definition a strongly connected component of Γ, and a *simple part* of Γ is a connected component of the graph obtained from Γ by removing all arcs and all edges of weight greater than 1. A *simple component* of Γ is the full subgraph of Γ spanned by a simple part. If Γ is an admissible W-graph, simple components of Γ are also called *molecules*.

Admissible W-cells and admissible simple components are by definition cells and simple components of admissible W-graphs.

In [4], Chmutov established the first step towards the the proof of Stembridge's conjecture, showing that the simple part of an admissible molecule of type A_{n-1} is isomorphic to the simple part of a Kazhdan–Lusztig left cell. The proof made use of the axiomatisation of dual

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equivalence graphs on standard tableaux generated by dual Knuth equivalence relations, given in an earlier paper by Assaf [1]. Our proof makes use of Chmutov's result.

We organize the paper in the following sections. Section 2 and Section 3 deal with the background on Coxeter groups and the corresponding Hecke algebras. In Section 4 the definition and properties of W-graphs are recalled. In Section 5, we recall the definitions of admissible W-graphs and molecules and how these can be characterised combinatorially. Section 6 presents combinatorics of tableaux and the relationship between Kazhdan–Lusztig left cells, dual Knuth equivalence classes and admissible molecules. We introduce the paired dual Knuth equivalence relation in Section 7. In Section 8, we prove the first main result, namely that admissible W-graphs in type A_{n-1} are ordered. The proof that type A-admissible cells are isomorphic to Kazhdan–Lusztig left cells is completed in Section 9.

2. Coxeter groups

Let (W,S) be a Coxeter system and l the length function on W. The Coxeter group W comes equipped with the left weak order, the right weak order and the Bruhat order, respectively denoted by $\leqslant_{\mathsf{L}}$, $\leqslant_{\mathsf{R}}$ and $\leqslant$, and defined as follows.

Definition 2.1. (*i*) The left weak order is the partial order generated by the relations $x \leqslant_{\mathsf{L}} y$ for all $x, y \in W$ with $l(x) < l(y)$ and $yx^{-1} \in S$.

(*ii*) The right weak order is the partial order generated by the relations $x \leqslant_{\mathsf{R}} y$ for all $x y \in W$ with $l(x) < l(y)$ and $x^{-1}y \in S$.

(*iii*) The Bruhat order is the partial order generated by the relations $x \leqslant y$ for all $x, y \in W$ with $l(x) < l(y)$ and yx^{-1} conjugate to an element of S.

Observe that the weak orders are characterized by the property that $x \leqslant_{\mathsf{R}} xy$ and $y \leqslant_{\mathsf{L}} xy$ whenever $l(xy) = l(x) + l(y)$.

For each $J \subseteq S$ let W_J be the (standard parabolic) subgroup of W generated by J, and let D_J the set of distinguished (or minimal) representatives of the left cosets of W_J in W. Thus each $w \in W$ has a unique factorization $w = du$ with $d \in D_J$ and $u \in W_J$, and $l(du) = l(d) + l(u)$ holds for all $d \in D_J$ and $u \in W_J$. It is easily seen that D_J is an ideal of $(W, \leqslant_{\mathsf{L}})$, in the sense that if $w \in D_J$ and $v \in W$ with $v \leqslant_{\mathsf{L}} w$ then $v \in D_J$.

If W_J is finite then we denote the longest element of W_J by w_J. By [7, Lemma 2.2.1], if W is finite then $D_J = \{d \in W \mid d \leqslant_{\mathsf{L}} d_J\}$, where d_J is the unique element in $D_J \cap w_S W_J$.

3. Hecke algebras

Let $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q, and let $\mathcal{A}^+ = \mathbb{Z}[q]$. The Hecke algebra of a Coxeter system (W,S), denoted by $\mathcal{H}(W)$ or simply by $\mathcal{H}$, is an associative $\mathcal{A}$-algebra with $\mathcal{A}$-basis $\{ H_w \mid w \in W \}$ satisfying

$$H_s^2 = 1 + (q - q^{-1})H_s \quad \text{for all } s \in S,$$
$$H_{xy} = H_x H_y \quad \text{for all } x, y \in W \text{ with } l(xy) = l(x) + l(y).$$

We let $a \mapsto \overline{a}$ be the involutory automorphism of $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$ defined by $\overline{q} = q^{-1}$. It is well known that this extends to an involutory automorphism of $\mathcal{H}$ satisfying

$$\overline{H_s} = H_s^{-1} = H_s - (q - q^{-1}) \quad \text{for all } s \in S.$$

If $J \subseteq S$ then $\mathcal{H}(W_J)$, the Hecke algebra associated with the Coxeter system (W_J, J), is isomorphic to the subalgebra of $\mathcal{H}(W)$ generated by $\{ H_s \mid s \in J \}$. We shall identify $\mathcal{H}(W_J)$ with this subalgebra.

4. W-GRAPHS

Given a set S, we define an *S-coloured graph* to be a triple $\Gamma = (V, \mu, \tau)$ consisting of a set V, a function $\mu\colon V \times V \to \mathbb{Z}$ and a function τ from V to $\mathcal{P}(S)$, the power set of S. The elements of V are the *vertices* of Γ, and if $v \in V$ then $\tau(v)$ is the *colour* of the vertex. To interpret Γ as a (directed) graph, we adopt the convention that if $v, u \in V$ then (v,u) is an arc of Γ if and only if $\mu(u,v) \neq 0$ and $\tau(u) \nsubseteq \tau(v)$, and $\{v,u\}$ is an edge of Γ if and only if (v,u) and (u,v) are both arcs. We call $\mu(u,v)$ the *weight* of the arc (v,u). An edge $\{u,v\}$ is said to be *symmetric* if $\mu(u,v) = \mu(v,u)$, and *simple* if $\mu(u,v) = \mu(v,u) = 1$.

If (W,S) is a Coxeter system, then a *W-graph* is an S-coloured graph $\Gamma = (V, \mu, \tau)$ such that the free $\mathcal{A}$-module with basis V admits an $\mathcal{H}$-module structure satisfying

$$H_s v = \begin{cases} -q^{-1}v & \text{if } s \in \tau(v) \\ qv + \sum_{\{u \in V \mid s \in \tau(u)\}} \mu(u,v)u & \text{if } s \notin \tau(v), \end{cases} \tag{1}$$

for all $s \in S$ and $v \in V$.

We shall write M_Γ for the $\mathcal{H}$-module afforded by the W-graph Γ in the manner described above. Since M_Γ is $\mathcal{A}$-free with basis V it admits an $\mathcal{A}$-semilinear involution $\alpha \mapsto \overline{\alpha}$, uniquely determined by the condition that $\overline{v} = v$ for all $v \in V$. We call this the *bar involution* on M_Γ. It is a consequence of (1) that $\overline{h\alpha} = \overline{h}\overline{\alpha}$ for all $h \in \mathcal{H}$ and $\alpha \in M_\Gamma$.

We shall sometimes write $\Gamma(V)$ for the W-graph with vertex set V, if the functions μ and τ are clear from the context.

Following [8], define a preorder $\leqslant_\Gamma$ on V as follows: $u \leqslant_\Gamma v$ if there exists a sequence of vertices $u = x_0, x_1, \dots, x_m = v$ such that $\tau(x_{i-1}) \nsubseteq \tau(x_i)$ and $\mu(x_{i-1}, x_i) \neq 0$ for all $i \in [1,m]$. That is, $u \leqslant_\Gamma v$ if there is a directed path from v to u in Γ. Let $\sim_\Gamma$ be the equivalence relation determined by this preorder. The equivalence classes with respect to $\sim_\Gamma$ are called the *cells* of Γ. That is, the cells are the strongly connected components of the directed graph Γ. Each equivalence class, regarded as a full subgraph of Γ, is itself a W-graph, with the μ and τ functions being the restrictions of those for Γ. The preorder $\leqslant_\Gamma$ induces a partial order on the set of cells: if $\mathcal{C}$ and $\mathcal{C}'$ are cells, then $\mathcal{C} \leqslant_\Gamma \mathcal{C}'$ if $u \leqslant_\Gamma v$ for some $u \in \mathcal{C}$ and $v \in \mathcal{C}'$.

It follows readily from (1) that a subset of V spans a $\mathcal{H}(W)$-submodule of M_Γ if and only if it is Γ-closed, in the sense that for every vertex v in the subset, each $u \in V$ satisfying $\mu(u,v) \neq 0$ and $\tau(u) \nsubseteq \tau(v)$ is also in the subset. Thus $U \subseteq V$ is a Γ-closed subset of V if and only if $U = \bigcup_{v \in U} \{\, u \in V \mid u \leqslant_\Gamma v \,\}$. Clearly, a subset of V is Γ-closed if and only if it is the union of cells that form an ideal with respect to the partial order $\leqslant_\Gamma$ on the set of cells.

Suppose that U is a Γ-closed subset of V, and let $\Gamma(U)$ and $\Gamma(V \setminus U)$ be the full subgraphs of Γ induced by U and $V \setminus U$, with edge weights and vertex colours inherited from Γ. Then $\Gamma(U)$ and $\Gamma(V \setminus U)$ are themselves W-graphs, and

$$M_{\Gamma(V \setminus U)} \cong M_{\Gamma(V)}/M_{\Gamma(U)}$$

as $\mathcal{H}(W)$-modules.

It is clear that if $J \subseteq S$ and $\Gamma = (V, \mu, \tau)$ is a W-graph then $\Gamma_J = (V, \mu, \tau_J)$ is a W_J-graph, where the function $\tau_J\colon V \to \mathcal{P}(J)$ is given by $\tau_J(v) = \tau(v) \cap J$.

We end this section by recalling the original Kazhdan–Lusztig W-graph for the regular representation of $\mathcal{H}(W)$. For each $w \in W$, define the sets

$$\mathcal{L}(w) = \{s \in S \mid l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) = \{s \in S \mid l(ws) < l(w)\},$$

the elements of which are called the left descents of w and the right descents of w, respectively. Kazhdan and Lusztig give a recursive procedure that defines polynomials $P_{y,w}$ whenever $y, w \in W$ and $y < w$. These polynomials satisfy $\deg P_{y,w} \leqslant \frac{1}{2}(l(w) - l(y) - 1)$, and $\mu_{y,w}$ is defined to be the leading coefficient of $P_{y,w}$ if the degree is $\frac{1}{2}(l(w) - l(y) - 1)$, or 0 otherwise.

Define $W^{\mathrm{o}} = \{\, w^{\mathrm{o}} \mid w \in W \,\}$ to be the group opposite to W, and observe that $(W \times W^{\mathrm{o}}, S \sqcup S^{\mathrm{o}})$ is a Coxeter system. Kazhdan and Lusztig show that if μ and τ are defined by the formulas

$$\mu(w,y) = \mu(y,w) = \begin{cases} \mu_{y,w} & \text{if } y < w \\ \mu_{w,y} & \text{if } w < y \end{cases}$$
$$\bar{\tau}(w) = \mathcal{L}(w) \sqcup \mathcal{R}(w)^{\mathrm{o}}$$

then $(W, \mu, \bar{\tau})$ is a $(W \times W^{\mathrm{o}})$-graph. Thus $M = \mathcal{A}W$ may be regarded as an $(\mathcal{H},\mathcal{H})$-bimodule. Furthermore, the construction produces an explicit $(\mathcal{H},\mathcal{H})$-bimodule isomorphism $M \cong \mathcal{H}$.

It follows easily from the definition of $\mu_{y,w}$ that $\mu(y,w) \neq 0$ only if $l(w) - l(y)$ is odd; thus $(W, \mu, \bar{\tau})$ is a bipartite graph. The non-negativity of all coefficients of the Kazhdan–Lusztig polynomials, conjectured in [8], has been proved by Elias and Williamson in [5].

Since W and W^{o} are standard parabolic subgroups of $W \times W^{\mathrm{o}}$, it follows that $\Gamma = (W, \mu, \tau)$ is a W-graph and $\Gamma^{\mathrm{o}} = (W, \mu, \tau^{\mathrm{o}})$ is a W^{o}-graph, where τ and τ^{o} are defined by $\tau(w) = \mathcal{L}(w)$ and $\tau^{\mathrm{o}}(w) = \mathcal{R}(w)^{\mathrm{o}}$, for all $w \in W$.

In accordance with the theory described above, there are preorders on W determined by the $(W \times W^{\mathrm{o}})$-graph structure, the W-graph structure and the W^{o}-graph structure. We call these the *two-sided preorder* (denoted by $\preceq_{\mathsf{LR}}$), the *left preorder* ($\preceq_{\mathsf{L}}$) and the *right preorder* ($\preceq_{\mathsf{R}}$). The corresponding cells are the *two-sided cells*, the *left cells* and the *right cells*.

5. ADMISSIBLE W-GRAPHS

Let (W,S) be a Coxeter system, not necessarily finite. For $s, t \in S$, let $m(s,t)$ be the order of st in W. Thus $\{s,t\}$ is a bond in the Coxeter diagram if and only if $m(s,t) > 2$.

DEFINITION 5.1. [12, Definition 2.1] An S-coloured graph $\Gamma = (V, \mu, \tau)$ is *admissible* if the following three conditions are satisfied:

(i) $\mu(V \times V) \subseteq \mathbb{N}$;
(ii) Γ is symmetric, that is, $\mu(u,v) = \mu(v,u)$ if $\tau(u) \nsubseteq \tau(v)$ and $\tau(v) \nsubseteq \tau(u)$;
(iii) Γ has a bipartition.

REMARK 5.2. As we have seen in Sec. 4, the Kazhdan–Lusztig graph $\Gamma_W = \Gamma(W, \varnothing)$ is admissible. So its cells are admissible.

Let (W,S) be a braid finite Coxeter system. (That is, $m(s,t) < \infty$ for all $s,t \in S$.)

DEFINITION 5.3. [14, Definition 2.1] An S-coloured graph $\Gamma = (V, \mu, \tau)$ is said to satisfy the *W-Compatibility Rule* if for all $u, v \in V$ with $\mu(u,v) \neq 0$, each $i \in \tau(u) \setminus \tau(v)$ and each $j \in \tau(v) \setminus \tau(u)$ are joined by a bond in the Coxeter diagram of W.

By [12, Proposition 4.1], every W-graph satisfies the W-Compatibility Rule.

DEFINITION 5.4. [14, Definition 2.3] An admissible S-coloured graph $\Gamma = (V, \mu, \tau)$ satisfies the *W-Simplicity Rule* if for all $u, v \in V$ with $\mu(u,v) \neq 0$, either $\tau(v) \subsetneqq \tau(u)$ and $\mu(v,u) = 0$ or $\tau(u)$ and $\tau(v)$ are not comparable and $\mu(u,v) = \mu(v,u) = 1$.

The Simplicity Rule implies that if $\mu(u,v) \neq 0$ and $\mu(v,u) \neq 0$ then $\mu(u,v) = \mu(v,u) = 1$. That is, all edges are simple. Furthermore if $\{u,v\}$ is an edge then $\tau(u)$ and $\tau(v)$ are not comparable, so that there exist at least one $i \in \tau(u) \setminus \tau(v)$ and at least one $j \in \tau(v) \setminus \tau(u)$. If the Compatibility Rule is also satisfied, then $\{i, j\}$ must be a bond in the Coxeter diagram.

If (W,S) is simply-laced then every W-graph with non-negative integer edge weights satisfies the Simplicity Rule, even if it fails to be admissible: see [12, Remark 4.3].

DEFINITION 5.5. [14, Definition 2.4] An admissible S-coloured graph $\Gamma = (V, \mu, \tau)$ satisfies the *W-Bonding Rule* if for all $i, j \in S$ with $m_{i,j} > 2$, the vertices v of Γ with $i \in \tau(v)$ and $j \notin \tau(v)$ or $i \notin \tau(v)$ and $j \in \tau(v)$, together with edges of Γ that include the label $\{i, j\}$, form a disjoint union of Dynkin diagrams of types A, D or E with Coxeter numbers that divide $m(i, j)$.

REMARK 5.6. In the case $m(i, j) = 3$, the W-Bonding Rule becomes the *W-Simply-Laced Bonding Rule*: for every vertex u such that $i \in \tau(u)$ and $j \notin \tau(u)$, there exists a unique adjacent vertex v such that $j \in \tau(v)$ and $i \notin \tau(v)$.

By [12, Proposition 4.4], admissible W-graphs satisfy the W-Bonding Rule.

Let $\Gamma = (V, \mu, \tau)$ be an S-coloured graph. Let $i, j \in S$ with $m(i, j) = p \geqslant 2$. Suppose that $u, v \in V$ with $i, j \notin \tau(u)$ and $i, j \in \tau(v)$. For $2 \leqslant r \leqslant p$, a directed path $(u, v_1, \dots, v_{r-1}, v)$ in Γ is said to be alternating of type (i, j) if $i \in \tau(v_k)$ and $j \notin \tau(v_k)$ for odd k and $j \in \tau(v_k)$ and $i \notin \tau(v_k)$ for even k. Define

$$N^r_{i,j}(\Gamma; u, v) = \sum_{v_1,\dots,v_{r-1}} \mu(v, v_{r-1})\mu(v_{r-1} v_{r-2}) \cdots \mu(v_2, v_1)\mu(v_1, u), \tag{2}$$

where the sum extends over all paths $(u, v_1, \dots, v_{r-1}, v)$ that are alternating of type (i, j).

Note that if Γ is admissible then all terms in (2) are positive.

DEFINITION 5.7. [14, Definition 2.9] An admissible S-coloured graph $\Gamma = (V, \mu, \tau)$ satisfies the *W-Polygon Rule* if for all $i, j \in S$ and all $u, v \in V$ such that $i, j \in \tau(v) \setminus \tau(u)$, we have

$$N^r_{i,j}(\Gamma; u, v) = N^r_{j,i}(\Gamma; u, v) \quad \text{for all } r \text{ such that } 2 \leqslant r \leqslant m(i, j).$$

By [12, Proposition 4.7], all W-graphs with integer edge weights satisfy the Polygon Rule.

The following result provides a necessary and sufficient condition for an admissible S-coloured graph to be a W-graph.

THEOREM 5.8. [12, Theorem 4.9] *An admissible S-coloured graph $\Gamma = (V, \mu, \tau)$ is a W-graph if and only if it satisfies the W-Compatibility Rule, the W-Simplicity Rule, the W-Bonding Rule and the W-Polygon Rule.*

It is convenient to introduce a weakened version of the W-polygon rule.

DEFINITION 5.9. [14, Definition 2.9] An admissible S-coloured graph $\Gamma = (V, \mu, \tau)$ satisfies the *W-Local Polygon Rule* if for all $i, j \in S$, all r such that $2 \leqslant r \leqslant m(i, j)$, and all u, v such that $i, j \in \tau(v) \setminus \tau(u)$, we have $N^r_{i,j}(\Gamma; u, v) = N^r_{j,i}(\Gamma; u, v)$ under any of the following conditions:

(*i*) $r = 2$, and $\tau(u) \setminus \tau(v) \neq \varnothing$;

(*ii*) $r = 3$, and there exist $k, l \in \tau(u) \setminus \tau(v)$ (not necessarily distinct) such that $\{k, i\}$ and $\{j, l\}$ are not bonds in the Dynkin diagram of W;

(*iii*) $r \geqslant 4$, and there is $k \in \tau(u) \setminus \tau(v)$ such that $\{k, i\}$ and $\{j, k\}$ are not bonds in the Dynkin diagram of W.

DEFINITION 5.10. [14, Definition 3.3] An admissible S-coloured graph is called a *W-molecular graph* if it satisfies the W-Compatibility Rule, the W-Simplicity Rule, the W-Bonding Rule and W-Local Polygon Rules.

A *simple part* of an S-coloured graph Γ is a connected component of the graph obtained by removing all arcs and all non-simple edges, and a *simple component* of Γ is the full subgraph spanned by a simple part.

DEFINITION 5.11. A *W-molecule* is a W-molecular graph that has only one simple part.

REMARK 5.12. If Γ is an admissible W-graph then its simple components are W-molecules, by [14, Fact 3.1.]. More generally, by [14, Fact 3.2.], the full subgraph of Γ induced by any union of simple parts is a W-molecular graph.

If $M = (V, \mu, \tau)$ is an S-coloured graph and $J \subseteq S$ then the W_J-restriction of M is defined to be the J-coloured graph $M{\downarrow_J} = (V, \underline{\mu}, \underline{\tau})$ where $\underline{\tau}(v) = \tau(v) \cap J$ for all $v \in V$ and

$$\underline{\mu}(u,v) = \begin{cases} \mu(u,v) & \text{if } \underline{\tau}(u) \nsubseteq \underline{\tau}(v), \\ 0 & \text{if } \underline{\tau}(u) \subseteq \underline{\tau}(v). \end{cases}$$

It is easy to check that if $M = (V, \mu, \tau)$ is a W-molecular graph $M{\downarrow_J}$ is a W_J-molecular graph. The W_J-molecules of $M{\downarrow_J}$ are called W_J-submolecules of M.

PROPOSITION 5.13. [4, Proposition 2.7] *Let (W,S) be a Coxeter system and $M = (V, \mu, \tau)$ a W-molecular graph, and let $J = \{r,s,t\} \subseteq S$ with $m(s,t) = 3$ and $r \notin \{s,t\}$. Suppose that $v, v', u, u' \in V$, and that $\{v, v'\}$ and $\{u, u'\}$ are simple edges with*

$$\begin{aligned} \tau(v) \cap J &= \{s\}, & \qquad \tau(u) \cap J &= \{s, r\}, \\ \tau(v') \cap J &= \{t\}, & \qquad \tau(u') \cap J &= \{t, r\}. \end{aligned}$$

Then $\mu(u,v) = \mu(u',v')$.

6. TABLEAUX, LEFT CELLS AND ADMISSIBLE MOLECULES OF TYPE A

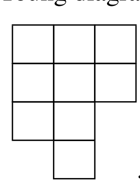
For the remainder of this paper we shall focus attention on Coxeter systems of type A.

A sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2 \dots, \lambda_k)$ is called a *composition* of n if $\sum_{i=1}^{k} \lambda_i = n$. The λ_i are called the *parts* of λ. We adopt the convention that $\lambda_i = 0$ for all $i > k$. A composition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is called a *partition* of n if $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$. We define $C(n)$ and $P(n)$ to be the sets of all compositions of n and all partitions of n, respectively.

For each $\lambda = (\lambda_1, \dots, \lambda_k) \in C(n)$ we define

$$[\lambda] = \{\, (i,j) \mid 1 \leqslant i \leqslant \lambda_j \text{ and } 1 \leqslant j \leqslant k \,\},$$

and refer to this as the Young diagram of λ. Pictorially $[\lambda]$ is represented by a top-justified array of boxes with λ_j boxes in the j-th column; the pair $(i,j) \in [\lambda]$ corresponds to the i-th box in the j-th column. Thus for us the Young diagram of $\lambda = (3,4,2)$ looks like this:


.

If $\lambda \in P(n)$ then λ^* denotes the *conjugate* of λ, defined to be the partition whose diagram is the transpose of $[\lambda]$; that is, $[\lambda^*] = \{(j,i) \mid (i,j) \in [\lambda]\}$.

Let $\lambda \in P(n)$. If $(i,j) \in [\lambda]$ and $[\lambda] \setminus \{(i,j)\}$ is still the Young diagram of a partition, we say that the box (i,j) is λ-*removable*. Similarly, if $(i,j) \notin [\lambda]$ and $[\lambda] \cup \{(i,j)\}$ is again the Young diagram of a partition, we say that the box (i,j) is λ-*addable*.

If $\lambda \in C(n)$ then a λ-*tableau* is a bijection $t\colon [\lambda] \to \mathcal{T}$, where $\mathcal{T}$ is a totally ordered set with n elements. We call $\mathcal{T}$ the *target* of t. In this paper the target will always be an interval $[m+1, m+n]$, with $m = 0$ unless otherwise specified. The composition λ is called the *shape* of t, and we write $\lambda = \mathrm{Shape}(t)$. For each $i \in [1,n]$ we define $\mathrm{row}_t(i)$ and $\mathrm{col}_t(i)$ to be the row index and column index of i in t (so that $t^{-1}(i) = (\mathrm{row}_t(i), \mathrm{col}_t(i))$). We define $\mathrm{Tab}_m(\lambda)$ to be the set of all λ-tableaux with target $\mathcal{T} = [m+1, m+n]$, and $\mathrm{Tab}(\lambda) = \mathrm{Tab}_0(\lambda)$. If $h \in \mathbb{Z}$ and $t \in \mathrm{Tab}_m(\lambda)$ then we define $t + h \in \mathrm{Tab}_{m+h}(\lambda)$ to be the tableau obtained by adding h to all entries of t.

We define $\tau_\lambda \in \mathrm{Tab}(\lambda)$ to be the specific λ-tableau given by $\tau_\lambda(i,j) = i + \sum_{h=1}^{j-1} \lambda_h$ for all $(i,j) \in [\lambda]$. That is, in τ_λ the numbers $1, 2, \dots, \lambda_1$ fill the first column of $[\lambda]$ in order from top to bottom, then the numbers $\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2$ similarly fill the second column, and so on. If $\lambda \in P(n)$ then we also define τ^λ to be the λ-tableau that is the transpose of τ_{λ^*}. Whenever $\lambda \in P(n)$ and $t \in \mathrm{Tab}_m(\lambda)$ we define $t^* \in \mathrm{Tab}_m(\lambda^*)$ to be the transpose of t.

Let $\lambda \in C(n)$ and $t \in \mathrm{Tab}(\lambda)$. We say that t is *column standard* if the entries increase down each column. That is, t is column standard if $t(i,j) < t(i+1,j)$ whenever (i,j) and $(i+1,j)$ are both in $[\lambda]$. We define $\mathrm{CStd}(\lambda)$ to be the set of all column standard λ-tableaux. In the case $\lambda \in P(n)$ we say that t is *row standard* if its transpose is column standard (so that $t(i,j) < t(i,j+1)$ whenever (i,j) and $(i,j+1)$ are both in $[\lambda]$), and we say that t is *standard* if it is both row standard and column standard. For each $\lambda \in P(n)$ we define $\mathrm{Std}(\lambda)$ to be the set of all standard λ-tableaux. We also define $\mathrm{Std}(n) = \bigcup_{\lambda\in P(n)} \mathrm{Std}(\lambda)$.

Let W_n be the symmetric group on the set $\{1,2,\dots,n\}$, and let $S_n = \{s_i \mid i \in [1,n-1]\}$, where s_i is the transposition $(i,i+1)$. Then (W_n,S_n) is a Coxeter system of type A_{n-1}. If $1 \leqslant h \leqslant k \leqslant n$ then we write $W_{[h,k]}$ for the standard parabolic subgroup of W_n generated by $\{\, s_i \mid i \in [h,k-1]\,\}$. We adopt a left operator convention for permutations, writing wi for the image of i under the permutation w.

It is clear that for any fixed composition $\lambda \in C(n)$ the group W_n acts on $\mathrm{Tab}(\lambda)$, via $(wt)(i,j) = w(t(i,j))$ for all $(i,j) \in [\lambda]$, for all λ-tableaux t and all $w \in W_n$. Moreover, the map from W_n to $\mathrm{Tab}(\lambda)$ defined by $w \mapsto w\tau_\lambda$ for all $w \in W_n$ is bijective. We define the map $\mathrm{perm}\colon \mathrm{Tab}(\lambda) \mapsto W_n$ to be the inverse of $w \mapsto w\tau_\lambda$, and use this to transfer the left weak order and the Bruhat order from W_n to $\mathrm{Tab}(\lambda)$. Thus if t_1 and t_2 are arbitrary λ-tableaux, we write $t_1 \leqslant_{\mathsf{L}} t_2$ if and only if $\mathrm{perm}(t_1) \leqslant_{\mathsf{L}} \mathrm{perm}(t_2)$, and $t_1 \leqslant t_2$ if and only if $\mathrm{perm}(t_1) \leqslant \mathrm{perm}(t_2)$. Similarly, we define the length of $t \in \mathrm{Tab}(\lambda)$ by $l(t) = l(\mathrm{perm}(t))$.

REMARK 6.1. If $\lambda \in C(n)$ and $t \in \mathrm{Tab}(\lambda)$ then the *reading word* of t is defined to be the sequence $b_1,\dots,b_n$ obtained by concatenating the columns of t in order from left to right, with the entries of each column read from bottom to top. This produces a bijection $\mathrm{Tab}(\lambda) \mapsto W_n$ that maps each t to the permutation $\mathrm{word}(t)$ given by $i \mapsto b_i$ for all $i \in \{1,\dots,n\}$. It is obvious that $\mathrm{perm}(t) = \mathrm{word}(t)w_\lambda^{-1}$, where $w_\lambda = \mathrm{word}(\tau_\lambda)$.

Given $\lambda \in C(n)$ we define J_λ to be the subset of S consisting of those s_i such that i and $i+1$ lie in the same column of τ_λ, and W_λ to be the standard parabolic subgroup of W_n generated by J_λ. Note that the longest element of W_λ is the element $w_\lambda = \mathrm{word}(\tau_\lambda)$ defined in Remark 6.1 above. We write D_λ for the set of minimal length representatives of the left cosets of W_λ in W_n. Since $l(ds_i) > l(d)$ if and only if $di < d(i+1)$, it follows that $D_\lambda = \{\, d \in W_n \mid di < d(i+1) \text{ whenever } s_i \in W_\lambda \,\}$, and the set of column standard λ-tableaux is precisely $\{\, d\tau_\lambda \mid d \in D_\lambda \,\}$.

We shall also need to work with tableaux defined on skew diagrams.

DEFINITION 6.2. A *skew partition* of n is an ordered pair (λ,μ), denoted by λ/μ, such that $\lambda \in P(m+n)$ and $\mu \in P(m)$ for some $m \geqslant 0$, and $\lambda_i \geqslant \mu_i$ for all i. We write $\lambda/\mu \vdash n$ to mean that λ/μ is a skew partition of n. In the case $m = 0$ we identify λ/μ with λ, and say that λ/μ is a *normal tableau*.

DEFINITION 6.3. The *skew diagram* $[\lambda/\mu]$ corresponding to a skew partition λ/μ is defined to be the complement of $[\mu]$ in $[\lambda]$:

$$[\lambda/\mu] = \{(i,j) \mid (i,j) \in [\lambda] \text{ and } (i,j) \notin [\mu]\}.$$

DEFINITION 6.4. A *skew tableau of shape* λ/μ, or (λ/μ)-tableau, where λ/μ is a skew partition of n, is a bijective map $t\colon [\lambda/\mu] \to \mathcal{T}$, where $\mathcal{T}$ is a totally ordered set with n elements. We write $\mathrm{Tab}_m(\lambda/\mu)$ for the set of all (λ/μ)-tableaux for which the target set $\mathcal{T}$ is the interval $[m+1,m+n]$. We shall omit the subscript m if $m = 0$.

Let λ/μ be a skew partition of n. We define $\tau_{\lambda/\mu} \in \mathrm{Tab}(\lambda/\mu)$ by

$$\tau_{\lambda/\mu}(i,j) = i - \mu_j + \sum_{h=1}^{j-1} (\lambda_h - \mu_h) \tag{3}$$

for all $(i,j) \in [\lambda/\mu]$, and define $\tau^{\lambda/\mu} \in \mathrm{Tab}(\lambda/\mu)$ to be the the transpose of τ_{λ^*/μ^*}.

If $\lambda/\mu \vdash n$ and $m \in \mathbb{Z}$ then $W_{[m+1,m+n]}$ acts naturally on $\mathrm{Tab}_m(\lambda/\mu)$, and we can define perm: $\mathrm{Tab}_m(\lambda/\mu) \to W_{[m+1,m+n]}$ and use it to transfer the Bruhat order and the left weak order from $W_{[m+1,m+n]}$ to $\mathrm{Tab}_m(\lambda/\mu)$ in exactly the same way as above.

All of our notation and terminology for partitions and Young tableaux extends naturally to skew partitions and tableaux, and will be used without further comment.

Let $\lambda \in C(n)$ and t a column standard λ-tableau. For each $m \in \mathbb{Z}$ we define $t\Downarrow m$ to be the tableau obtained by removing from t all boxes with entries greater than m. Thus if $\mu = \mathrm{Shape}(t\Downarrow m)$ then $\mu \in C(m)$ and $[\mu] = \{\, b \in [\lambda] \mid t(b) \leqslant m \,\}$, and $t\Downarrow m\colon [\mu] \to [1,m]$ is the restriction of t. It is clear that $t\Downarrow m$ is column standard. Moreover, if $\lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$ then $\mu \in P(m)$ and $t\Downarrow m \in \mathrm{Std}(\mu)$.

Similarly, if $\lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$ then for each $m \in \mathbb{Z}$ we define $t\uparrow m$ to be the skew tableau obtained by removing from t all boxes with entries less than or equal to m. Observe that $\{\, b \in [\lambda] \mid t(b) \leqslant m \,\}$ is the Young diagram of a partition $\nu \in P(n)$, and λ/ν is a skew partition of $n-m$. Clearly $t\uparrow m$ is the restriction of t to $[\lambda/\nu]$, and $t\uparrow m \in \mathrm{Std}_m(\lambda/\nu)$.

We also define $t\downarrow m = t\Downarrow(m-1)$ and $t\Uparrow m = t\uparrow(m-1)$.

The *dominance order* is defined on $C(n)$ as follows.

DEFINITION 6.5. Let $\lambda, \mu \in C(n)$. We say that λ *dominates* μ, and write $\lambda \geqslant \mu$ or $\mu \leqslant \lambda$, if $\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i$ for each positive integer k.

The *lexicographic order* on compositions is defined as follows.

DEFINITION 6.6. Let $\lambda, \mu \in C(n)$. We write $\lambda >_{\mathrm{lex}} \mu$ (or $\mu <_{\mathrm{lex}} \lambda$) if there exists a positive integer k such that $\lambda_k < \mu_k$ and $\lambda_i = \mu_i$ for all $i < k$. We say that λ *leads* μ, and write $\lambda \geqslant_{\mathrm{lex}} \mu$, if $\lambda = \mu$ or $\lambda >_{\mathrm{lex}} \mu$.

It is clear that the lexicographic order is a refinement of the dominance order.

PROPOSITION 6.7. *If $\lambda, \mu \in C(n)$ with $\lambda \geqslant \mu$, then $\lambda \geqslant_{\mathrm{lex}} \mu$.*

For a fixed $\lambda \in C(n)$ the *dominance order on* $\mathrm{CStd}(\lambda)$ is defined as follows.

DEFINITION 6.8. Let u and t be column standard λ-tableaux. We say that t dominates u if $\mathrm{Shape}(t\Downarrow m) \geqslant \mathrm{Shape}(u\Downarrow m)$ for all $m \in [1,n]$.

REMARK 6.9. Let $\lambda \in C(n)$ and let $u,t \in \mathrm{CStd}(\lambda)$ with $u \neq t$. Since $u\Downarrow 0 = t\Downarrow 0$ and $u\Downarrow n \neq t\Downarrow n$, we can choose $i \in [0,n-1]$ with $u\Downarrow i = t\Downarrow i$ and $u\Downarrow(i+1) \neq t\Downarrow(i+1)$. Let $\mu = \mathrm{Shape}(u\Downarrow(i+1))$ and $\lambda = \mathrm{Shape}(t\Downarrow(i+1))$, and let $k = \mathrm{col}_u(i+1)$ and $l = \mathrm{col}_t(i+1)$. Then $k \neq l$, and $\mu_j = \lambda_j$ for all $j < m = \min(k,l)$. Furthermore, $\mu_m = \lambda_m + 1$ if $m = k$, and $\lambda_m = \mu_m + 1$ if $m = l$. Thus $\lambda >_{\mathrm{lex}} \mu$ if and only if $l > k$.

Now suppose that t dominates u. Since $u\Downarrow i = t\Downarrow i$ we have $\mathrm{Shape}(u\Downarrow m) = \mathrm{Shape}(t\Downarrow m)$ for all $m \leqslant i$, and by Definition 6.8 we must have $\mathrm{Shape}(t\Downarrow(i+1)) \geqslant \mathrm{Shape}(u\Downarrow(i+1))$. That is, $\lambda \geqslant \mu$. By 6.7 it follows that $\lambda \geqslant_{\mathrm{lex}} \mu$, and so $\mathrm{col}_u(i+1) = k < l = \mathrm{col}_t(i+1)$.

The following theorem shows that the dominance order on $\mathrm{CStd}(\lambda)$ is the restriction of the Bruhat order on $\mathrm{Tab}(\lambda)$. That is, if $u,t \in \mathrm{CStd}(\lambda)$ then t dominates u if and only if $t \geqslant u$.

THEOREM 6.10. *Let $\lambda \in C(n)$, and let u and t be column standard λ-tableaux. Then t dominates u if and only if* $\mathrm{perm}(t) \geqslant \mathrm{perm}(u)$.

Proof. This is exactly [9, Theorem 3.8], except that we use columns where [9] uses rows. □

Let $\lambda = (\lambda_1, \dots, \lambda_k) \in C(n)$. For each $t \in \mathrm{Tab}(\lambda)$, we define $\mathrm{cp}(t)$ to be the composition of the number $\sum_{i=1}^{k} i\lambda_i$ given by $\mathrm{cp}(t)_i = \mathrm{col}_t(n+1-i)$, the column index of $n+1-i$ in t. We can now define the *lexicographic order* on $\mathrm{CStd}(\lambda)$, a total order that refines the Bruhat order.

DEFINITION 6.11. Let λ be a composition of n and let u and t be column standard λ-tableaux. We say that t leads u, and write $t \geqslant_{\mathrm{lex}} u$, if $\mathrm{cp}(t) \geqslant_{\mathrm{lex}} \mathrm{cp}(u)$.

REMARK 6.12. It is immediate from Definitions 6.6 and 6.11 that if $u, t \in \mathrm{CStd}(\lambda)$ then $t >_{\mathrm{lex}} u$ if and only if there exists $l \in [1,n]$ such that $\mathrm{col}_t(l) < \mathrm{col}_u(l)$ and $\mathrm{col}_t(i) = \mathrm{col}_u(i)$ for all $i \in [l+1,n]$. Since u and t are column standard and of the same shape, the latter condition is equivalent to $t{\uparrow}l = u{\uparrow}l$.

LEMMA 6.13. *Let $\lambda \in C(n)$, and let $u, t \in \mathrm{Tab}(\lambda)$. If $t \geqslant u$ then $t \geqslant_{\mathrm{lex}} u$.*

Proof. By Theorem 6.10 and the definition of the Bruhat order, it suffices to show that if $u = (i,j)t$ for some $i, j \in [1,n]$, then $t > u$ implies $t >_{\mathrm{lex}} u$. Without loss of generality we may assume that $j > i$, and then $t > u$ means that $\mathrm{col}_t(j) < \mathrm{col}_t(i) = \mathrm{col}_u(j)$. Since $\{\, k \mid \mathrm{col}_t(k) \neq \mathrm{col}_u(k)\,\} = \{i, j\}$, and j is the maximum element of this set, it follows from Remark 6.12 that $t >_{\mathrm{lex}} u$, as required. □

COROLLARY 6.14. *Let $\lambda \in C(n)$, and let $u, t \in \mathrm{CStd}(\lambda)$. If t dominates u then $t >_{\mathrm{lex}} u$.*

Let $\lambda \in P(n)$. For each $t \in \mathrm{Std}(\lambda)$ we define the following subsets of $[1, n-1]$:

$$\begin{aligned}
\mathrm{SA}(t) &= \{\, i \in [1,n-1] \mid \mathrm{row}_t(i) > \mathrm{row}_t(i+1)\,\},\\
\mathrm{SD}(t) &= \{ i \in [1,n-1] \mid \mathrm{col}_t(i) > \mathrm{col}_t(i+1)\},\\
\mathrm{WA}(t) &= \{ i \in [1,n-1] \mid \mathrm{row}_t(i) = \mathrm{row}_t(i+1)\},\\
\mathrm{WD}(t) &= \{ i \in [1,n-1] \mid \mathrm{col}_t(i) = \mathrm{col}_t(i+1)\}.
\end{aligned}$$

REMARK 6.15. It is easily checked that $i \in \mathrm{SA}(t)$ if and only if $s_i t \in \mathrm{Std}(\lambda)$ and $s_i t > t$, while $i \in \mathrm{SD}(t)$ if and only if $s_i t < t$ (which implies that $s_i t \in \mathrm{Std}(\lambda)$). Note also that if $w = \mathrm{perm}(t)$ then $i \in \mathrm{D}(t)$ if and only if $s_i \in \mathcal{L}(ww_\lambda)$; this is proved in [10, Lemma 5.2].

REMARK 6.16. It is clear that if $\lambda/\mu \vdash n$ and $m \in \mathbb{Z}$ then $m + \tau_{\lambda/\mu}$ is the unique minimal element of $\mathrm{Std}_m(\lambda/\mu)$ with respect to the Bruhat order and the left weak order. Accordingly, we call $m + \tau_{\lambda/\mu}$ the *minimal element of* $\mathrm{Std}_m(\lambda/\mu)$. It is easily shown that if $t \in \mathrm{Std}_m(\lambda/\mu)$ then $t = m + \tau_{\lambda/\mu}$ if and only if $\mathrm{SD}(t) = \varnothing$. That is, t is minimal if and only if $\mathrm{D}(t) = \mathrm{WD}(t)$.

For technical reasons it is convenient to make the following definition.

DEFINITION 6.17. Let $\lambda/\mu \vdash n > 1$ and $m \in \mathbb{Z}$. Let i be minimal such that $\lambda_i > \mu_i$, and assume that $\lambda_{i+1} > \mu_{i+1}$. The m-critical tableau of shape λ/μ is the tableau $t \in \mathrm{Std}_{m-1}(\lambda/\mu)$ such that $\mathrm{col}_t(m) = i$ and $\mathrm{col}_t(m+1) = i+1$, and $t{\uparrow}(m+1)$ is the minimal tableau of its shape.

If t is m-critical then, with i as in the definition, $\mathrm{col}_t(m+2) = i$ if and only if $\lambda_i - \mu_i > 1$.

REMARK 6.18. Let $\lambda \in P(n)$ and $m \in \mathbb{Z}$, and let $t \in \mathrm{Std}(\lambda)$ satisfy $\mathrm{col}_t(m+1) = \mathrm{col}_t(m) + 1$. We claim that $t{\Uparrow}m$ is m-critical if and only if the following two conditions both hold:

1) either $\mathrm{col}_t(m) = \mathrm{col}_t(m+2)$ or $m+1 \notin \mathrm{SD}(t)$,
2) every $j \in \mathrm{D}(t)$ with $j > m+1$ is in $\mathrm{WD}(t)$.

Let $\mathrm{Shape}(t{\Uparrow}m) = \lambda/\mu$, and put $i = \mathrm{col}_t(m)$. Note that since $m+1$ is in column $i+1$ of $t{\Uparrow}m$, it follows that $\lambda_{i+1} > \mu_{i+1}$.

Given that $\mathrm{col}_t(m+1) = \mathrm{col}_t(m) + 1$, the second alternative in condition (1) is equivalent to $\mathrm{col}_t(m) + 1 \leqslant \mathrm{col}_t(m+2)$. Hence condition (1) is equivalent to $\mathrm{col}_t(m) \leqslant \mathrm{col}_t(m+2)$. But by Remark 6.16, condition (2) holds if and only if $t{\uparrow}(m+1)$ is minimal, which in turn is equivalent to $\mathrm{col}_t(m+2) \leqslant \mathrm{col}_t(m+3) \leqslant \cdots \leqslant \mathrm{col}_t(n)$. So (1) and (2) both hold if and only if $t\uparrow(m+1)$ is minimal and $\mathrm{col}_t(j) \geqslant \mathrm{col}_t(m)$ for all $j \geqslant m$.

Since $\mathrm{col}_t(m+1) = i+1$, it follows from the definition that $t\Uparrow m$ is m-critical if and only if $t\uparrow(m+1)$ is minimal and $i = \mathrm{col}_t(m)$ is equal to $\min\{\, j \mid \lambda_j > \mu_j \,\}$. But this last condition holds if and only if m is in the first nonempty column of $t\Uparrow m$, and since this holds if and only if $\mathrm{col}_t(j) \geqslant \mathrm{col}_t(m)$ for all $j \geqslant m$, the claim is established.

Recall that if $w \in W_n$ then applying the Robinson–Schensted algorithm to the sequence $(w1, w2, \dots, wn)$ produces a pair $\mathrm{RS}(w) = (P(w), Q(w))$, where $P(w), Q(w) \in \mathrm{Std}(\lambda)$ for some $\lambda \in P(n)$. Details of the algorithm can be found (for example) in [11, Section 3.1]. The first component of $\mathrm{RS}(w)$ is called the *insertion* tableau and the second component is called the *recording* tableau.

The following theorem is proved, for example, in [11, Theorem 3.1.1].

THEOREM 6.19. *The Robinson–Schensted map is a bijection from W_n to $\bigcup_{\lambda\in P(n)} \mathrm{Std}(\lambda)^2$.*

The following property of the Robinson–Schensted map is also proved, for example, in [11, Theorem 3.6.6].

THEOREM 6.20. *Let $w \in W_n$. If $RS(w) = (t,x)$ then $RS(w^{-1}) = (x,t)$.*

The following lemma will be used below in the discussion of dual Knuth equivalence classes.

LEMMA 6.21. [10, Lemma 6.3] *Let $\lambda \in \mathcal{P}(n)$ and let $w \in W_n$. Then $RS(w) = (t, \tau_\lambda)$ for some $t \in \mathrm{Std}(\lambda)$ if and only if $w = vw_\lambda$ for some $v \in W_n$ such that $v\tau_\lambda \in \mathrm{Std}(\lambda)$. When these conditions hold, $t = v\tau_\lambda$.*

DEFINITION 6.22. The *dual Knuth equivalence relation* is the equivalence relation $\approx$ on W_n generated by the requirements that for all $x \in W_n$ and $k \in [1, n-2]$,

1) $x \approx s_{k+1}x$ whenever $\mathcal{L}(x) \cap \{s_k, s_{k+1}\} = \{s_k\}$ and $\mathcal{L}(s_{k+1}x) \cap \{s_k, s_{k+1}\} = \{s_{k+1}\}$,
2) $x \approx s_k x$ whenever $\mathcal{L}(x) \cap \{s_k, s_{k+1}\} = \{s_{k+1}\}$ and $\mathcal{L}(s_k x) \cap \{s_k, s_{k+1}\} = \{s_k\}$.

The relations 1) and 2) above are known as the dual Knuth relations of the first kind and second kind, respectively.

REMARK 6.23. It is not hard to check that 1) and 2) above can be combined to give an alternative formulation of Definition 6.22, as follows: $\approx$ is the equivalence relation on W_n generated by the requirement that $x \approx sx$ for all $x \in W_n$ and $s \in S_n$ such that $x < sx$ and $\mathcal{L}(x) \nsubseteq \mathcal{L}(sx)$. In [8] Kazhdan and Lusztig show that whenever this holds then x and sx are joined by a simple edge in the Kazhdan–Lusztig W-graph $\Gamma = \Gamma(W_n)$. Furthermore, they show that the dual Knuth equivalence classes coincide with the left cells in $\Gamma(W_n)$.

The following result is well-known.

THEOREM 6.24. [11, Theorem 3.6.10] *Let $x, y \in W_n$. Then $x \approx y$ if and only if $Q(x) = Q(y)$.*

Let $\lambda \in P(n)$, and for each $t \in \mathrm{Std}(\lambda)$ define $C(t) = \{\, w \in W_n \mid Q(w) = t \,\}$. Theorem 6.24 says that these sets are the dual Knuth equivalence classes in W_n. It follows from Lemma 6.21 that $C(\tau_\lambda) = \{\, vw_\lambda \mid v\tau_\lambda \in \mathrm{Std}(\lambda) \,\} = \{\, \mathrm{perm}(t)w_\lambda \mid t \in \mathrm{Std}(\lambda) \,\} = \{\, \mathrm{word}(t) \mid t \in \mathrm{Std}(\lambda) \,\}$.

Let t, $u \in \mathrm{Std}(\lambda)$, and suppose that $t = s_k u$ for some $k \in [2, n-1]$. By Remark 6.15 above, if $x = \mathrm{word}(u)$ then $\mathcal{L}(x) \cap \{s_{k-1}, s_k\} = \{s_{k-1}\}$ and $\mathcal{L}(s_k x) \cap \{s_{k-1}, s_k\} = \{s_k\}$ if and only if $\mathrm{D}(u) \cap \{k-1, k\} = \{k-1\}$ and $\mathrm{D}(t) \cap \{k-1, k\} = \{k\}$. Under these circumstances we write $u \to^{*1} t$, and say that there is a *dual Knuth move of the first kind* from u to t. Similarly, if $t = s_k u$ for some $k \in [1, n-2]$ such that $\mathrm{D}(u) \cap \{k, k+1\} = \{k+1\}$ and $\mathrm{D}(t) \cap \{k, k+1\} = \{k\}$ then we write $u \to^{*2} t$, and say that there is a *dual Knuth move of the second kind* from u to t.

Since $C(\tau_\lambda)$ is a single dual Knuth equivalence class, any standard tableau of shape λ can be transformed into any other by a sequence of dual Knuth moves or their inverses.

We call the integer k above the *index* of the corresponding dual Knuth move, and denote it by $\operatorname{ind}(u,t)$.

REMARK 6.25. Dual Knuth moves are also defined for standard skew tableaux; the definitions are exactly the same as for tableaux of normal shape. If $\lambda/\mu \vdash n$ and $u, t \in \operatorname{Std}(\lambda/\mu)$ then we write $u \approx t$ if and only if u and t are related by a sequence of dual Knuth moves.

DEFINITION 6.26. For each $J \subseteq S_n$ let $\approx_J$ be the equivalence relation on W_n generated by the requirement that $x \approx_J sx$ for all $s \in J$ and $x \in W_n$ such that $x < sx$ and $\mathcal{L}(x) \cap J \nsubseteq \mathcal{L}(sx)$.

REMARK 6.27. Let $J \subseteq S_n$, let $(W,S) = (W_n, S_n)$ and let Γ be the regular Kazhdan–Lusztig W-graph. By the results of Section 4 we know that a simple edge $\{x,y\}$ of Γ remains a simple edge of $\Gamma{\downarrow_J}$ provided that $\mathcal{L}(x) \cap J \nsubseteq \mathcal{L}(y) \cap J$ and $\mathcal{L}(y) \cap J \nsubseteq \mathcal{L}(x) \cap J$. Recall that the simple edges of Γ all have the form $\{x, sx\}$, where $s \in S$ and $x < sx \in W$. Given that $x < sx$, the condition $\mathcal{L}(sx) \cap J \nsubseteq \mathcal{L}(x) \cap J$ holds if and only if $s \in J$, and so $\{x, sx\}$ is a simple edge of $\Gamma{\downarrow_J}$ if and only if $s \in J$ and $\mathcal{L}(x) \cap J \nsubseteq \mathcal{L}(sx)$. Thus $\approx_J$ is the equivalence relation on W generated by the requirement that $x \approx_J y$ whenever $\{x,y\}$ is a simple edge of $\Gamma{\downarrow_J}$.

DEFINITION 6.28. Let $\lambda \in P(n)$ and $1 \leqslant m \leqslant n$. Let $\approx_m$ be the equivalence relation on $\operatorname{Std}(\lambda)$ defined by the requirement that $u \approx_m t$ whenever there is a dual Knuth move of index at most $m-1$ from u to t and $\mathrm{D}(u) \cap [1, m-1] \nsubseteq \mathrm{D}(t)$. We call such a move a *$(\leqslant m)$-dual Knuth move*. The $\approx_m$ equivalence classes in $\operatorname{Std}(\lambda)$ will be called the *$(\leqslant m)$-subclasses* of $\operatorname{Std}(\lambda)$, and we shall say that $u, t \in \operatorname{Std}(\lambda)$ are *$(\leqslant m)$-dual Knuth equivalent* whenever $u \approx_m t$.

REMARK 6.29. Assume that $\lambda \in P(n)$ and $1 \leqslant m \leqslant n$, and let $u, t \in \operatorname{Std}(\lambda)$. If $u \to^{*2} t$ and $\operatorname{ind}(u,t) \leqslant m-1$ then $\mathrm{D}(u) \cap [1, m-1] \nsubseteq \mathrm{D}(t)$ if and only if $\operatorname{ind}(u,t) \in [1, m-2]$. Clearly this holds if and only if $u{\uparrow}m = t{\uparrow}m$ and $u{\Downarrow}m \to^{*2} t{\Downarrow}m$. If $u \to^{*1} t$ and $\operatorname{ind}(u,t) \leqslant m-1$ then $\operatorname{ind}(u,t) \in [2, m-1]$, and $\mathrm{D}(u) \cap [1, m-1] \nsubseteq \mathrm{D}(t)$ is automatically satisfied. Clearly this holds if and only if $u{\uparrow}m = t{\uparrow}m$ and $u{\Downarrow}m \to^{*1} t{\Downarrow}m$. It follows that $u \approx_m t$ if and only if $u{\uparrow}m = t{\uparrow}m$, since $\operatorname{Shape}(u{\Downarrow}m) = \operatorname{Shape}(t{\Downarrow}m)$ guarantees that $u{\Downarrow}m$ and $t{\Downarrow}m$ are related by a sequence of dual Knuth moves. So in fact $u \approx_m t$ if and only if $t = wu$ for some $w \in W_m$.

It is a consequence of Definitions 6.26 and 6.28 that if $u, t \in \operatorname{Std}(\lambda)$ then $u \approx_m t$ if and only if $\operatorname{word}(u) \approx_J \operatorname{word}(t)$, where $J = S_m$. The set of all $(\leqslant m)$-subclasses of $\operatorname{Std}(\lambda)$ is in bijective correspondence with the set $\{\, v \in \operatorname{Std}_m(\lambda/\mu) \mid \mu \in P(m) \text{ and } [\mu] \subseteq [\lambda] \,\}$, and each $(\leqslant m)$-subclass of $\operatorname{Std}(\lambda)$ is in bijective correspondence with $\operatorname{Std}(\mu)$ for some $\mu \in P(m)$ with $[\mu] \subseteq [\lambda]$. If $t \in \operatorname{Std}(\lambda)$ then the $(\leqslant m)$-subclass that contains t is denoted by $C_m(t)$ and is given by $C_m(t) = \{\, u \in \operatorname{Std}(\lambda) \mid u{\uparrow}m = t{\uparrow}m \,\}$.

In view of Remark 6.23 and Theorem 6.24, the following theorem follows from the results of Kazhdan and Lusztig [8].

THEOREM 6.30. *If $t, t' \in \operatorname{Std}(n)$ then the W_n-graphs $\Gamma(C(t))$ and $\Gamma(C(t'))$ are isomorphic if and only if $\operatorname{Shape}(t) = \operatorname{Shape}(t')$. In particular, if $\lambda \in P(n)$ then $\Gamma(C(t)) \cong \Gamma(C(\tau_\lambda))$ whenever $t \in \operatorname{Std}(\lambda)$.*

COROLLARY 6.31. *Let Γ be the W_n-graph of a Kazhdan–Lusztig left cell of W_n. Then Γ is isomorphic to $\Gamma(C(\tau_\lambda))$ for some $\lambda \in P(n)$.*

Clearly for each $\lambda \in P(n)$ the bijection $t \mapsto \operatorname{word}(t)$ from $\operatorname{Std}(\lambda)$ to $C(\tau_\lambda)$ can be used to create a W_n-graph isomorphic to $\Gamma(C(\tau_\lambda)$ with $\operatorname{Std}(\lambda)$ as the vertex set.

NOTATION 6.32. For each $\lambda \in P(n)$ we write $\Gamma_\lambda = \Gamma(\operatorname{Std}(\lambda), \mu^{(\lambda)}, \tau^{(\lambda)})$ for the W_n-graph just described.

REMARK 6.33. Let $\lambda \in P(n)$ and let $J = S_m \subseteq S_n$. It follows from Remark 6.27 and Definition 6.28 that the J-submolecules of Γ_λ are spanned by the $(\leqslant m)$-subclasses of $\mathrm{Std}(\lambda)$.

Now let $\lambda \in P(n)$ and $1 \leqslant m \leqslant n$, and put $J = S_n \smallsetminus S_m$. The J-submolecules of Γ_λ can be determined by an analysis similar to that used above. We define $\approx^m$ to be the equivalence relation on $\mathrm{Std}(\lambda)$ generated by the requirement that $u \approx^m t$ whenever there is a dual Knuth move of index at least m from u to t and $\mathrm{D}(u) \cap [m, n-1] \nsubseteq \mathrm{D}(t)$. The $\approx^m$ equivalence classes in $\mathrm{Std}(\lambda)$ will be called the *$(\geqslant m)$-subclasses* of $\mathrm{Std}(\lambda)$. If $u, t \in \mathrm{Std}(\lambda)$ then $u \approx^m t$ if and only if $\mathrm{word}(u) \approx_J \mathrm{word}(t)$, with $J = S_n \smallsetminus S_m$. An equivalent condition is that $u{\downarrow}m = t{\downarrow}m$ and $u{\Uparrow}m \approx t{\Uparrow}m$. It follows that if $t \in \mathrm{Std}(\lambda)$ then the $(\geqslant m)$-subclass that contains t is the set $C^m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u{\downarrow}m = t{\downarrow}m \text{ and } u{\Uparrow}m \approx t{\Uparrow}m \,\}$.

REMARK 6.34. Let $\lambda \in P(n)$ and $m \in [1, n]$, and put $J = S_n \smallsetminus S_m$. By the discussion above, the J-submolecules of Γ_λ are spanned by the $(\geqslant m)$-subclasses of $\mathrm{Std}(\lambda)$.

We shall need to use some properties of the well-known "jeu-de-taquin" operation on skew tableaux, which we now describe.

Fix a positive integer n and a target set $\mathcal{T} = [m+1, m+n]$. It is convenient to define a *partial tableau* to be a bijection t from a subset of $\{\, (i, j) \mid i, j \in \mathbb{Z}^+ \,\}$ to $\mathcal{T}$. We shall also assume that the domain of t is always of the form $[\kappa/\xi] \smallsetminus \{(i, j)\}$, where κ/ξ is a skew partition of $n+1$ and $(i, j) \in [\kappa/\xi]$. If (i, j) is ξ-addable then t is a (κ/μ)-tableau, with $[\mu] = [\xi] \cup \{(i, j)\}$, and if (i, j) is κ-removable then t is a (λ/ξ)-tableau, with $[\lambda] = [\kappa] \smallsetminus \{(i, j)\}$.

Now suppose that λ/μ is a skew partition of n and $t \in \mathrm{Std}(\lambda/\mu)$, and suppose also that $c = (i, j)$ is a μ-removable box. Note that t may be regarded as a partial tableau, since $[\lambda/\mu] = [\kappa/\xi] \smallsetminus \{(i, j)\}$, where $[\kappa] = [\lambda]$ and $[\xi] = [\mu] \smallsetminus \{(i, j)\}$. The *jeu de taquin slide on t into c* is the process $j(c, t)$ given as follows.

Start by defining $t_0 = t$ and $b_0 = (i, j)$. Proceeding recursively, suppose that $k \geqslant 0$ and that t_k and b_k are defined, with t_k a partial tableau whose domain is $[\kappa/\xi] \smallsetminus \{b_k\}$. If b_k is λ-removable then the process terminates, we define $t' = t_k$ and put $m = k$. If $b_k = (g, h)$ is not λ-removable we put $x = \min(t_k(g+1, h), t_k(g, h+1))$, define $b_{k+1} = t_k^{-1}(x)$, and define t_{k+1} to be the partial tableau with domain $[\kappa/\xi] \smallsetminus \{b_{k+1}\}$ given by

$$t_{k+1}(b) = \begin{cases} t_k(b) & \text{whenever } b \text{ is in the domain of } t_k \text{ and } b \neq b_{k+1}, \\ x & \text{if } b = b_k. \end{cases}$$

(We say that x slides from b_{k+1} into b_k.) The tableau t' obtained by the above process is denoted by $j^{(c)}(t)$. The sequence of boxes $b_0 = c, b_1, \ldots, b_m$ is called the *slide path* of $j(c, t)$, and the box b_m is said to be *vacated* by $j(c, t)$.

The following observation follows immediately from the definition of a slide path.

LEMMA 6.35. *Let $b_0 = c, b_1, \ldots, b_m$ be the slide path of a jeu de taquin slide, as described above. If $0 \leqslant i < j \leqslant m$ then $b_i(1) \leqslant b_j(1)$ and $b_i(2) \leqslant b_j(2)$.*

We also have the following trivial result.

LEMMA 6.36. *Let $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_l^{m_l}) \in P(n)$ and $\mu = (1) \in P(1)$, and put $t = (\tau_\lambda \uparrow 1) - 1$. Then (λ_1, m_1) is vacated by the slide $j((1,1), t)$. Similarly, if $u = (\tau^\lambda \uparrow 1) - 1$, where $\lambda \in P(n)$ and $\lambda^* = \mu = (\mu_1^{n_1}, \ldots, \mu_k^{n_k})$, then (n_1, μ_1) is vacated by the slide $j((1,1), u)$.*

A sequence of boxes $\beta = (b_1, \ldots, b_l)$ called a *slide sequence* for a standard skew tableau t if there exists a sequence of skew tableaux $t_0 = t, t_1, \ldots, t_l$ such that the jeu de taquin slide $j(b_i, t_{i-1})$ is defined for each $i \in [1, l]$, and $t_i = j^{(b_i)}(t_{i-1})$. We write $t_l = j_\beta(t)$. Clearly the slide sequence $\beta = (b_1, \ldots, b_l)$ can be extended to a longer slide sequence $b_1, \ldots, b_{l+1}$ if the skew tableau t_l is not of normal shape. If t_l is of normal shape then we write $t_l = j(t)$.

Theorem 6.37 below says that $j(t)$ is independent of the slide sequence and is the insertion tableau of word(t).

THEOREM 6.37. [11, Theorem 3.7.7] *Let λ/μ be a skew partition of n and $t \in \mathrm{Std}(\lambda/\mu)$. If β is any maximal length slide sequence for t then $j_\beta(t) = P(\mathrm{word}(t))$.*

Skew tableaux u and t are said to be *dual equivalent* if the skew tableaux $j_\beta(u)$ and $j_\beta(t)$ are of the same shape whenever β is a slide sequence for both u and t. Dual equivalent skew tableaux are necessarily of the same shape, since the slide sequence β is allowed to have length zero. It is easily shown that if u and t are dual equivalent then every slide sequence for u is also a slide sequence for t, from which it follows that dual equivalence is indeed an equivalence relation. The following result says that this equivalence relation coincides with dual Knuth equivalence.

THEOREM 6.38. [11, Theorem 3.8.8] *Let λ/μ be a skew partition, and let u and t be standard λ/μ-tableaux. Then u is dual equivalent to t if and only if $u \approx t$.*

Note that Theorem 6.38 generalizes the fact that the set of standard tableaux of a given normal shape form a single dual Knuth equivalence class.

If λ/μ is a skew partition of n then the corresponding *dual equivalence graph* has vertex set $\mathrm{Std}(\lambda/\mu)$ and edge set $\{\,\{u,t\} \mid u,t \in \mathrm{Std}(\lambda/\mu) \text{ and } u \to^{*1} t \text{ or } u \to^{*2} t\,\}$.

If $k \in [1, n-1]$ then each $v \in \mathrm{Std}(\lambda/\mu)$ with $\mathrm{D}(v) \cap \{k, k+1\} = \{k\}$ is adjacent in the dual equivalence graph to a unique v' with $\mathrm{D}(v') \cap \{k, k+1\} = \{k+1\}$, and each v with $\mathrm{D}(v) \cap \{k, k+1\} = \{k+1\}$ is adjacent to a unique v' with $\mathrm{D}(v') \cap \{k, k+1\} = \{k\}$. In fact, $v' = s_k v$ if $\mathrm{col}_v(k) < \mathrm{col}_v(k+2) \leqslant \mathrm{col}_v(k+1)$ or $\mathrm{col}_v(k+1) < \mathrm{col}_v(k+2) \leqslant \mathrm{col}_v(k)$, and $v' = s_{k+1} v$ if $\mathrm{col}_v(k+1) \leqslant \mathrm{col}_v(k) < \mathrm{col}_v(k+2)$ or $\mathrm{col}_v(k+2) \leqslant \mathrm{col}_v(k) < \mathrm{col}_v(k+1)$.

DEFINITION 6.39. We call the above tableau v' the *k-neighbour* of v, and write $v' = k\text{-neb}(v)$.

It follows from Remark 6.23 that if μ is the empty partition then the dual equivalence graph is isomorphic to the simple part of each Kazhdan–Lusztig left cell $\Gamma(C(t))$ for $t \in \mathrm{Std}(\lambda)$; in this case we call the dual equivalence graph the *standard dual equivalence graph* corresponding to $\lambda \in P(n)$. Extending earlier work of Assaf [1], Chmutov showed in [4] that the simple part of an admissible W_n-molecule is isomorphic to a standard dual equivalence graph. The following result is the main theorem of [4].

THEOREM 6.40. *The simple part of an admissible molecule of type A_{n-1} is isomorphic to the simple part of a Kazhdan–Lusztig left cell.*

It is worth noticing that Stembridge has shown that there are A_{15}-molecules that cannot occur in Kazhdan–Lusztig left cells [14, Remark 3.8].

REMARK 6.41. It follows that if $M = (V, \mu, \tau)$ is a molecule then there exists $\lambda \in P(n)$ and a bijection $t \mapsto c_t$ from $\mathrm{Std}(\lambda)$ to V such that $\tau(c_t) = \mathrm{D}(t)$ and the simple edges of M are the pairs $\{c_u, c_t\}$ such that $u, t \in \mathrm{Std}(\lambda)$ and there is a dual Knuth move from u to t or from t to u. The molecule M is said to be of type λ.

Let $M = (V, \mu, \tau)$ be an arbitrary S_n-coloured molecular graph, and for each $\lambda \in P(n)$ let m_λ be the number of molecules of type λ in M. For each λ such that $m_\lambda \neq 0$ let $\mathcal{I}_\lambda$ be some indexing set of cardinality m_λ. Then we can write

$$V = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} V_{\alpha,\lambda}, \tag{4}$$

where Λ consists of all $\lambda \in P(n)$ such that $m_\lambda \neq 0$, each $V_{\alpha,\lambda} = \{\, c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\,\}$ is the vertex set of a molecule of type λ, $\tau(c_{\alpha,t}) = \mathrm{D}(t)$, and the simple edges of M are the pairs $\{c_{\alpha,u}, c_{\beta,t}\}$ such that $\alpha = \beta \in \mathcal{I}_\lambda$ for some $\lambda \in P(n)$ and $u, t \in \mathrm{Std}(\lambda)$ are related by a dual Knuth move. We shall call the set Λ the set of *molecule types* for the molecular graph M.

Note that if $\Gamma = (V, \mu, \tau)$ is an admissible W_n-graph then Γ is an S_n-coloured molecular graph, by Remark 5.12, and hence Eq. (4) can be used to describe the vertex set of Γ.

REMARK 6.42. We know from Remark 5.2 and Corollary 6.31 that, for each $\lambda \in P(n)$, the W_n-graph $\Gamma_\lambda = (\mathrm{Std}(\lambda), \mu^{(\lambda)}, \tau^{(\lambda)})$ is admissible. Since $\{u,t\}$ is a simple edge in Γ_λ when $u, t \in \mathrm{Std}(\lambda)$ are related by a dual Knuth move, and $\mathrm{Std}(\lambda)$ is a single dual Knuth equivalence class, we see that Γ_λ consists of a single molecule (of type λ).

REMARK 6.43. Let $\Gamma = (V, \mu, \tau)$ be an admissible W_n-graph, and continue with the notation and terminology of Remark 6.41 above. Let $m \in [1,n]$, and let $K = S_m$ and $L = S_n \setminus S_m$.

Let $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$, and let Θ be the molecule of Γ whose vertex set is $V_{\alpha,\lambda}$. Write $\Gamma{\downarrow_K} = (V, \mu, \underline{\tau})$ (where $\underline{\tau} = \tau_K$ in the notation of Section 4 above). By Remark 6.41 applied to $\Theta{\downarrow_K}$, we may write

$$V_{\alpha,\lambda} = \bigsqcup_{\kappa \in \Lambda_{K,\alpha,\lambda}} \ \bigsqcup_{\beta \in \mathcal{I}_{K,\alpha,\lambda,\kappa}} V_{\alpha,\lambda,\beta,\kappa},$$

where $\Lambda_{K,\alpha,\lambda}$ is the set of all $\kappa \in P(m)$ such that Θ contains a K-submolecule of type κ, and $\mathcal{I}_{K,\alpha,\lambda,\kappa}$ is an indexing set whose size is the number of such K-submolecules. Each $V_{\alpha,\lambda,\beta,\kappa}$ is the vertex set of a K-submolecule of Θ of type κ. Writing $V_{\alpha,\lambda,\beta,\kappa} = \{\, c'_{\beta,u} \mid u \in \mathrm{Std}(\kappa) \,\}$, we see that each $c_{\alpha,t} \in V_{\alpha,\lambda}$ coincides with some $c'_{\beta,v}$ with $\beta \in \mathcal{I}_{K,\alpha,\lambda,\kappa}$ and $v \in \mathrm{Std}(\kappa)$. It follows from Remark 6.33 above that the K-submolecule of Θ containing a given vertex $c_{\alpha,t}$ is spanned by the $(\leqslant m)$-subclass $C_m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u{\uparrow}m = t{\uparrow}m \,\}$. Thus when we write $c_{\alpha,t} = c'_{\beta,v}$ as above, we can identify v with $t{\Downarrow}k$.

Similarly, applying Remark 6.41 to $\Theta{\downarrow_L}$, we may write

$$V_{\alpha,\lambda} = \bigsqcup_{\theta \in \Lambda_{L,\alpha,\lambda}} \ \bigsqcup_{\gamma \in \mathcal{I}_{L,\alpha,\lambda,\theta}} V_{\alpha,\lambda,\gamma,\theta},$$

where $\Lambda_{L,\alpha,\lambda}$ is the set of all $\theta \in P(n-m+1)$ such that Θ contains an L-submolecule of type θ, and $\mathcal{I}_{L,\alpha,\lambda,\theta}$ is a set whose size is the number of such L-submolecules. Each $V_{\alpha,\lambda,\gamma,\theta}$ is the vertex set of an L-submolecule of Θ of type θ. Writing $V_{\alpha,\lambda,\gamma,\theta} = \{\, c''_{\gamma,v} \mid v \in \mathrm{Std}_{m-1}(\theta) \,\}$ (where $\mathrm{Std}_{m-1}(\theta)$ is the set of standard θ-tableaux with target $[m,n]$), we see that each $c_{\alpha,t} \in V_{\alpha,\lambda}$ coincides with some $c''_{\gamma,v}$ with $\gamma \in \mathcal{I}_{L,\alpha,\lambda,\theta}$ and $v \in \mathrm{Std}_{m-1}(\theta)$. By Remark 6.34 above we see that the L-submolecule of Θ containing a given vertex $c_{\alpha,t}$ is spanned by the $(\geqslant m)$-subclass $C^m(t) = \{\, u \in \mathrm{Std}(\lambda) \mid u{\downarrow}m = t{\downarrow}m \text{ and } u{\Uparrow}m \approx t{\Uparrow}m \,\}$. Since the condition $u{\Uparrow}m \approx t{\Uparrow}m$ is satisfied if and only if $\mathrm{word}(1-m+(u{\Uparrow}m)) \approx \mathrm{word}(1-m+(t{\Uparrow}m))$, and $j(1-m+(t{\Uparrow}m)) = P(\mathrm{word}(1-m+(t{\Uparrow}m)))$ by Theorem 6.37, it follows that when we write $c_{\alpha,t} = c''_{\beta,v}$ as above we can identify v with $j(t{\Uparrow}m) = m-1+j(1-m+(t{\Uparrow}m))$.

7. EXTENDED DOMINANCE ORDER ON $\mathrm{Std}(n)$ AND PAIRED DUAL KNUTH EQUIVALENCE RELATION

Let $n \geqslant 1$, and let (W_n, S_n) be the Coxeter group of type A_{n-1} and $\mathcal{H}_n$ the corresponding Hecke algebra. We shall need the following partial order, called the *extended dominance order*, on $\mathrm{Std}(n)$.

DEFINITION 7.1. Let $\lambda, \mu \in P(n)$, and let $u \in \mathrm{Std}(\lambda)$ and $t \in \mathrm{Std}(\mu)$. Then t is said to dominate u if $\mathrm{Shape}(u{\Downarrow}m) \leqslant \mathrm{Shape}(t{\Downarrow}m)$ for all $m \in [1,n]$. When this holds we write $u \leqslant t$.

This is obviously a partial order on $\mathrm{Std}(n) = \bigcup_{\lambda \in P(n)} \mathrm{Std}(\lambda)$, and it is also clear that $u \leqslant t$ if and only if $\mathrm{Shape}(u) \leqslant \mathrm{Shape}(t)$ and $u{\Downarrow}(n-1) \leqslant t{\Downarrow}(n-1)$. The terminology and the $\leqslant$ notation is justified since it extends the dominance order on $\mathrm{Std}(\lambda)$ for each fixed $\lambda \in P(n)$. For example, for $n = 2$, we have $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} < \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$, and for $n = 3$, we have $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} < \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} < \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$.

We remark that in [2] this order was used in the context of the representation theory of symmetric groups, while in [3] it was used in the context of combinatorics of permutations.

LEMMA 7.2. *Let $\mu,\lambda \in P(n)$, let $u \in \mathrm{Std}(\mu)$ and $t \in \mathrm{Std}(\lambda)$, and let $\eta = \mathrm{Shape}(u\!\downarrow\! n)$ and $\theta = \mathrm{Shape}(t\!\downarrow\! n)$. Suppose that $\eta \leqslant \theta$ and $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$. Then $\mu \leqslant \lambda$.*

Proof. Let $\mathrm{col}_u(n) = p$ and $\mathrm{col}_t(n) = q$, and assume that $p \leqslant q$. We are given that $\eta \leqslant \theta$, and so $\sum_{m=1}^{l} \theta_m \leqslant \sum_{m=1}^{l} \eta_m$ holds for all l. Hence for all $l \in [1, p-1]$ we have

$$\sum_{m=1}^{l} \lambda_m = \sum_{m=1}^{l} \theta_m \leqslant \sum_{m=1}^{l} \eta_m = \sum_{m=1}^{l} \mu_m,$$

while for all $l \in [p, q-1]$ we have

$$\sum_{m=1}^{l} \lambda_m = \sum_{m=1}^{l} \theta_m < (\eta_p + 1) + \sum_{\substack{m=1\\ m\neq p}}^{l} \eta_m = \sum_{m=1}^{l} \mu_m,$$

and for all $l > q$ we have

$$\sum_{m=1}^{l} \lambda_m = (\theta_q + 1) + \sum_{\substack{m=1\\ m\neq q}}^{l} \theta_m \leqslant (\eta_p + 1) + \sum_{\substack{m=1\\ m\neq p}}^{l} \eta_m = \sum_{m=1}^{l} \mu_m.$$

Hence $\mu \leqslant \lambda$. □

LEMMA 7.3. *Let $\lambda \in P(n)$ and $t \in \mathrm{Std}(\lambda)$. Suppose that $i \in \mathrm{SD}(t)$, and let $p = \mathrm{col}_t(i)$ and $j = \mathrm{col}_t(i+1)$. For all $h \in [1, n-1]$ let $\lambda^{(h)} = \mathrm{Shape}(t\Downarrow h)$ and $\theta^{(h)} = \mathrm{Shape}(s_i t\Downarrow h)$. Then*

$$\sum_{m=1}^{l} \theta_m^{(i)} = \begin{cases} \sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \lambda_m^{(i+1)} = \sum_{m=1}^{l} \lambda_m^{(i-1)} & \text{if } l < j \\ \sum_{m=1}^{l} \lambda_m^{(i)} + 1 = \sum_{m=1}^{l} \lambda_m^{(i+1)} = \sum_{m=1}^{l} \lambda_m^{(i-1)} + 1 & \text{if } j \leqslant l < p \\ \sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \lambda_m^{(i+1)} - 1 = \sum_{m=1}^{l} \lambda_m^{(i-1)} + 1 & \text{if } p < l \end{cases} \tag{5}$$

and

$$\sum_{m=1}^{l} \lambda_m^{(i)} = \begin{cases} \sum_{m=1}^{l} \theta_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i+1)} = \sum_{m=1}^{l} \theta_m^{(i-1)} & \text{if } l < j \\ \sum_{m=1}^{l} \theta_m^{(i)} - 1 = \sum_{m=1}^{l} \theta_m^{(i+1)} - 1 = \sum_{m=1}^{l} \theta_m^{(i-1)} & \text{if } j \leqslant l < p \\ \sum_{m=1}^{l} \theta_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i+1)} - 1 = \sum_{m=1}^{l} \theta_m^{(i-1)} + 1 & \text{if } p < l. \end{cases} \tag{6}$$

Proof. The results given by Eq. (5) and Eq. (6) are readily obtained from the following formulae

$$\theta_m^{(i)} = \begin{cases} \lambda_m^{(i)} + 1 = \lambda_m^{(i+1)} = \lambda_m^{(i-1)} + 1 & \text{if } m = j \\ \lambda_m^{(i)} - 1 = \lambda_m^{(i+1)} - 1 = \lambda_m^{(i-1)} & \text{if } m = p \\ \lambda_m^{(i)} = \lambda_m^{(i+1)} = \lambda_m^{(i-1)} & \text{if } m \neq j, p \end{cases} \tag{7}$$

and

$$\lambda_m^{(i)} = \begin{cases} \theta_m^{(i)} - 1 = \theta_m^{(i+1)} - 1 = \theta_m^{(i-1)} & \text{if } m = j \\ \theta_m^{(i)} + 1 = \theta_m^{(i+1)} = \theta_m^{(i-1)} + 1 & \text{if } m = p \\ \theta_m^{(i)} = \theta_m^{(i+1)} = \theta_m^{(i-1)} & \text{if } m \neq j, p, \end{cases} \tag{8}$$

respectively. □

LEMMA 7.4. *Let $\mu,\lambda \in P(n)$, let $u \in \mathrm{Std}(\mu)$ and $t \in \mathrm{Std}(\lambda)$. Suppose that $i \in \mathrm{SD}(u) \cap \mathrm{SD}(t)$. Then $u \leqslant t$ if and only if $s_i u \leqslant s_i t$.*

Proof. Let $j = \mathrm{col}_t(i+1)$, let $p = \mathrm{col}_t(i)$, let $k = \mathrm{col}_u(i+1)$ and let $q = \mathrm{col}_u(i)$. For all $h \in [1,n]$ let $\lambda^{(h)} = \mathrm{Shape}(t \Downarrow h)$, let $\theta^{(h)} = \mathrm{Shape}(s_i t \Downarrow h)$, let $\mu^{(h)} = \mathrm{Shape}(u \Downarrow h)$ and let $\eta^{(h)} = \mathrm{Shape}(s_i u \Downarrow h)$.

Suppose that $u \leqslant t$. Since $s_i u$ and $s_i t$ differ from u and t respectively only in the positions of i and $i+1$, we have $\mu^{(h)} = \eta^{(h)}$ and $\lambda^{(h)} = \theta^{(h)}$ for all $h \neq i$. But since $\mu^{(h)} \leqslant \lambda^{(h)}$ for all h by our assumption, it follows that $\eta^{(h)} \leqslant \theta^{(h)}$ for all $h \neq i$. Hence to show that $s_i u \leqslant s_i t$ it suffices to show that $\eta^{(i)} \leqslant \theta^{(i)}$. Let $l \in \mathbb{Z}^+$ be arbitrary.

Case 1.
Suppose that $l \geqslant k$. By Lemma 7.3 applied to u, we have $\sum_{m=1}^{l} \eta_m^{(i)} = \sum_{m=1}^{l} \mu_m^{(i-1)} + 1$, by the last two formulae of Eq.(5). Since $\mu^{(i-1)} \leqslant \lambda^{(i-1)}$ gives $\sum_{m=1}^{l} \mu_m^{(i-1)} \geqslant \sum_{m=1}^{l} \lambda_m^{(i-1)}$, it follows that $\sum_{m=1}^{l} \eta_m^{(i)} \geqslant \sum_{m=1}^{l} \lambda_m^{(i-1)} + 1$. But by Lemma 7.3 applied to t, in each case in Eq.(5) we have $\sum_{m=1}^{l} \lambda_m^{(i-1)} + 1 \geqslant \sum_{m=1}^{l} \theta_m^{(i)}$. Hence $\sum_{m=1}^{l} \eta_m^{(i)} \geqslant \sum_{m=1}^{l} \theta_m^{(i)}$.

Case 2.
Suppose that $l < k$. By Lemma 7.3 applied to u, we have $\sum_{m=1}^{l} \eta_m^{(i)} = \sum_{m=1}^{l} \mu_m^{(i)} = \sum_{m=1}^{l} \mu_m^{(i+1)}$, by the first formula of Eq. (5). Since $\mu^{(i)} \leqslant \lambda^{(i)}$ and $\mu^{(i+1)} \leqslant \lambda^{(i+1)}$, for each $h \in \{i, i+1\}$ we obtain $\sum_{m=1}^{l} \mu_m^{(h)} \geqslant \sum_{m=1}^{l} \lambda_m^{(h)}$, and hence $\sum_{m=1}^{l} \eta_m^{(i)} \geqslant \sum_{m=1}^{l} \lambda_m^{(h)}$. By Lemma 7.3 applied to t, in each case in Eq.(5) there exists $h \in \{i, i+1\}$ such that $\sum_{m=1}^{l} \lambda_m^{(h)} = \sum_{m=1}^{l} \theta_m^{(i)}$. Hence $\sum_{m=1}^{l} \eta_m^{(i)} \geqslant \sum_{m=1}^{l} \theta_m^{(i)}$.

Conversely, suppose that $s_i u \leqslant s_i t$. As above, it suffices to show that $\mu^{(i)} \leqslant \lambda^{(i)}$. Let $l \in \mathbb{Z}^+$ be arbitrary.

Case 1.
Suppose that $l \geqslant j$. By Lemma 7.3 applied to t, we have $\sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i+1)} - 1$, by the last two formulae of Eq.(6). Since $\eta^{(i+1)} \leqslant \theta^{(i+1)}$ gives $\sum_{m=1}^{l} \eta_m^{(i+1)} \geqslant \sum_{m=1}^{l} \theta_m^{(i+1)}$, it follows that $\sum_{m=1}^{l} \eta_m^{(i+1)} - 1 \geqslant \sum_{m=1}^{l} \lambda_m^{(i)}$. But by Lemma 7.3 applied to u, in each case in Eq.(6) we have $\sum_{m=1}^{l} \mu_m^{(i)} \geqslant \sum_{m=1}^{l} \eta^{(i+1)} - 1$. Hence $\sum_{m=1}^{l} \mu_m^{(i)} \geqslant \sum_{m=1}^{l} \lambda_m^{(i)}$.

Case 2.
Suppose that $l < j$. By Lemma 7.3 applied to t, we have $\sum_{m=1}^{l} \lambda_m^{(i)} = \sum_{m=1}^{l} \theta_m^{(i-1)} = \sum_{m=1}^{l} \theta_m^{(i)}$, by the first formula of Eq. (6). Since $\theta^{(i-1)} \geqslant \eta^{(i-1)}$ and $\theta^{(i)} \geqslant \eta^{(i)}$, for each $h \in \{i-1, i\}$ we obtain $\sum_{m=1}^{l} \theta_m^{(h)} \leqslant \sum_{m=1}^{l} \eta_m^{(h)}$, and hence $\sum_{m=1}^{l} \lambda_m^{(i)} \leqslant \sum_{m=1}^{l} \eta_m^{(h)}$. By Lemma 7.3 applied to u, in each case in Eq.(6) there exists $h \in \{i-1, i\}$ such that $\sum_{m=1}^{l} \eta_m^{(h)} = \sum_{m=1}^{l} \mu_m^{(i)}$. Hence $\sum_{m=1}^{l} \lambda_m^{(i)} \leqslant \sum_{m=1}^{l} \mu_m^{(i)}$. □

DEFINITION 7.5. Let $\lambda, \mu \in P(n)$ and let $1 \leqslant m \leqslant n$. Let $u, v \in \mathrm{Std}(\mu)$ and $t, x \in \mathrm{Std}(\lambda)$, and let $i \in \{1,2\}$. We say there is a *paired $(\leqslant m)$-dual Knuth move of the i-th kind* from (u,t) to (v,x) if there exists $k \leqslant m-1$ such that $u \to^{*i} v$ and $t \to^{*i} x$ are $(\leqslant m)$-dual Knuth moves of index k. When this holds we write $(u,t) \to^{*i} (v,x)$, and call k the index of the paired move.

We have the following equivalence relation on $\mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$.

DEFINITION 7.6. Let $\lambda, \mu \in P(n)$. The *paired $(\leqslant m)$-dual Knuth equivalence relation* is the equivalence relation $\approx_m$ on $\mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ generated by paired $(\leqslant m)$-dual Knuth moves. When $m = n$ we write $\approx$ for $\approx_n$, and call it the paired dual Knuth equivalence relation.

We denote by $C_m(u,t)$ the $\approx_m$ equivalence class that contains (u,t). By Remark 6.29 we see that if $(v,x) \in C_m(u,t)$ then $(v,x) = (wu, wt)$ for some $w \in W_m$; furthermore, $v \uparrow m = u \uparrow m$ and $x \uparrow m = t \uparrow m$.

REMARK 7.7. It is clear that $(u,t) \approx_m (v,x)$ implies $(u,t) \approx_{m'} (v,x)$ whenever $m \leqslant m'$. In particular, $(u,t) \approx_m (v,x)$ implies $(u,t) \approx (v,x)$. For this reason, $C_m(u,t)$ will be called the $(\leqslant m)$-*subclass* of $C(u,t) = C_n(u,t)$.

REMARK 7.8. Let $\lambda \in P(n)$. Since $\mathrm{Std}(\lambda)$ is a single dual Knuth equivalence class, it follows that $C(u,u) = \{(t,t) \mid t \in \mathrm{Std}(\lambda)\}$ holds for all $u \in \mathrm{Std}(\lambda)$.

For example, consider $\mu = (3,1)$ and $\lambda = (2,1,1)$. Then set $\mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ has 9 elements. It is easily shown that there are seven paired dual Knuth equivalence classes, of which two classes have 2 elements and five classes have 1 element only. The two non-trivial classes are $\{(\young(14,2,3), \young(124,3)), (\young(13,2,4), \young(123,4))\}$ and $\{(\young(13,2,4), \young(134,2)), (\young(12,3,4), \young(124,3))\}$.

Let $\mu, \lambda \in P(n)$ and $(u,t), (v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. and suppose that $(v,x) = (s_i u, s_i t)$ for some $i \in [1, n-1]$. If $i \in \mathrm{SD}(u) \cap SD(t)$ then $u \leqslant t$ if and only if $v \leqslant x$, by Lemma 7.4, and it follows by interchanging the roles of (u,t) and (v,x) that the same is true if $i \in \mathrm{SA}(u) \cap SA(t)$. In particular, if there is a paired dual Knuth move from (u,t) to (v,x) or from (v,x) to (u,t) then $u \leqslant t$ if and only if $v \leqslant x$. An obvious induction now yields the following result.

PROPOSITION 7.9. *Let $\mu, \lambda \in P(n)$. Let $(u,t), (v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ and suppose that $(u,t) \approx (v,x)$. Then $u \leqslant t$ if and only if $v \leqslant x$.*

Let $\mu, \lambda \in P(n)$ and $(u,t), (v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. and suppose that $(v,x) = (s_i u, s_i t)$ for some $i \in [1, n-1]$. If $i \in \mathrm{SD}(u) \cap SD(t)$ then $l(v) - l(x) = (l(u) - 1) - (l(t) - 1) = l(u) - l(t)$, and if $i \in \mathrm{SA}(u) \cap \mathrm{SA}(t)$ then $l(v) - l(x) = (l(u) + 1) - (l(t) + 1) = l(u) - l(t)$. In particular, $l(v) - l(x) = l(u) - l(t)$ if there is a paired dual Knuth move from (u,t) to (v,x) or from (v,x) to (u,t). It clearly follows that $l(x) - l(v)$ is constant for all $(v,x) \in C(u,t)$. Hence we obtain the following result.

PROPOSITION 7.10. *Let $\mu, \lambda \in P(n)$. Let $(u,t), (v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ and suppose that $(u,t) \approx (v,x)$. Then $u \leqslant_{\mathsf{L}} v$ if and only if $t \leqslant_{\mathsf{L}} x$.*

Proof. Since $(u,t) \approx (v,x)$ there exists $w \in W_m$ such that $v = wu$ and $x = wt$. By the definition of the left weak order it follows that $u \leqslant_{\mathsf{L}} v$ if and only if $l(v) - l(u) = l(w)$, and $t \leqslant_{\mathsf{L}} x$ if and only if $l(x) - l(t) = l(w)$. Since $(u,t) \approx (v,x)$ implies that $l(v) - l(u) = l(x) - l(t)$, the result follows. $\square$

DEFINITION 7.11. Let $\mu, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. If $j \in [1,n]$ and $u \Downarrow j = t \Downarrow j$ then we say that the pair (u,t) is *j-restrictable*.

REMARK 7.12. It is clear that the set $R(u,t) = \{\, j \in [1,n] \mid (u,t) \text{ is } j\text{-restrictable}\}$ is always nonempty, since $1 \in R(u,t)$. Moreover, $R(u,t) = [1,k]$ for some $k \in [1,n]$.

DEFINITION 7.13. Let $\mu, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. We shall call the number k satisfying $R(u,t) = [1,k]$ the *restriction number* of the pair (u,t). If k is the restriction number of (u,t) then we say that (u,t) is *k-restricted*.

REMARK 7.14. With (u,t) as above, the restriction number of (u,t) is at least 1 and at most n. If $k \in [1,n]$ then (u,t) is k-restricted if and only if it is k-restrictable and not $(k+1)$-restrictable. If (u,t) is k-restricted then $k = n$ if and only if $u = t$, and if $k < n$ then $\mathrm{col}_u(k+1) \neq \mathrm{col}_t(k+1)$ and $\mathrm{row}_u(k+1) \neq \mathrm{row}_t(k+1)$.

LEMMA 7.15. *Let $\mu, \lambda \in P(n)$, and let $u \in \mathrm{Std}(\mu)$ and $t \in \mathrm{Std}(\lambda)$. If $n < 4$ then $\mathrm{D}(u) = \mathrm{D}(t)$ implies $u = t$.*

Proof. This is trivially proved by listing all the standard tableaux. $\square$

DEFINITION 7.16. Let $\mu, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. We say that the pair (u,t) is *favourable* if the restriction number of (u,t) lies in $\mathrm{D}(u) \oplus \mathrm{D}(t)$, the symmetric difference of the descent sets of u and t.

REMARK 7.17. Let $\mu, \lambda \in P(n)$, and suppose that $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ is k-restricted. Then no element of $[1,k-1]$ can belong to $\mathrm{D}(u) \oplus \mathrm{D}(t)$, since $u \Downarrow k = t \Downarrow k$ implies that $\mathrm{D}(u) \cap [1,k-1] = \mathrm{D}(t) \cap [1,k-1]$. So if (u,t) is favourable then $k = \min(\mathrm{D}(u) \oplus \mathrm{D}(t))$, and if (u,t) is not favourable then $k < \min(\mathrm{D}(u) \oplus \mathrm{D}(t))$.

Let $\mu, \lambda \in P(n)$, and let $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. Let i be the restriction number of (u,t), and suppose that $i \neq n$. Let $w = u \Downarrow i = t \Downarrow i \in \mathrm{Std}(\xi)$, where $\xi = \mathrm{Shape}(w)$, and let also $(g,p) = u^{-1}(i+1)$ and $(h,q) = t^{-1}(i+1)$, the boxes of u and t that contain $i+1$. Thus (g,p) and (h,q) are ξ-addable, and $(g,p) \neq (h,q)$ since (u,t) is not $(i+1)$-restrictable. Clearly there is at least one ξ-removable box (d,m) that lies between (g,p) and (h,q) (in the sense that either $g > d \geqslant h$ and $p \leqslant m < q$, or $h > d \geqslant g$ and $q \leqslant m < p$), and note that $i \in \mathrm{D}(u) \oplus \mathrm{D}(t)$ if and only if the ξ-removable box $w^{-1}(i)$ is such a box.

With (d,m) as above, suppose that $w' \in \mathrm{Std}(\xi)$ satisfies $w'(d,m) = i$. Since $\mathrm{Std}(\xi)$ is a single dual Knuth equivalence class there must be a sequence of dual Knuth moves of index at most $i-1$ taking w to w'. This same sequence of dual Knuth moves takes (u,t) to (v,x), where v satisfies $v \Downarrow i = w'$ and $v \uparrow i = u \uparrow i$, and x satisfies $x \Downarrow i = w'$ and $x \uparrow i = t \uparrow i$. Thus (v,x) is i-restricted and favourable, and $(v,x) \approx_i (u,t)$.

We denote by $F(u,t)$ the set of all (v,x) obtained by the above construction, as (d,m) and w' vary. Clearly every $(v,x) \in F(u,t)$ is k-restricted and favourable, and satisfies$(v,x) \approx_i (u,t)$. Note also that $(u,t) \in F(u,t)$ if and only if (u,t) is favourable.

Since $\mathrm{col}_v(i+1) = \mathrm{col}_u(i+1)$ and $\mathrm{col}_x(i+1) = \mathrm{col}_t(i+1)$, we can now deduce the following result.

LEMMA 7.18. *Let $\mu, \lambda \in P(n)$ and let $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ with $u \neq t$. Let i be the restriction number of (u,t), and assume that $i \notin \mathrm{D}(u) \oplus \mathrm{D}(t)$. Let $(v,x) \in F(u,t)$. Then either $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$ and $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, this alternative occurring in the case that $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$, or else $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \{i\} \cup (\mathrm{D}(t) \smallsetminus \mathrm{D}(u))$ and $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ (in the case that $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$).*

Proof. The construction of (v,x) is given in the preamble above. Since (v,x) and (u,t) are both i-restricted, $\mathrm{D}(v) \cap [1,i-1] = \mathrm{D}(x) \cap [1,i-1]$ and $\mathrm{D}(u) \cap [1,i-1] = \mathrm{D}(t) \cap [1,i-1]$. That is, $(\mathrm{D}(v) \oplus \mathrm{D}(x)) \cap [1,i-1] = (\mathrm{D}(u) \oplus \mathrm{D}(t)) \cap [1,i-1] = \varnothing$. Furthermore, since $v \uparrow i = u \uparrow i$ and $x \uparrow i = t \uparrow i$ it follows that $(\mathrm{D}(v) \smallsetminus \mathrm{D}(x)) \cap [i+1,n-1] = (\mathrm{D}(u) \smallsetminus \mathrm{D}(t)) \cap [i+1,n-1]$ and $(\mathrm{D}(x) \smallsetminus \mathrm{D}(v)) \cap [i+1,n-1] = (\mathrm{D}(t) \smallsetminus \mathrm{D}(u)) \cap [i+1,n-1]$. It remains to observe that if $p = \mathrm{col}_v(i+1) \leqslant m = \mathrm{col}_v(i) = \mathrm{col}_x(i) < q = \mathrm{col}_x(i+1)$ then $i \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$, while if $q \leqslant m < p$ then $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$. $\square$

LEMMA 7.19. *Let $\mu, \lambda \in P(n)$ and let $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. Assume that the restriction number of (u,t) lies in $\mathrm{D}(u) \oplus \mathrm{D}(t)$, and let $(v,x) \in F(u,t)$. Then $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$.*

Proof. The proof is the same as the proof of Lemma 7.18, except that it can be seen now that $i \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ if $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$ if $i \in \mathrm{D}(t) \smallsetminus \mathrm{D}(u)$. $\square$

LEMMA 7.20. *Let $\mu, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$, and i the restriction number of (u,t). Suppose that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $i < j$, where $j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$. Let $(v,x) \in F(u,t)$. If $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ then $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \varnothing$, while if $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$ then $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \{i\}$. In the former case $\mathrm{D}(v) \cap \{i,j\} = \{i,j\}$ and $\mathrm{D}(x) \cap \{i,j\} = \varnothing$, while in the latter case $\mathrm{D}(v) \cap \{i,j\} = \{j\}$ and $\mathrm{D}(x) \cap \{i,j\} = \{i\}$.*

Proof. Since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ we have $\mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t) \neq \varnothing$; so $j = \min(\mathrm{D}(u) \oplus \mathrm{D}(t))$, and since $j > i$, we have $i \notin \mathrm{D}(u) \oplus \mathrm{D}(t)$, hence $(v,x) \in F(u,t)$ satisfies the further properties specified in Lemma 7.18.

If $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ then Lemma 7.18 gives $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \varnothing$, since $\mathrm{D}(t) \smallsetminus \mathrm{D}(u) = \varnothing$ by hypothesis. In particular, since $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, we see that $\mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ contains both i and j.

If $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$ then Lemma 7.18 combined together with $\mathrm{D}(t) \smallsetminus \mathrm{D}(u) = \varnothing$ gives $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \{i\}$. In particular it follows that $j \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ and $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$. □

Let $\Gamma = \Gamma(C, \mu, \tau)$ be a W_n-molecular graph, and let Λ be the set of molecule types for Γ. For each $\lambda \in \Lambda$ let m_λ be the number of molecules of type λ in Γ, and $\mathcal{I}_\lambda$ some indexing set of cardinality m_λ. As in Remark 6.41, the vertex set of Γ can be expressed in the form

$$C = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda},$$

where $C_{\alpha,\lambda} = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ for each $\alpha \in \mathcal{I}_\lambda$, and the simple edges of Γ are the pairs $\{c_{\beta,u}, c_{\alpha,t}\}$ such that $\alpha = \beta \in \mathcal{I}_\lambda$ for some $\lambda \in \Lambda$ and $u, t \in \mathrm{Std}(\lambda)$ are related by a dual Knuth move.

Now let $\lambda, \mu \in \Lambda$, and let $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$, so that $c_{\alpha,t}$ and $c_{\beta,u}$ are vertices of Γ. Suppose that $\mathrm{D}(u) \smallsetminus \mathrm{D}(t) \neq \varnothing$, and let $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$.

Suppose that there exist $i < j$ and $(v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ such that (u,t) and (v,x) are related by a paired $(\leqslant i)$-dual Knuth move. Then $j \in \mathrm{D}(u{\uparrow}i) \smallsetminus \mathrm{D}(t{\uparrow}i)$, since $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j > i$. Thus $j \in \mathrm{D}(v{\uparrow}i) \smallsetminus \mathrm{D}(x{\uparrow}i)$, since $(v,x) \approx_i (u,t)$ gives $v{\uparrow}i = u{\uparrow}i$ and $x{\uparrow}i = t{\uparrow}i$. Hence $j \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$. Moreover, since (u,t) and (v,x) are related by a paired $(\leqslant i)$-dual Knuth move, there are $k, l \leqslant i-1$ with $|k-l| = 1$ such that

$$\begin{aligned} \mathrm{D}(x) \cap \{k,l,j\} &= \{k\}, \qquad & \mathrm{D}(v) \cap \{k,l,j\} &= \{k,j\}, \\ \mathrm{D}(t) \cap \{k,l,j\} &= \{l\}, \qquad & \mathrm{D}(u) \cap \{k,l,j\} &= \{l,j\}, \end{aligned}$$

and it follows from Proposition 5.13 that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.

More generally, suppose that $i < j$ and $(v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ satisfy $(v,x) \approx_i (u,t)$, so that for some $m \in \mathbb{N}$ there exist (u_0,t_0), (u_1,t_1), $\ldots$, (u_m,t_m) in $\mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$, with (u_{h-1},t_{h-1}) and (u_h,t_h) related by a paired $(\leqslant i)$-dual Knuth move for each $h \in [1,m]$, and $(u_0,t_0) = (u,t)$ and $(u_m,t_m) = (v,x)$. Applying the argument in the preceding paragraph and a trivial induction, we deduce that $j \in \mathrm{D}(u_h) \smallsetminus \mathrm{D}(t_h)$ and $\mu(c_{\beta,u_h}, c_{\alpha,t_h}) = \mu(c_{\beta,u}, c_{\alpha,t})$ for all $h \in [0,m]$. Thus we obtain the following result.

LEMMA 7.21. *Let Γ be a W_n-molecular graph. Using the notation as above, let $\lambda, \mu \in \Lambda$, and let $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$. Suppose that $\mathrm{D}(u) \smallsetminus \mathrm{D}(t) \neq \varnothing$, and let $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$. Then for all $i < j$ and all $(v,x) \in C_i(u,t)$ we have $j \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.*

COROLLARY 7.22. *Let Γ be a W_n-molecular graph as above. Let $\lambda \in \Lambda$, and $u, t \in \mathrm{Std}(\lambda)$, and suppose that $u = s_j t > t$ for some $j \in [1, n-1]$. Then $\mu(c_{\alpha,u}, c_{\alpha,t}) = 1$, for all $\alpha \in \mathcal{I}_\lambda$.*

Proof. Since $t < s_j t = u$, it follows from Remark 6.23 that if $\mathrm{D}(t) \nsubseteq \mathrm{D}(u)$ then there is a dual Knuth move from t to u, and $\{c_{\alpha,u}, c_{\alpha,t}\}$ is a simple edge. Thus $\mu(c_{\alpha,u}, c_{\alpha,t}) = 1$ in this case, and so we may assume that $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$.

Since $u = s_j t$ it is clear that $t{\downarrow}j = u{\downarrow}j$, and hence $\mathrm{D}(t) \cap [1, j-2] = \mathrm{D}(u) \cap [1, j-2]$. If $j-1 \in \mathrm{D}(u)$ then $j-1 \in \mathrm{D}(t)$, as $\mathrm{col}_t(j-1) = \mathrm{col}_u(j-1) \geqslant \mathrm{col}_u(j) \geqslant \mathrm{col}_u(j+1) = \mathrm{col}_t(j)$. Moreover, since $t < s_j t$ gives $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, it follows that $j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$. Note also that $j-1$ is the restriction number of (u,t).

Writing i for $j-1$, we see that u and t satisfy the hypotheses of Lemma 7.20, since $i < j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$. Since $\mathrm{col}_t(i+1) < \mathrm{col}_u(i+1)$, it follows that $(u,t) \approx_i (v,x)$, where $(v,x) \in F(u,t)$ satisfies $\mathrm{D}(v) \cap \{i,j\} = \{j\}$ and $\mathrm{D}(x) \cap \{i,j\} = \{i\}$. Since $(u,t) \approx_i (v,x)$ there exists $w \in W_i$ with $v = wu$ and $x = wt$, and since $j > i$ it follows that $s_j w = ws_j$. Thus $s_j x = s_j wt = ws_j t = wu = v$. Furthermore $s_j x > x$, since $j \notin \mathrm{D}(x)$, and $\mathrm{D}(x) \nsubseteq \mathrm{D}(v)$ since $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$. So there is a dual Knuth move indexed by j from x to v, and so $\{c_{\alpha,v}, c_{\alpha,x}\}$ is a simple edge. Thus $\mu(c_{\alpha,v}, c_{\alpha,x}) = 1$, and so $\mu(c_{\alpha,u}, c_{\alpha,t}) = 1$ by Lemma 7.21. □

LEMMA 7.23. *Let Γ be a W_n-molecular graph as above. Let μ, $\lambda \in \Lambda$, let $u \in \mathrm{Std}(\mu)$ and let $t \in \mathrm{Std}(\lambda)$. Suppose that $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$ and $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ for some $\beta \in \mathcal{I}_\mu$ and $\alpha \in \mathcal{I}_\lambda$. Suppose further that the restriction number of (u,t) is $i < n-2$. Then $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$, and $(u,t) \approx_i (v,x)$ for some $(v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ such that $\mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$ and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$.*

Proof. Since clearly $u \neq t$, the set $F(u,t)$ is defined and nonempty. Let $(v,x) \in F(u,t)$. Then it follows by Lemmas 7.20 and 7.21 that $(u,t) \approx_i (v,x)$ and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Moreover, if $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ then Lemma 7.20 gives $\mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$. Thus it remains to show that $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$. Suppose otherwise. Then Lemma 7.20 shows that $n-1 \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$ and $i \in \mathrm{D}(x) \smallsetminus \mathrm{D}(v)$, and now the W-Compatibility Rule says that i and $n-1$ must be joined by a bond in the Coxeter diagram of W_n. This contradicts the assumption that $i < n-2$. □

LEMMA 7.24. *Suppose that $u, t \in \mathrm{Std}(n)$ are such that the restriction number of (u,t) is $n-1$ and $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$. Then $\mathrm{col}_u(n) < \mathrm{col}_t(n)$, $\mathrm{Shape}(u) < \mathrm{Shape}(t)$, and $u < t$.*

Proof. Clearly $n \geqslant 2$. Since $u{\Downarrow}(n-1) = t{\Downarrow}(n-1)$ we have $\mathrm{Shape}(u{\downarrow}n) = \mathrm{Shape}(t{\downarrow}n)$, and since $n-1 \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ we have $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) = \mathrm{col}_t(n-1) < \mathrm{col}_t(n)$. Hence $\mathrm{Shape}(u) < \mathrm{Shape}(t)$ by Lemma 7.2, and $u < t$ by Definition 7.1. □

LEMMA 7.25. *Let Γ be a W_n-molecular graph as above. Let μ, $\lambda \in \Lambda$, let $u \in \mathrm{Std}(\mu)$, and let $t \in \mathrm{Std}(\lambda)$. Suppose that $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$ and that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ for some $\beta \in \mathcal{I}_\mu$ and $\alpha \in \mathcal{I}_\lambda$, and suppose that the restriction number of (u,t) is $n-2$. Then $(u,t) \approx_{n-2} (v,x)$ for some $(v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ with $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$, and either $u < t$ and $\mathrm{D}(v) = \{n-2, n-1\} \cup \mathrm{D}(x)$ (in the case $\mathrm{col}_u(n-1) < \mathrm{col}_t(n-1)$), or else $(\lambda, \alpha) = (\mu, \beta)$ and $u = s_{n-1}t > t$, and $\mu(c_{\beta,u}, c_{\alpha,t}) = 1$ (in the case $\mathrm{col}_t(n-1) < \mathrm{col}_u(n-1)$).*

Proof. Clearly $n \geqslant 3$. Since (u,t) is $(n-2)$-restricted, we have $u{\Downarrow}(n-2) = t{\Downarrow}(n-2)$ and $\mathrm{col}_u(n-1) \neq \mathrm{col}_t(n-1)$. Observe that (u,t) satisfies the hypotheses of Lemma 7.20 with $i = n-2$ and $j = n-1$. Thus letting $(v,x) \in F(u,t)$, it follows that $(u,t) \approx_{n-2} (v,x)$, and furthermore, $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ by Lemma 7.21.

Case 1.
Suppose that $\mathrm{col}_u(n-1) < \mathrm{col}_t(n-1)$. Then since $\mathrm{Shape}(u{\Downarrow}(n-2)) = \mathrm{Shape}(t{\Downarrow}(n-2))$ it follows from Lemma 7.2 that $\mathrm{Shape}(u{\Downarrow}(n-1)) < \mathrm{Shape}(t{\Downarrow}(n-1))$. Furthermore, since $n-1 \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, it follows that $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) < \mathrm{col}_t(n-1) < \mathrm{col}_t(n)$. Hence $\mu < \lambda$ by Lemma 7.2, and $u < t$ by Definition 7.1. Moreover, since $\mathrm{col}_u(n-1) < \mathrm{col}_t(n-1)$ and $\mathrm{D}(u) \smallsetminus \mathrm{D}(t) = \{n-1\}$, it follows from Lemma 7.18 that $\mathrm{D}(v) = \mathrm{D}(x) \cup \{n-2, n-1\}$.

Case 2.
Suppose that $\mathrm{col}_t(n-1) < \mathrm{col}_u(n-1)$. Lemma 7.18 gives $\mathrm{D}(x) \cap \{n-2, n-1\} = \{n-2\}$ and $\mathrm{D}(v) \cap \{n-2, n-1\} = \{n-1\}$, and since $\mu(c_{\beta,v}, c_{\alpha,x}) \neq 0$ it follows from W_n-Simplicity Rule that $\mu(c_{\beta,v}, c_{\alpha,x}) = 1$. Hence $\mu(c_{\beta,u}, c_{\alpha,t}) = 1$. Moreover, since $\{c_{\beta,v}, c_{\alpha,x}\}$ is a simple edge, it follows that from Theorem 6.40 and Remark 6.41 that $\lambda = \mu$ and $\alpha = \beta$. Hence $u = s_{n-1}t$, since $u{\Downarrow}(n-2) = t{\Downarrow}(n-2)$, and $u > t$ since $\mathrm{col}_t(n-1) < \mathrm{col}_u(n-1)$. □

REMARK 7.26. Let Γ be a W_n-molecular graph as above. Let $\mu, \lambda \in \Lambda$, let $u \in \mathrm{Std}(\mu)$, and let $t \in \mathrm{Std}(\lambda)$. Suppose that $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$ and that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ for some $\beta \in \mathcal{I}_\mu$ and $\alpha \in \mathcal{I}_\lambda$. Let i be the restriction number of (u,t), and note that $i \leqslant n-1$. If $i < n-2$ then $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ by Lemma 7.24, and if $i = n-1$ then $\mathrm{col}_u(n) < \mathrm{col}_t(n)$ by Lemma 7.23. In the remaining case $i = n-2$, if $\mathrm{col}_u(n-1) > \mathrm{col}_t(n-1)$ then $u = s_{n-1}t > t$ by Lemma 7.25. Thus it can be deduced that if $u \neq s_{n-1}t > t$ then $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$.

REMARK 7.27. Let Γ be a W_n-molecular graph as above. Let $\mu, \lambda \in \Lambda$, and let $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ and $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$. Let $j = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, and i the restriction number of (u,t), and note that $i \leqslant j$.

Let $K = S \setminus \{s_{j+1}, \ldots, s_{n-1}\}$, and let $\Gamma_K = \Gamma{\downarrow_K}$, the W_K-graph obtained by restricting Γ to W_K. As in Remark 6.43, for each $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$ we define $\Lambda_{K,\alpha,\lambda}$ to be the set of all $\kappa \in P(j+1)$ such that the molecule of Γ with the vertex set $C_{\alpha,\lambda}$ contains a K-submolecule of type κ, and let $\mathcal{I}_{K,\alpha,\lambda,\kappa}$ index these submolecules. Let $\Lambda_K = \bigcup_{\alpha,\lambda} \Lambda_{K,\alpha,\lambda}$, the set of molecule types for Γ_K, and for each $\kappa \in \Lambda_K$ let $\mathcal{I}_{K,\kappa} = \bigsqcup_{\{(\alpha,\lambda) \mid \kappa \in \Lambda_{K,\alpha,\lambda}\}} \mathcal{I}_{K,\alpha,\lambda,\kappa}$. For each $\beta \in \mathcal{I}_{K,\kappa}$ we write $\{c'_{\beta,u} \mid u \in \mathrm{Std}(\kappa)\}$ for the vertex set of the corresponding K-submolecule of Γ.

Let $v = u\Downarrow(j+1)$ and $x = t\Downarrow(j+1)$, and write $\eta = \mathrm{Shape}(v)$ and $\theta = \mathrm{Shape}(x)$. By Remark 6.43, we can identify the vertex $c_{\beta,u}$ of Γ_K with $c'_{\delta,v}$ for some $\delta \in \mathcal{I}_{K,\beta,u,\eta}$, and the vertex $c_{\alpha,t}$ of Γ_K with $c'_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{K,\alpha,\lambda,\theta}$. It is clear that $\mathrm{D}(v) = \mathrm{D}(x) \cup \{j\}$, so it follows that $\mu(c'_{\delta,v}, c'_{\gamma,x}) = \underline{\mu}(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Moreover, since $i \leqslant j$, the restriction number of (v,x) is also i. Thus Lemma 7.24, Lemma 7.23 and Lemma 7.25 are applicable to Γ_K and (v,x) subject to hypotheses $i = j$, $i < j-1$ and $i = j-1$, respectively. In particular, Remark 7.26 says that if $u \neq s_{i+1}t > t$ then $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$.

We end this section with two technical lemmas that will be used throughout the rest of the paper. They are concerned with descent sets and the lexicographic order on standard tableaux. The first of these lemmas is needed for future applications of the polygon rule. Recall that if $t \in \mathrm{Std}(n)$ and $i \in [1, n-1]$ then $s_i t \in \mathrm{Std}(n)$ if and only if either $i \in \mathrm{SA}(t)$ or $i \in \mathrm{SD}(t)$.

LEMMA 7.28. *Let $t \in \mathrm{Std}(n)$ and let $i \in \mathrm{A}(t)$ and $j \in \mathrm{SD}(t)$. Put $v = s_j t$.*

(i) *Suppose that $i < j-1$. Then $i \notin \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$.*
Additionally, if $i \in \mathrm{SA}(v)$ then $i \in \mathrm{D}(s_i v)$ and $j \notin \mathrm{D}(s_i v)$.

(ii) *Suppose that $i = j-1$ and $\mathrm{col}_t(j+1) > \mathrm{col}_t(j-1)$. Then $j-1 \notin \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$.*
Additionally, if $j-1 \in \mathrm{SA}(v)$ then $j-1 \in \mathrm{D}(s_{j-1}v)$ and $j \notin \mathrm{D}(s_{j-1}v)$.

(iii) *Suppose that $i = j-1$ and $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$. Then $j-1 \in \mathrm{SD}(v)$. Writing $w = s_{j-1}v$, we have $j-1 \in \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$, and $j-1 \notin \mathrm{D}(w)$ and $j \notin \mathrm{D}(w)$.*
Additionally, if $j \in \mathrm{SA}(w)$, then $j-1 \in \mathrm{SA}(s_j w)$, and we have $j \in \mathrm{D}(s_j w)$ and $j-1 \notin \mathrm{D}(s_j w)$, and $j-1 \in \mathrm{D}(s_{j-1}s_j w)$ and $j \notin \mathrm{D}(s_{j-1}s_j w)$.

Proof. (i) Since $v = s_j t$ and $j \in \mathrm{SD}(t)$, it follows that $j \in \mathrm{SA}(v)$, whence $j \notin \mathrm{D}(v)$. Since v is obtained from t by switching the positions of j and $j+1$, and since $i+1 < j$, it follows that i and $i+1$ have the same row and column index in v as they have in t. Since $i \notin \mathrm{D}(t)$, this shows that $i \notin \mathrm{D}(v)$.

If $i \in \mathrm{SA}(v)$ then $s_i v$ is standard and $i \in \mathrm{D}(s_i v)$. Since $s_i v$ is obtained from v by switching i and $i+1$, and since $j > i+1$, it follows that j and $j+1$ have the same row and column index in $s_i v$ as in v. Since $j \notin \mathrm{D}(v)$ it follows that $j \notin \mathrm{D}(s_i v)$.

(ii) Since $v = s_j t$ and $j \in \mathrm{SD}(t)$, it follows that $j \in \mathrm{SA}(v)$, whence $j \notin \mathrm{D}(v)$. Now since $\mathrm{col}_v(j-1) = \mathrm{col}_t(j-1)$ and $\mathrm{col}_v(j) = \mathrm{col}_t(j+1)$, and $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$ by assumption, it follows that $\mathrm{col}_v(j-1) < \mathrm{col}_v(j)$. That is, $j-1 \notin \mathrm{D}(v)$.

If $j-1 \in \mathrm{SA}(v)$ then $s_{j-1}v$ is standard and $j-1 \in \mathrm{D}(s_{j-1}v)$. Since $j-1$ and j are both ascents of v, we have $\mathrm{col}_v(j-1) < \mathrm{col}_v(j) < \mathrm{col}_v(j+1)$, and since $s_{j-1}v$ is obtained from v

by switching $j-1$ and j, we have $\mathrm{col}_{s_{j-1}v}(j)=\mathrm{col}_v(j-1)$ and $\mathrm{col}_{s_{j-1}v}(j+1)=\mathrm{col}_v(j+1)$, and it follows that $\mathrm{col}_{s_{j-1}v}(j)<\mathrm{col}_{s_{j-1}v}(j+1)$. Thus $j\notin \mathrm{D}(s_{j-1}v)$.

(iii) As in (i) and (ii) we have $j\notin \mathrm{D}(v)$. The assumption $\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)$ gives $\mathrm{col}_v(j)<\mathrm{col}_v(j-1)$, and so $j-1\in \mathrm{SD}(v)$. Hence $w=s_{j-1}v$ is standard, and $j-1\in \mathrm{SA}(w)$. Since $\mathrm{col}_w(j+1)=\mathrm{col}_v(j+1)=\mathrm{col}_t(j)$ and $\mathrm{col}_w(j)=\mathrm{col}_v(j-1)=\mathrm{col}_t(j-1)$, and since $j-1\in \mathrm{A}(t)$ by assumption, it follows that $j\in \mathrm{A}(w)$. Thus $j-1\in \mathrm{D}(v)$ and $j\notin \mathrm{D}(v)$, and $j-1\notin \mathrm{D}(w)$ and $j\notin \mathrm{D}(w)$, as required.

If $j\in \mathrm{SA}(w)$ then $s_jw\in \mathrm{Std}(\lambda)$. Since $j-1$ and j are both strong ascents of w, we have $\mathrm{row}_w(j-1)>\mathrm{row}_w(j)>\mathrm{row}_w(j+1)$, and since s_jw is obtained from w by switching j and $j+1$, we have $\mathrm{row}_{s_jw}(j-1)=\mathrm{row}_w(j-1)$ and $\mathrm{row}_{s_jw}(j)=\mathrm{row}_w(j+1)$, and it follows that $\mathrm{row}_{s_jw}(j-1)>\mathrm{row}_{s_jw}(j)$. Thus $j-1\in \mathrm{SA}(s_jw)$.

Now $j-1\in \mathrm{SA}(s_jw)$ gives $j-1\notin \mathrm{D}(s_jw)$, and gives $j-1\in \mathrm{D}(s_{j-1}s_jw)$. Similarly, $j\in \mathrm{SA}(w)$ gives $j\in \mathrm{D}(s_jw)$. Finally, the assumption $\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)$ gives $\mathrm{col}_{s_{j-1}s_jw}(j)=\mathrm{col}_{s_jw}(j-1)=\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)=\mathrm{col}_{s_jw}(j+1)=\mathrm{col}_{s_{j-1}s_jw}(j+1)$, and $j\notin \mathrm{D}(s_{j-1}s_jw)$. □

Recall from Remark 6.12 that if $\lambda\in P(n)$ and $u,t\in \mathrm{Std}(\lambda)$ then $t>_{\mathrm{lex}}u$ if and only if there exists $l\in[1,n]$ such that $\mathrm{col}_t(l)<\mathrm{col}_u(l)$ and $t\uparrow l=u\uparrow l$.

LEMMA 7.29. *Let $\lambda\in P(n)$ and $0\leqslant i\leqslant n-1$. Let $t,t'\in \mathrm{Std}(\lambda)$ satisfy $t\uparrow i=t'\uparrow i$. Let $j\in \mathrm{SD}(t)$ and put $v=s_jt$, and suppose that $i\in \mathrm{A}(t)$ and $i<j$. Then $v<_{\mathrm{lex}}t'$, and the following all hold.*

(i) *If $i\in \mathrm{SA}(v)$ then $s_iv\in \mathrm{Std}(\lambda)$ and $s_iv<_{\mathrm{lex}}t'$.*
(ii) *If $y\in \mathrm{Std}(\lambda)$ and $y<v$ then $y<_{\mathrm{lex}}t'$.*
(iii) *Suppose that $i=j-1$ and that $\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)$, and let $w=s_{j-1}v$. Then $w\in \mathrm{Std}(\lambda)$ and $w<_{\mathrm{lex}}t'$. If $j\in \mathrm{SA}(w)$ then $s_{j-1}s_jw\in \mathrm{Std}(\lambda)$ and $s_{j-1}s_jw<_{\mathrm{lex}}t'$.*
(iv) *Suppose that $i=j-1$ and that $\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)$, and let $w=s_{j-1}v$. Let $x\in \mathrm{Std}(\lambda)$ be such that $x<w$ and $\mathrm{D}(x)$ contains exactly one of $j-1$ or j, and let y be the $(j-1)$-neighbour of x (see Definition 6.39). Then $y<_{\mathrm{lex}}t'$.*

Proof. Since $j\in \mathrm{SD}(t)$ we have $t>s_jt=v$, and hence $t>_{\mathrm{lex}}v$ by Corollary 6.14. Indeed, $\mathrm{col}_t(j+1)<\mathrm{col}_t(j)=\mathrm{col}_v(j+1)$ and $t\uparrow(j+1)=v\uparrow(j+1)$. Since $t\uparrow i=t'\uparrow i$ and $j+1>i$ it follows that $\mathrm{col}_{t'}(j+1)<\mathrm{col}_v(j+1)$ and $t'\uparrow(j+1)=v\uparrow(j+1)$, giving $t'>_{\mathrm{lex}}v$.

(i) The assumption $i\in \mathrm{SA}(v)$ gives $s_iv\in \mathrm{Std}(\lambda)$, and since $j+1>i+1$ it follows that $\mathrm{col}_{t'}(j+1)<\mathrm{col}_v(j+1)=\mathrm{col}_{s_iv}(j+1)$ and $t'\uparrow(j+1)=s_iv\uparrow(j+1)$. So $t'>_{\mathrm{lex}}s_iv$.

(ii) If $y<v$ then $y<_{\mathrm{lex}}v$, by Corollary 6.14, and since $v<_{\mathrm{lex}}t'$ this gives $y<_{\mathrm{lex}}t'$.

(iii) Since $\mathrm{col}_v(j)=\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)=\mathrm{col}_v(j-1)$, we have $j-1\in \mathrm{SD}(v)$, and since this gives $s_{j-1}v\in \mathrm{Std}(\lambda)$, an argument similar to that for (i) yields $w<_{\mathrm{lex}}t'$.

If $j\in \mathrm{SA}(w)$ then $s_jw\in \mathrm{Std}(\lambda)$. Since $j-1\in \mathrm{SA}(s_jw)$ by Lemma 7.28 (iii), we have $s_{j-1}s_jw\in \mathrm{Std}(\lambda)$. Since $\mathrm{col}_t(j+1)<\mathrm{col}_t(j-1)=\mathrm{col}_{s_{j-1}s_jw}(j+1)$, and since $j+1>i+1$, it follows that $\mathrm{col}_{t'}(j+1)<\mathrm{col}_{s_{j-1}s_jw}(j+1)$ and $t'\uparrow j+1=s_{j-1}s_jw\uparrow j+1$. This gives $t'>_{\mathrm{lex}}s_{j-1}s_jw$.

(iv) There are two cases to consider.

Case 1.
Suppose that $\mathrm{D}(x)\cap\{j-1,j\}=\{j-1\}$ and $\mathrm{D}(y)\cap\{j-1,j\}=\{j\}$. Then either $y=s_jx>x$ or $y=s_{j-1}x<x$.

Suppose first that $y=s_jx>x$. Since $x<s_jx=y$ and $w<s_jw$, the assumption $x<w$ gives $y<s_jw$ by Lemma 7.4. Since $s_jw<s_{j-1}s_jw$, it follows that $y<s_{j-1}s_jw$, and this gives $y<_{\mathrm{lex}}s_{j-1}s_jw$ by Corollary 6.14. But $s_{j-1}s_jw<_{\mathrm{lex}}t'$ by (v), this yields $y<_{\mathrm{lex}}t'$.

Suppose now that $y=s_{j-1}x<x$. Since $x<w$, we have $y<w$, whence $y<_{\mathrm{lex}}w$ by Corollary 6.14. But $w<_{\mathrm{lex}}t'$ by (v), this yields $y<_{\mathrm{lex}}t'$.

Case 2.
Suppose that $\mathrm{D}(x) \cap \{j-1, j\} = \{j\}$ and $\mathrm{D}(y) \cap \{j-1, j\} = \{j-1\}$. Then either $y = s_j x < x$ or $y = s_{j-1}x > x$.

Suppose first that $y = s_{j-1}x > x$. Since $x < s_{j-1}x = y$ and $w < s_{j-1}w = v$, the assumption $x < w$ gives $y < v$ by Lemma 7.4. Thus $y <_{\mathrm{lex}} t'$ by (ii).

Suppose now that $y = s_j x < x$. Since $x < w$, we have $y < w$, whence $y <_{\mathrm{lex}} w$ by Corollary 6.14. But $w <_{\mathrm{lex}} t'$ by (v), this yields $y <_{\mathrm{lex}} t'$. □

8. Ordered admissible W-graphs in type A

Let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible W_n-graph, and let $\Lambda \subseteq P(n)$ be the set of molecule types for Γ. As in Remark 6.41 we write

$$C = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda},$$

where for each $\lambda \in \Lambda$ the set $\mathcal{I}_\lambda$ indexes the molecules of Γ of type λ, and for each $\lambda \in \Lambda$ and $\alpha \in \mathcal{I}_\lambda$ the set $C_{\alpha,\lambda} = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ is the vertex set of a molecule of type λ. Fix $\lambda \in \Lambda$ and let $C'_\lambda = C \setminus \left(\bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda}\right)$, the set of vertices of Γ belonging to molecules of type different from λ. We define $\mathrm{Ini}_\lambda(\Gamma)$ to be the set of $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ such that there exists an arc from $c_{\alpha,t}$ to some vertex in C'_λ. That is,

$$\mathrm{Ini}_\lambda(\Gamma) = \Big\{ (\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda) \,\Big|\, \mu(c_{\beta,u}, c_{\alpha,t}) \neq 0 \text{ for some } (\beta, u) \in \bigsqcup_{\mu \in \Lambda \setminus \{\lambda\}} (\mathcal{I}_\mu \times \mathrm{Std}(\mu)) \Big\}.$$

For each $\alpha \in \mathcal{I}_\lambda$ we also define $\mathrm{Ini}_{(\alpha,\lambda)}(\Gamma) = \{ t \in \mathrm{Std}(\lambda) \mid (\alpha, t) \in \mathrm{Ini}_\lambda(\Gamma) \}$.

Note that, by Theorem 5.8, Γ satisfies the W_n-Compatibility Rule, the W_n-Simplicity Rule, the W_n-Bonding Rule and the W_n-Polygon Rule.

Now since Γ satisfies the W_n-Simplicity Rule, it follows by Definition 5.4 that whenever vertices $c_{\alpha,t}$ and $c_{\beta,u}$ belong to different molecules and $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, we must have $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $\mu(c_{\alpha,t}, c_{\beta,u}) = 0$.

Suppose that $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$. We define $t_{\Gamma,\lambda}$ to be the element of $\bigcup_{\alpha \in \mathcal{I}_\lambda} \mathrm{Ini}_{(\alpha,\lambda)}(\Gamma)$ that is minimal in the lexicographic order on $\mathrm{Std}(\lambda)$. If Γ is clear from the context then we will simply write t_λ for $t_{\Gamma,\lambda}$.

We make the following definition.

DEFINITION 8.1. Let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible W_n-graph, and let

$$C = \bigsqcup_{\lambda \in \Lambda} \bigsqcup_{\alpha \in \mathcal{I}_\lambda} C_{\alpha,\lambda},$$

as above. Then Γ is said to be *ordered* if for all vertices $c_{\alpha,t}$ and $c_{\beta,u}$ with $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, either $u < t$ (in the extended dominance order) or else $\alpha = \beta$ and $u = st > t$ for some $s \in S_n$.

Note that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ implies that $\mathrm{D}(u) \nsubseteq \mathrm{D}(t)$. In particular, since $S_1 = \varnothing$, the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ can never be satisfied in the case $n = 1$. Thus it is vacuously true that any W_1-graph is ordered.

Our objective in this section is to prove Theorem 8.18, which states that all admissible W_n-graphs are ordered. The proof will proceed by induction on n.

REMARK 8.2. In particular, it will follow from Theorem 8.18 that the Kazhdan–Lusztig W_n-graph corresponding to the regular representation of $\mathcal{H}(W_n)$ is ordered in the sense of Definition 8.1. In this case the vertex set of $\Gamma = (C, \mu, \tau)$ is $C = W_n$, the set of molecule types is $\Lambda = P(n)$, for each $\lambda \in P(n)$ the set of molecules of type λ is indexed by $\mathcal{I}_\lambda = \mathrm{Std}(\lambda)$, and for each $\lambda \in \Lambda$ and $x \in \mathcal{I}_\lambda$ the set $C_{x,\lambda}$ consists of those $w \in W_n$ such that $Q(w) = x$, where $Q(w)$ is the recording tableau in the Robinson–Schensted process.

Now let $v, w \in W_n$ and put $\mathrm{RS}(w) = (t,x) \in \mathrm{Std}(\lambda)^2$ and $\mathrm{RS}(v) = (u,y) \in \mathrm{Std}(\nu)^2$, where $\lambda, \mu \in P(n)$. The conclusion of Theorem 8.18, applied in this case, is that if $\mu(v,w) \neq 0$ and $\tau(v) \nsubseteq \tau(w)$ then either $u < t$ or else $\mu = \lambda$ and $(u,y) = (st,x)$ for some $s \in S_n$.

If Γ is replaced by $\Gamma^{\mathrm{o}} = (C, \mu, \tau^{\mathrm{o}})$, then since $\mathrm{RS}(w^{-1}) = (x,t)$ and $\mathrm{RS}(v^{-1}) = (y,u)$ by Theorem 6.20, the conclusion of Theorem 8.18 is that if $\mu(v,w) \neq 0$ and $\tau^{\mathrm{o}}(v) \nsubseteq \tau^{\mathrm{o}}(w)$ then either $y < x$ or else $\mu = \lambda$ and $(u,y) = (t,sx)$ for some $s \in S_n$.

Thus, in particular, if $\mu(v,w) \neq 0$ and $\tau(v) \nsubseteq \tau(w)$ or $\tau^{\mathrm{o}}(v) \nsubseteq \tau^{\mathrm{o}}(w)$ then $\mu \leqslant \lambda$.

It follows from the definition of the preorder $\preceq_{\mathsf{LR}}$ (in Section 4 above) that if $v, w \in W_n$ and $v \preceq_{\mathsf{LR}} w$ then there is a sequence of elements $z_0 = v, z_1, \ldots, z_{m-1}, z_m = w$ such that $\mu(z_{i-1}, z_i) \neq 0$ and $\bar\tau(z_{i-1}) \nsubseteq \bar\tau(z_i)$ for each $i \in [1,m]$. Since $\bar\tau(z_{i-1}) \nsubseteq \bar\tau(z_i)$ is equivalent to $\tau(z_{i-1}) \nsubseteq \tau(z_i)$ or $\tau^{\mathrm{o}}(z_{i-1}) \nsubseteq \tau^{\mathrm{o}}(z_i)$, it follows that $\mu \leqslant \lambda$.

We now commence the proof of Theorem 8.18. We assume that n is a positive integer and that all admissible W_m-graphs are ordered for $1 \leqslant m < n$. We let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible W_n-graph, and use the notation introduced in the preamble to this section: Λ is the set of molecule types of Γ, and for each $\lambda \in \Lambda$ the set $\mathcal{I}_\lambda$ indexes the molecules of type λ. We fix $K = S_n \smallsetminus \{s_{n-1}\}$ and $L = S_n \smallsetminus \{s_1\}$, and we let $\Gamma_K = \Gamma{\downarrow_K}$ and $\Gamma_L = \Gamma{\downarrow_L}$, the W_K-graph and W_L-graph obtained by restricting Γ to W_K and W_L. Since $|K| = |L| = n-1$, the inductive hypothesis tells us that Γ_K and Γ_L are ordered.

By Remark 6.43, the set of molecule types for Γ_K is $\Lambda_K = \bigcup_{\alpha,\lambda} \Lambda_{K,\alpha,\lambda}$, where $\Lambda_{K,\alpha,\lambda}$ is the set of all $\kappa \in P(n-1)$ such that the molecule with the vertex set $C_{\alpha,\lambda}$ contains a K-submolecule of type κ, and for each $\kappa \in \Lambda_K$, the indexing set for those molecules of type κ is $\mathcal{I}_{K,\kappa} = \bigsqcup_{\{\alpha,\lambda \mid \kappa \in \Lambda_{K,\alpha,\lambda}\}} \mathcal{I}_{K,\alpha,\lambda,\kappa}$, where $\mathcal{I}_{K,\alpha,\lambda,\kappa}$ indexes the K-submolecules of type κ in the molecule with the vertex set $C_{\alpha,\lambda}$. The vertex set of Γ_K is

$$C = \bigsqcup_{\kappa \in \Lambda_K} \{c'_{\gamma,x} \mid (\gamma,x) \in \mathcal{I}_{K,\kappa} \times \mathrm{Std}(\kappa)\}.$$

By Remark 6.43, the set of molecule types for Γ_L is $\Lambda_L = \bigcup_{\alpha,\lambda} \Lambda_{L,\alpha,\lambda}$, where $\Lambda_{L,\alpha,\lambda}$ is the set of all $\theta \in P(n-1)$ such that the molecule with the vertex set $C_{\alpha,\lambda}$ contains an L-submolecule of type θ, and for each $\theta \in \Lambda_L$, the indexing set for those molecules of type θ is $\mathcal{I}_{L,\theta} = \bigsqcup_{\{\alpha,\lambda \mid \theta \in \Lambda_{L,\alpha,\lambda}\}} \mathcal{I}_{L,\alpha,\lambda,\theta}$, where $\mathcal{I}_{L,\alpha,\lambda,\theta}$ indexes the L-submolecules of type θ in the molecule with the vertex set $C_{\alpha,\lambda}$. The vertex set of Γ_L is

$$C = \bigsqcup_{\theta \in \Lambda_L} \{c''_{\varepsilon,y} \mid (\varepsilon,y) \in \mathcal{I}_{L,\theta} \times \mathrm{Std}(\kappa)\}.$$

LEMMA 8.3. *Let $\mu, \lambda \in \Lambda$ with $\mu \leqslant \lambda$, and let $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ and $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. Let $j = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and assume that $j < n-1$. Then $u < t$ unless $\alpha = \beta$ and $u = s_j t > t$.*

Proof. Since j is at least 1, the requirement that $n-1 > j$ implies that $n \geqslant 3$. Let $v = u{\Downarrow}(n-1)$ and $x = t{\Downarrow}(n-1)$, and write $\eta = \mathrm{Shape}(v)$ and $\theta = \mathrm{Shape}(x)$. We shall need the restriction of Γ to W_K constructed earlier.

By Remark 6.43, we can identify the vertex $c_{\beta,u}$ of Γ_K with $c'_{\delta,v}$ for some $\delta \in \mathcal{I}_{K,\beta,u,\eta}$, and the vertex $c_{\alpha,t}$ of Γ_K with $c'_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{K,\alpha,\lambda,\theta}$. Now since $j \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j < n-1$, we have $j \in (\mathrm{D}(u) \cap [1,n-2]) \smallsetminus (\mathrm{D}(t) \cap [1,n-2]) = \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$, and it follows that $\mu(c'_{\delta,v}, c'_{\gamma,x}) = \underline{\mu}(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Since Γ_K is ordered, we have either $v < x$ or $\gamma = \delta$ and $v = s_i x > x$ for some $i \in [1,n-2]$. In the former case, since $\mathrm{Shape}(u) = \mu \leqslant \lambda = \mathrm{Shape}(t)$ by hypothesis and since $u{\Downarrow}(n-1) = v < x = t{\Downarrow}(n-1)$, we have $u < t$ by the remark following Definition 7.1 In the latter case, we have $\alpha = \beta$, and since it is clear that $i = j$, it follows that $u = s_j t > t$. ☐

PROPOSITION 8.4. *Let $\mu, \lambda \in \Lambda$ with $\mu \leqslant \lambda$, and suppose that $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ and $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Then $u < t$ unless $\alpha = \beta$ and $u = s_i t > t$ for some $i \in [1, n-1]$.*

Proof. Since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $\mathrm{D}(u) \nsubseteq \mathrm{D}(t)$. If $\mathrm{D}(t) \nsubseteq \mathrm{D}(u)$ then the W_n-Simplicity Rule shows that $\{c_{\beta,s}, c_{\alpha,t}\}$ is a simple edge, thus $\alpha = \beta$ and $u = s_i t$ for some $i \in [1, n-1]$. Thus we may assume that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. If $\min(\mathrm{D}(s) \smallsetminus \mathrm{D}(t)) < n-1$ then the result is given by Lemma 8.3.

It remains to consider the case $\mathrm{D}(u) = \mathrm{D}(t) \cup \{n-1\}$. Let i be the restriction number of the pair (u,t) and note that $i < n$ by Remark 7.14. If $i = n-1$ or $i = n-2$ then the results are given by Lemma 7.24 and Lemma 7.25, respectively. We may assume that $i < n-2$. It follows by Lemma 7.23 that $(u,t) \approx_i (v,x)$ for some $(v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ satisfying the condition $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,v}, c_{\alpha,x}) \neq 0$ and $\mathrm{D}(x) \subsetneqq \mathrm{D}(v) = \mathrm{D}(x) \cup \{i, n-1\}$. Since it is clear that $v \neq s_k x > x$ for all $k \in [1, n-1]$, Lemma 8.3 shows that $v < x$, equivalently, $u < t$ by Proposition 7.9. □

The following definitions are motivated by the structure of $t_{\Gamma,\lambda}$.

DEFINITION 8.5. Let $\mu, \lambda \in P(n)$. Let $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$, and let k be the restriction number of (u,t). The pair (u,t) is said to be *k-minimal*, and t is said to be *k-minimal* with respect to u, if $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $t \Uparrow k$ is k-critical, and $t \downarrow k$ is the minimal tableau of its shape.

For example, $\left(\begin{smallmatrix}1&2&5\\3\\4\end{smallmatrix}, \begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}\right)$ is 2-restricted but not 2-minimal, $\left(\begin{smallmatrix}1&2&3\\4\\5\end{smallmatrix}, \begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}\right)$ is 4-minimal, and $\left(\begin{smallmatrix}1&2&5\\3\\4\end{smallmatrix}, \begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}\right)$ is 3-minimal.

Let $\mu, \lambda \in P(n)$, and let $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$. Let k be the restriction number of (u,t), and assume that $k \in [1, n-1]$ (or, equivalently, $u \neq t$). Recall that

$$F(u,t) = \{(v,x) \in C_k(u,t) \mid v^{-1}(k) = x^{-1}(k) \text{ lies between } u^{-1}(k+1) \text{ and } t^{-1}(k+1)\}.$$

DEFINITION 8.6. Let $\mu, \lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ with $u \neq t$, and let k be the restriction number of (u,t). We define $A(u,t) = \{(v,x) \in F(u,t) \mid \mathrm{col}_x(k) = \mathrm{col}_t(k+1) - 1\}$ and call any element of $A(u,t)$ an *approximate* of (u,t).

Note that $A(u,t) \neq \varnothing$ if and only if $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$.

REMARK 8.7. Let u, t as above and assume that $A(u,t) \neq \varnothing$. It is clear from Definition 8.6 that every approximate (v,x) of (u,t) is k-restricted and satisfies $(v,x) \approx_k (u,t)$, and that $A(u,t) = \{(v,x) \in C_k(u,t) \mid \mathrm{col}_v(k) = \mathrm{col}_x(k) = \mathrm{col}_t(k+1) - 1\}$, which is a (non-empty) $(k-1)$-subclass of $C_k(u,t)$. It follows that if $\kappa = \mathrm{Shape}(x \Downarrow (k-1)) = \mathrm{Shape}(v \Downarrow (k-1))$, where $(v,x) \in A(u,t)$, then the bijection from $\mathrm{Std}(\kappa)$ to $A(u,t)$ given by $w \mapsto (v,x)$ such that $v \Downarrow (k-1) = x \Downarrow (k-1) = w$ transfers the partial order $\leqslant$ from $\mathrm{Std}(\kappa)$ to $A(u,t)$. The minimal element of $A(u,t)$, called the *minimal approximate* of (u,t), is the pair (v,x) given by $w = \tau_\kappa$, and the maximal element of $A(u,t)$, called the *maximal approximate* of (u,t), is the pair (v,x) given by $w = \tau^\kappa$.

REMARK 8.8. Let $\mu, \lambda \in \Lambda$, and let $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ and $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. Let $k \in [1, n-1]$ be the restriction number of the pair (u,t). Let $l = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, and let $L = \{s_1, \ldots, s_l\}$. Remark 7.27 applied to $\Gamma \downarrow_L$, the W_L-graph obtained by restricting Γ to W_L, shows that if $u \neq s_{k+1} t > t$ then $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$. Thus if $u \neq s_{k+1} t > t$ then the set $A(u,t) \neq \varnothing$. In particular, if $\alpha \neq \beta$ then, since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ implies that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, and since $\mu = \lambda$ and $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ imply that $u < t$ by Proposition8.4, the condition $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ is sufficient for the set $A(u,t) \neq \varnothing$.

LEMMA 8.9. *Let $\mu,\lambda \in P(n)$ and $(u,t) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ with $u \neq t$, and let k be the restriction number of (u,t). Assume that $A(u,t) \neq \varnothing$, and let $(v,x) \in A(u,t)$. Then (v,x) is k-restricted and satisfies $(v,x) \approx_k (u,t)$. If, moreover, $D(t) \subsetneq \mathrm{D}(u)$, then we have $\mathrm{D}(x) \subsetneq \mathrm{D}(v)$ with $k = \min(\mathrm{D}(v) \setminus \mathrm{D}(x))$.*

Proof. It follows from Remark 8.7 that (v,x) is k-restricted and satisfies $(v,x) \approx_k (u,t)$. It remains to show that $\mathrm{D}(x) \subsetneq \mathrm{D}(v)$ with $k = \min(\mathrm{D}(v) \setminus \mathrm{D}(x))$ if $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$. So suppose further that $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$

Since $\mathrm{col}_u(k+1) < \mathrm{col}_t(k+1)$ (since $A(u,t) \neq \varnothing$), Lemma 7.18 and Lemma 7.19 show that $\mathrm{D}(x) \setminus \mathrm{D}(v) = \mathrm{D}(t) \setminus \mathrm{D}(u)$ and $\mathrm{D}(v) \setminus \mathrm{D}(x) \supseteq \mathrm{D}(u) \setminus \mathrm{D}(t)$. Since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$, this yields $\mathrm{D}(x) \subsetneq \mathrm{D}(v)$. Now since (v,x) is favourable, we have $k = \min(\mathrm{D}(v) \oplus \mathrm{D}(x)$ by Remark 7.17, and it follows that $k = \min(\mathrm{D}(v) \setminus \mathrm{D}(x))$. ☐

LEMMA 8.10. *Let $\mu,\lambda \in \Lambda$ with $\mu \neq \lambda$, and let $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ and $(\alpha,t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. Let k be the restriction number of (u,t). Then $A(u,t) \neq \varnothing$, and for all $(v,x) \in A(u,t)$ the following three conditions hold:*

(i) *$(v,x) \approx (u,t)$,*
(ii) *$\mathrm{D}(x) \subsetneq \mathrm{D}(v)$ and $k = \min(\mathrm{D}(v) \setminus \mathrm{D}(x))$,*
(iii) *$\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.*

Proof. We have $A(u,t) \neq \varnothing$ by Remark 8.8. Let $(v,x) \in A(u,t)$, then by Lemma 8.9, we have $(u,t) \approx_k (v,x)$, whence $(u,t) \approx (v,x)$. Moreover, since $\mu \neq \lambda$, and since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, we have $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$, and it follows by Lemma 8.9 that $\mathrm{D}(x) \subsetneq \mathrm{D}(v)$ with $k = \min(\mathrm{D}(v) \setminus \mathrm{D}(x))$. It remains to show that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$. Let $l = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$. Since (u,t) is k-restricted, we have $k \leqslant l$.

Suppose first that $k < l$. Since $(u,t) \approx_k (v,x)$ and $k < l \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, the result follows from Lemma 7.21.

Suppose now that $k = l = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, in particular, this shows that $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$. Let $w = t \Downarrow k = u \Downarrow k \in \mathrm{Std}(\xi)$, where $\xi = \mathrm{Shape}(w)$, and let $(h,q) = t^{-1}(k+1)$ and $(g,p) = t^{-1}(k)$, the boxes of t that contain $k+1$ and k respectively. Since $k \notin \mathrm{D}(t)$, it follows that $g \geqslant h$ and $p < q$. If $p = q-1$ then we have $(u,t) \in A(u,t)$. Since $(u,t) \approx_{k-1} (v,x)$ by Remark 8.7 and since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, we have $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$ by Lemma 7.21. Thus, we can assume that $p < q-1$.

Let $(d,m) = (\xi_{q-1}, q-1)$, and note that the assumption implies that $g > d \geqslant h > \xi_q$. It is clear that (g,p) and (d,m) are ξ-removable, and $(g,p) \neq (d,m)$. Let $\zeta \in P(k-2)$ such that $[\zeta] = [\xi] \setminus \{(g,p),(d,m)\}$, and let (i,j) be a ζ-removable box that lies between (g,p) and (d,m) (in the sense that $g > i \geqslant d$ and $p \leqslant j < m$). We can choose $w' \in \mathrm{Std}(\xi)$ with $w'(i,j) = k-2$, $w'(d,m) = k-1$ and $w'(g,p) = k$, and define (u_1,t_1) by $u_1 \Downarrow k = w'$ and $u_1 \uparrow k = u \uparrow k$, and $t_1 \Downarrow k = w'$ and $t_1 \uparrow k = t \uparrow k$. Since

$$(u_1,t_1) \in \{(v,x) \in C_k(u,t) \mid \mathrm{col}_v(k) = \mathrm{col}_x(k) = g\},$$

the $(k-1)$-subclass of $C_k(u,t)$, it follows that $(u,t) \approx_{k-1} (u_1,t_1)$. and it follows by Lemma 7.21 that $\mu(c_{\beta,u_1}, c_{\alpha,t_1}) = \mu(c_{\beta,u}, c_{\alpha,t})$ and $k \in \mathrm{D}(u_1) \setminus \mathrm{D}(t_1)$.

Since $p < m$, we have $k-1 \in \mathrm{SD}(w')$, and so $k-1 \in \mathrm{SD}(u_1)$ and $k-1 \in \mathrm{SD}(t_1)$. It follows that we can define $(u_2,t_2) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ by $u_2 = s_{k-1}u_1$ and $t_2 = s_{k-1}t_1$, and we note that $w'' = u_2 \Downarrow k = t_2 \Downarrow k = s_{k-1}w'$, and $u_2 \uparrow k = u_1 \uparrow k$ and $t_2 \uparrow k = t_1 \uparrow k$. Since $w' = s_{k-1}w'' > w''$, and $\mathrm{D}(w'') \cap \{k-2,k-1\} = \{k-2\}$ and $\mathrm{D}(w') \cap \{k-2,k-1\} = \{k-1\}$, it follows that there is a dual Knuth move (of the first kind) of index $k-1$ taking w'' to w'. As the same dual Knuth move takes (u_2,t_2) to (u_1,t_1), we have (u_1,t_1) and (u_2,t_2) are related by a paired $\leqslant k$-dual Knuth relation indexed by $(k-1)$. Moreover, it can be verified easily that

$$\begin{aligned}
&\mathrm{D}(t_1) \cap \{k-2,k-1,k\} = \{k-1\}, \quad \mathrm{D}(u_1) \cap \{k-2,k-1,k\} = \{k-1,k\},\\
&\mathrm{D}(t_2) \cap \{k-2,k-1,k\} = \{k-2\}, \quad \mathrm{D}(u_2) \cap \{k-2,k-1,k\} = \{k-2,k\}. \qquad ,
\end{aligned}$$

and it follows by Proposition 5.13 that $\mu(c_{\beta,u_2}, c_{\alpha,t_2}) = \mu(c_{\beta,u_1}, c_{\alpha,t_1})$.

Finally, since it is clear that $(u_2,t_2) \in A(u,t)$, we have $(u_2,t_2) \approx_{k-1} (v,x)$ by Remark 8.7, and since $k \in \mathrm{D}(u_2) \setminus \mathrm{D}(t_2)$, it follows by Lemma 7.21 that $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u_2}, c_{\alpha,t_2})$. Thus, $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$. as required. □

PROPOSITION 8.11. *Let $\lambda \in \Lambda$ satisfy the condition that* $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$*, and let $t' = t_{\Gamma,\lambda}$. Let $(\alpha,t') \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta,u') \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$, where $\mu \in \Lambda \setminus \{\lambda\}$, satisfy the condition that $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Let k be the restriction number of (u',t'), and let $(u,t) \in A(u',t')$. Then $t \Uparrow k$ is k-critical. Thus if (u,t) is the minimal approximate of (u',t') then t is k-minimal with respect to u.*

Proof. Lemma 8.10 tells us that $(u,t) \approx (u',t')$, that $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $k = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, and that $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Note that $\mathrm{col}_t(k+1) = \mathrm{col}_t(k) + 1$, since (u,t) is an approximate of (u',t') (see Definition 8.6). Thus, by Remark 6.18, to show that $t \Uparrow k$ is k-critical it will suffice to show that every $j \in \mathrm{D}(t)$ with $j > k+1$ is in $\mathrm{WD}(t)$, and that either $\mathrm{col}_t(k+2) = \mathrm{col}_t(k)$ or $k+1 \notin \mathrm{SD}(t)$. We do both parts of this by contradiction.

For the first part, suppose that $j > k+1$ and $j \in \mathrm{SD}(t)$. Since $j \in \mathrm{D}(t)$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, we have $j \in \mathrm{D}(t) \cap \mathrm{D}(u)$, and since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, it follows that $j \in \mathrm{D}(t)$ and $k \notin \mathrm{D}(t)$, and $k, j \in \mathrm{D}(u)$. Let $v = s_j t$, which is standard since $j \in \mathrm{SD}(t)$. It follows by Lemma 7.28 (i) that $k, j \notin \mathrm{D}(v)$. Moreover, since $\mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.22, and since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $(c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating directed path of type (j,k).

Recall that if $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ then $\mu(c_{\beta,u}, c_{\alpha,t}) > 0$, because Γ is admissible. Thus $N^2_{j,k}(\Gamma; v, u) > 0$, whence $N^2_{k,j}(\Gamma; v, u) > 0$, as Γ satisfies the W_n-Bonding Rule. So there exists at least one $\nu \in \Lambda$ and $(\gamma,y) \in \mathcal{I}_\nu \times \mathrm{Std}(\nu)$ such that $(c_{\alpha,v}, c_{\gamma,y}, c_{\beta,u})$ is an alternating directed path of type (k,j). Since $t' \uparrow k = t \uparrow k$ and $k < j-1$, we have $v <_{\mathrm{lex}} t'$ by Lemma 7.29. Thus, if $\nu \neq \lambda$ then we have $(\alpha, v) \in \mathrm{Ini}_\lambda(\Gamma)$ and $v \in \bigcup_{\alpha \in \mathcal{I}_\lambda} \mathrm{Ini}_{(\alpha,\lambda)}(\Gamma)$, and so this contradicts the assumption that $t' = t_{\Gamma,\lambda}$. It follows that $\nu = \lambda$ and $y \in \mathrm{Std}(\lambda)$.

By Proposition 8.4, we must have either $\gamma = \alpha$ and $y = s_k v > v$ or $y < v$. Recall that $s_k v \in \mathrm{Std}(\lambda)$ and $s_k v > v$ if and only if $k \in \mathrm{SA}(v)$. Thus in the case $\gamma = \alpha$ and $y = s_k v > v$, then since $t' \uparrow k = t \uparrow k$ and $k < j-1$ we have $y = s_k v <_{\mathrm{lex}} t'$ by Lemma 7.29 (i), while in the case $y < v$, then since $t' \uparrow k = t \uparrow k$ and $k < j-1$ we have $y <_{\mathrm{lex}} t'$ by Lemma 7.29 (ii). In either case, since $(\gamma, y) \in \mathrm{Ini}_\lambda(\Gamma)$ and $y \in \bigcup_{\alpha \in \mathcal{I}_\lambda} \mathrm{Ini}_{(\alpha,\lambda)}(\Gamma)$, this contradicts the assumption that $t' = t_{\Gamma,\lambda}$.

For the second part, suppose that $k+1 \in \mathrm{SD}(t)$ and $\mathrm{col}_t(k+2) \neq \mathrm{col}_t(k)$.

Case 1.
Suppose that $\mathrm{col}_t(k) < \mathrm{col}_t(k+2)$. Since $(u,t) \in A(u',t')$, we have $\mathrm{col}_t(k) = \mathrm{col}_t(k+1) - 1$, and it follows that $\mathrm{col}_t(k+1) \leqslant \mathrm{col}_t(k+2)$. This contradicts the assumption that $k+1 \in \mathrm{SD}(t)$.

Case 2.
Suppose that $\mathrm{col}_t(k+2) < \mathrm{col}_t(k)$. Since $k+1 \in \mathrm{SD}(t) \subseteq \mathrm{D}(t)$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, it follows that $k+1 \in \mathrm{D}(t) \cap \mathrm{D}(u)$, and since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, it follows that $k+1 \in \mathrm{D}(t)$ and $k \notin \mathrm{D}(t)$, and $k, k+1 \in \mathrm{D}(u)$. Let $v = s_{k+1} t$. Since $k+1 \in \mathrm{SD}(t))$, we have $v \in \mathrm{Std}(\lambda)$. Let $w = s_k v$. Since $k \in \mathrm{SD}(v)$ by Lemma 7.28 (iii), we have $w \in \mathrm{Std}(\lambda)$. Now, it follows by Lemma 7.28 (iii) that $k \in \mathrm{D}(v)$ and $k+1 \notin \mathrm{D}(v)$, and $k \notin \mathrm{D}(w)$ and $k+1 \notin \mathrm{D}(w)$.

Moreover, since $\mu(c_{\alpha,v}, c_{\alpha,w}) = \mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.22, and since it is also true that $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $(c_{\alpha,w}, c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating directed path of type $(k, k+1)$.

As recalled above, if $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ then $\mu(c_{\beta,u}, c_{\alpha,t}) > 0$, because Γ is admissible. Thus $N^3_{k,k+1}(\Gamma; w, u) > 0$, whence $N^3_{k+1,k}(\Gamma; w, u) > 0$, as Γ satisfies the W_n-Bonding Rule.

So there exist $\xi \in \Lambda$ and $(\delta, x) \in \mathcal{I}_\xi \times \mathrm{Std}(\xi)$, and $\nu \in \Lambda$ and $(\gamma, y) \in \mathcal{I}_\nu \times \mathrm{Std}(\nu)$ such that $(c_{\alpha,w}, c_{\delta,x}, c_{\gamma,y}, c_{\beta,u})$ is an alternating directed path of type $(k+1, k)$.

Since $t'\uparrow k = t\uparrow k$, we have $w <_{\mathrm{lex}} t'$ by Lemma 7.29 (iii). Thus, if $\xi \neq \lambda$ then we have $(\alpha, w) \in \mathrm{Ini}_\lambda(\Gamma)$ and $w \in \bigcup_{\alpha\in\mathcal{I}_\lambda} \mathrm{Ini}_{(\alpha,\lambda)}(\Gamma)$, and so this contradicts the assumption that $t' = t_{\Gamma,\lambda}$. It follows that $\xi = \lambda$ and $x \in \mathrm{Std}(\lambda)$.

Since $\mu(c_{\gamma,y}, c_{\delta,x}) \neq 0$, and since $\mathrm{D}(x)\cap\{k,k+1\} = \{k+1\}$ and $\mathrm{D}(y)\cap\{k,k+1\} = \{k\}$, we have $\{c_{\delta,x}, c_{\gamma,y}\}$ is a simple edge by the W_n-Simplicity Rule. Thus $\nu = \lambda$ and $\gamma = \delta$, and y and x are related by a dual Knuth move. We have either $\delta = \alpha$ and $x = s_{k+1}w > w$ or $x < w$ by Proposition 8.4, and y is the unique k-neighbour of x. If $x = s_{k+1}w > w$, then since $x \in \mathrm{Std}(\lambda)$, this is equivalent to $k+1 \in \mathrm{SA}(w)$. It follows by Lemma 7.28 (iii) that $y = s_k x > x$ is the unique k-neighbour of x. In this case, since $t'\uparrow k = t\uparrow k$, we have $y <_{\mathrm{lex}} t'$ by Lemma 7.29 (iii). If $x < w$ and y is the unique k-neighbour of x, then since $t'\uparrow k = t\uparrow k$, we have $y <_{\mathrm{lex}} t'$ by Lemma 7.29 (iv). In either case, since $(\gamma, y) \in \mathrm{Ini}_\lambda(\Gamma)$ and $y \in \bigcup_{\alpha\in\mathcal{I}_\lambda} \mathrm{Ini}_{(\alpha,\lambda)}(\Gamma)$, this contradicts the assumption that $t' = t_{\Gamma,\lambda}$.

If (u,t) is the minimal approximate of (u',t'), then it is clear that t is k-minimal with respect to u in accordance with Definition 8.5. □

COROLLARY 8.12. *Let $\lambda \in \Lambda$ satisfy the condition that $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$, and let $t' = t_\lambda$. Let $(\alpha, t') \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u') \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$, where $\mu \in \Lambda \smallsetminus \{\lambda\}$, satisfy the condition that $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Let k be the restriction number of (u',t'). Then $t_\lambda\uparrow(k+1)$ is minimal and if $k+1 \in \mathrm{SD}(t_\lambda)$ then $\mathrm{col}_{t_\lambda}(k+1) = \mathrm{col}_{t_\lambda}(k+2)+1$.*

Proof. Let $(u,t) \in A(u',t')$. Then $t\Uparrow k$ is k-critical, by Proposition 8.11. Now since $t\uparrow k = t_\lambda\uparrow k$, this shows that $t_\lambda\uparrow(k+1)$ is minimal and if $k+1 \in \mathrm{SD}(t_\lambda)$ then $\mathrm{col}_{t_\lambda}(k+1) = \mathrm{col}_{t_\lambda}(k+2) + 1$. □

LEMMA 8.13. *Let $n \geqslant 2$, and let $\mu, \lambda \in P(n)$. Let $t \in \mathrm{Std}(\lambda)$ and $u \in \mathrm{Std}(\mu)$ and suppose that t is 1-minimal with respect to u. Then $\mu < \lambda$.*

Proof. Since (u,t) is 1-minimal, we have $t(1,1) = u(1,1) = 1$ and $t(1,2) = u(2,1) = 2$. So if $n = 2$, we have $\mu = (2) < (1,1) = \lambda$. We proceed inductively on $n \geqslant 3$. If $t(1,3) = 3$ then since t is 1-minimal, we have $t = \tau_\lambda$, where $\lambda = (1,\dots,1)$. Since $\lambda = \max((P(n),\leqslant))$, and since $\mu_1 > 1 = \lambda_1$, we deduce that $\mu < \lambda$. We may just assume that $t(2,1) = 3, \dots, t(\lambda_1,1) = \lambda_1+1$, and it follows that $2,\dots,\lambda_1 \in \mathrm{D}(t)$. Now since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, we have $2,\dots,\lambda_1 \in \mathrm{D}(u)$, and so, $u(3,1) = 3, \dots, u(\lambda_1+1,1) = \lambda_1+1$. In particular, this shows $\mu_1 > \lambda_1$.

Let $\eta = \mathrm{Shape}(u\Downarrow(n-1))$ and let $\theta = \mathrm{Shape}(t\Downarrow(n-1))$. It is clear that $t\Downarrow(n-1)$ is 1-minimal with respect to $u\Downarrow(n-1)$, whence $\eta < \theta$ by the inductive hypothesis. We shall show that $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$. Suppose to the contrary that $\mathrm{col}_t(n) < \mathrm{col}_u(n)$.

Suppose first that $\mathrm{col}_t(n-1) = 1$ so that $1,3,\dots,n-1$ fill column 1 of t. Since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, we have $1,2,3,\dots,n-1$ fill column 1 of u. Since $\mathrm{col}_u(n) > 1$, we have $u(1,2) = n$, and it follows that $n-1 \in \mathrm{A}(u)$. Now since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, we have $n-1 \in \mathrm{A}(t)$, consequently $\mathrm{col}_t(n) > \mathrm{col}_t(n-1) = 1$. It follows that $\mathrm{col}_t(n) \geqslant 2 = \mathrm{col}_u(n)$, contradicting our assumption.

Suppose now that $\mathrm{col}_t(n-1) > 1$. Let $1 < q = \mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(n)$. Since $\eta \leqslant \theta$, we have $n-1 = \sum_{m=1}^{q} \theta_m \leqslant \sum_{m=1}^{q} \eta_m$, and so, if $i < n$ then $\mathrm{col}_u(i) \leqslant \mathrm{col}_t(n-1)$. It follows that $\mathrm{col}_u(n) \leqslant q+1 = \mathrm{col}_t(n-1)+1 < \mathrm{col}_t(n)+1$, whence $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$, if $n-1 \in \mathrm{WA}(u)$, and $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) \leqslant \mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(n)$, if $n-1 \in \mathrm{D}(u)$. Either case contradicts our assumption.

Since $\eta < \theta$ and $\mathrm{col}_u(n) \leqslant \mathrm{col}_t(n)$, we have $\mu \leqslant \lambda$ by Lemma 7.2, and since $\mu_1 > \lambda_1$, we obtain $\mu < \lambda$. □

LEMMA 8.14. *Let $\lambda \in \Lambda$ satisfy the condition that $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$. Let $(\alpha, t') \in \mathrm{Ini}_\lambda(\Gamma)$ with $t' = t_\lambda$, and let $\mu \in \Lambda \smallsetminus \{\lambda\}$ and $(\beta, u') \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ such that $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Let $(u,t) \in \mathrm{Std}(\mu)\times\mathrm{Std}(\lambda)$, and let $k \geqslant 3$ be the restriction number of (u,t). Suppose that (u,t)*

satisfies

$$t \Downarrow k = u \Downarrow k \ = \ \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & k-2 & k-1 \\ \hline k \\ \cline{1-1} \end{array} \ ,$$

and t satisfies further properties that $\mathrm{col}_t(n) = k-1$, *and u satisfies further properties that* $u(1,k) = n$ *and* $(2,k-1) \notin [\mu]$. *Then* $(u,t) \notin A(u',t')$.

Proof. Assume to the contrary that $(u,t) \in A(u',t')$. By Remark 8.7, both (u',t') and (u,t) have the same restriction number, and $A(u',t')$ consists of $(v,x) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\lambda)$ such that $v \Downarrow (k-1) = x \Downarrow (k-1) \in \mathrm{Std}((1^{k-1}))$, and $v \Uparrow k = u \Uparrow k$ and $x \Uparrow k = t \Uparrow k$. Thus it follows that (u',t') is k-restricted, and $A(u',t') = \{(u,t)\}$.

It follows by Lemma 8.10 that $(u,t) \approx (u',t')$, $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ with $k = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, and $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$, and it follows by Lemma 8.11 that $t \Uparrow k$ is k-critical. Since $\mathrm{col}_t(k+1) = \mathrm{col}_t(k) + 1$, we have $t(2,2) = k+1$, and since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, we have $\mathrm{col}_u(k+1) \leqslant \mathrm{col}_u(k)$, hence $\mathrm{col}_u(k+1) = 1$, and it follows that $u(3,1) = k+1$. Since $u(1,k) = n$, it follows further that $k+1 < n$.

Case 1.
Suppose that $(u,t) = (u',t')$.

Since $k \geqslant 3$, we have $\mathrm{col}_u(k) = 1 < k-1 = \mathrm{col}_u(k-1)$, and so $k-1 \in \mathrm{SD}(u) \subseteq \mathrm{D}(u)$. Let $v = s_{k-1}u$. Since $k-1 \in \mathrm{SD}(u)$, it follows that $v \in \mathrm{Std}(\mu)$ and $k-1 \notin \mathrm{D}(v)$. Since $\mathrm{col}_u(k-2) = k-2 < k-1 = \mathrm{col}_u(k-1)$, it follows that $k-2 \notin \mathrm{D}(u)$. Moreover, since v is obtained from u by switching the positions of $k-1$ and k, and since $k \geqslant 3$, we have $\mathrm{col}_v(k-1) = \mathrm{col}_u(k) = 1 \leqslant k-2 = \mathrm{col}_u(k-2) = \mathrm{col}_v(k-2)$, and so $k-2 \in \mathrm{D}(v)$.Thus there is a dual Knuth move (of the first kind) of index $k-1$ taking v to u, which shows that $\{c_{\beta,u}, c_{\beta,v}\}$ is a simple edge in Γ.

Since $k \geqslant 3$, we have $1 \leqslant k-2 = \mathrm{col}_t(k-2) < k-1 = \mathrm{col}_t(k-1)$, and it follows that $k-2 \notin \mathrm{D}(t)$. Since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, we also have $k \notin \mathrm{D}(t)$. Similarly, since $u \Downarrow k = t \Downarrow k$, we have $k-2 \notin \mathrm{D}(u)$, but since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$, we have $k \in \mathrm{D}(u)$. We have shown that $k-2 \in \mathrm{D}(v)$. Now since $\mathrm{col}_v(k+1) = \mathrm{col}_u(k+1) = 1 < k-1 = \mathrm{col}_u(k-1) = \mathrm{col}_v(k)$, as v is obtained from u by switching the positions of $k-1$ and k, we also have $k \in \mathrm{D}(v)$. Moreover, since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ and $\mu(c_{\beta,v}, c_{\beta,u}) = 1$ (as $\{c_{\beta,u}, c_{\beta,v}\}$ is a simple edge), it follows that $(c_{\alpha,t}, c_{\beta,u}, c_{\beta,v})$ is an alternating directed path of type $(k, k-2)$.

Since $N^2_{k-2,k}(\Gamma; t, v) = N^2_{k,k-2}(\Gamma; t, v)$, as Γ satisfies the W_n-Polygon Rule, and since $N^2_{k,k-2}(\Gamma; t, v) \geqslant \mu(c_{\beta,u}, c_{\alpha,t})$, it follows that $N^2_{k-2,k}(\Gamma; t, v) > 0$, whence there are $\xi \in \Lambda$ and $(\gamma, x) \in \mathcal{I}_\xi \times \mathrm{Std}(\xi)$ such that $(c_{\alpha,t}, c_{\gamma,x}, c_{\beta,u})$ is an alternating directed path of type $(k-2, k)$.

Since $\mathrm{row}_t(k-2) = \mathrm{row}_t(k-1)$, we have $k-2 \in \mathrm{WA}(t)$, and so $k-2 \notin \mathrm{D}(t)$, and since $t \Downarrow k = u \Downarrow k$, we have $k-1 \in \mathrm{D}(t)$, since $k-1 \in \mathrm{D}(u)$. Thus if $k-1 \notin \mathrm{D}(x)$ then $\{c_{\alpha,t}, c_{\gamma,x}\}$ is a simple edge. That is, if $k-1 \notin \mathrm{D}(x)$ then $\alpha = \gamma$ and t and x are related by a dual Knuth move. Moreover, since $s_{k-2}t \notin \mathrm{Std}(\lambda)$, this shows that $x = s_{k-1}t$. But then since $k \geqslant 3$ and since x is obtained from t by switching the positions of $k-1$ and k, we have $\mathrm{col}_x(k+1) = \mathrm{col}_t(k+1) = 2 \leqslant k-1 = \mathrm{col}_t(k-1) = \mathrm{col}_x(k)$, and it follows that $k \in \mathrm{D}(x)$, contradicting the requirement that $k \notin \mathrm{D}(x)$. Thus $k-1 \in \mathrm{D}(x)$.

Since $\mu(c_{\beta,v}, c_{\gamma,x}) \neq 0$, and since $\mathrm{D}(x) \cap \{k-1, k\} = \{k-1\}$ and $\mathrm{D}(v) \cap \{k-1, k\} = \{k\}$, the W_n-Simplicity Rule shows that $\{c_{\beta,v}, c_{\gamma,x}\}$ is a simple edge. Equivalently, $\gamma = \beta$, and x and v are related by a dual Knuth move. Indeed, x is, in this case, the $(k-1)$-neighbour of v. But the $(k-1)$-neighbour of v is $s_k v$, since $\mathrm{col}_v(k+1) = \mathrm{col}_v(k-1) < \mathrm{col}_v(k)$. Therefore, $x = s_k v$.

It can be seen that $(\alpha, t) = (\alpha, t')$ and $(\beta, s_k v)$ satisfy the conditions of Corollary 8.12. Since it is clear that $(s_k v, t') = (s_k v, t)$ is $(k-2)$-restricted, and since $\mathrm{col}_t(k) < \mathrm{col}_t(k-1)$, we have by Corollary 8.12 that $k-1 = \mathrm{col}_t(k-1) = \mathrm{col}_t(k) + 1 = 2$. Thus, $k = 3$, and so $\mathrm{col}_t(n) = 2$. Since $t \uparrow (k-1)$ is the minimal tableau of its shape by Corollary 8.12, we have $\mathrm{col}_t(3) \leqslant \mathrm{col}_t(4) \leqslant \cdots \leqslant \mathrm{col}_t(n)$, and so $\mathrm{col}_t(3) = 1$ and $\mathrm{col}_t(4) = \cdots = \mathrm{col}_t(n) = 2$. Thus

$\lambda_1 = 2$ and $\lambda_2 = n-2$, and since $\lambda_1 \geqslant \lambda_2$, it follows that $n \leqslant 4$. This contradicts the fact that $n > k+1$ (as shown earlier).

Case 2.
Suppose that $(u,t) \neq (u',t')$.

Since $(u,t) \approx_k (u',t')$ by Lemma 8.9, there exists $z \in W_k \smallsetminus \{1\}$ with $u' = zu$ and $t' = zt$. Hence there is an $i \in [1,k-2]$ such that u' and t' satisfy

$$w' = t' \Downarrow k = u' \Downarrow k \ = \ \begin{array}{|c|c|c|c|c|c|} \hline 1 & \cdots & i & i+2 & \cdots & k \\ \hline i+1 \\ \cline{1-1} \end{array} \ ,$$

and, furthermore, $t' \uparrow k = t \uparrow k$ and $u' \uparrow k = u \uparrow k$.

If $k = 3$ then since $(2,2) \notin [\mu]$ and $u(1,3) = n$, we have $\mu_2 = 1$ and $\mu_3 = 1$. Therefore $\mu = (n-2,1,1)$ and the first row of u is $\begin{array}{|c|c|c|} \hline 1 & 2 & n \\ \hline \end{array}$, while $u(2,1) = 3$ and $u(i,1) = i+1$ for $i \in [3,n-2]$. Since $\mathrm{col}_u(n-1) = 1 < 3 = \mathrm{col}_u(n)$, we have $n-1 \notin \mathrm{D}(u)$, and since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ it follows that $n-1 \notin \mathrm{D}(t)$. That is, $\mathrm{col}_t(n-1) < \mathrm{col}_t(n)$. Now $t \uparrow (k+1)$ is the minimal tableau of it shape, by Corollary 8.12, and so

$$\mathrm{col}_t(5) \leqslant \mathrm{col}_t(6) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) < \mathrm{col}_t(n).$$

Thus $\mathrm{col}_t(k+2) = \mathrm{col}_t(5) = \cdots = \mathrm{col}_t(n-1) = 1$ and $\mathrm{col}_t(n) = 2$. Hence $\lambda = (n-3,3)$, the first three rows of t are $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline 5 & n \\ \hline \end{array}$ and $t(i,1) = i+2$ for $i \in [4,n-2]$. Now since $C_k(u,t) = \{(u,t),(s_2u,s_2t)\}$, we have $(u',t') = (s_2u,s_2t)$. It can be verified easily that $\mathrm{D}(s_2u) = \mathrm{D}(s_2t) = \{1,3,\ldots,n-2\}$, and it follows that $\mu(c_{\beta,s_2u}, c_{\alpha,s_2t}) = 0$. This is in contradiction to the assumption that $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Henceforth, we may assume that $k \geqslant 4$.

Since $t' \Downarrow k = u' \Downarrow k$, we have $\mathrm{D}(t') \cap [1,k-1] = \mathrm{D}(u') \cap [1,k-1]$. Since $k \geqslant 4$, one the one hand, we have $\mathrm{col}_{t'}(k+1) = \mathrm{col}_t(k+1) = 2 < k-1 = \mathrm{col}_t(k-1) = \mathrm{col}_{t'}(k)$, and on the other hand, we have $\mathrm{col}_{u'}(k+1) = \mathrm{col}_u(k+1) = 1 < k-1 = \mathrm{col}_u(k-1) = \mathrm{col}_{u'}(k)$. Thus, it follows that $k \in \mathrm{D}(t') \cap \mathrm{D}(u')$, and so $\mathrm{D}(t') \cap [1,k] = \mathrm{D}(u') \cap [1,k]$. Let $l \in \mathrm{D}(u') \smallsetminus \mathrm{D}(t')$, which is not an empty set since $\mathrm{D}(t') \subsetneqq \mathrm{D}(u')$. This shows that $l > k$.

We claim that $i = 1$. Suppose to the contrary that $i > 1$. Now since $i > 1$, it follows that $\mathrm{col}_{w'}(i+1) = 1 \leqslant i-1 = \mathrm{col}_{w'}(i-1) < \mathrm{col}_{w'}(i-1)+1 = \mathrm{col}_{w'}(i)$, and so $i \in \mathrm{SD}(w') \subseteq \mathrm{D}(w')$ and $i-1 \notin \mathrm{D}(w')$. Since $i \in \mathrm{SD}(w')$, we have s_iw' is standard and $i \notin \mathrm{D}(s_iw')$. Moreover, since $\mathrm{col}_{s_iw'}(i) = \mathrm{col}_{w'}(i+1) < \mathrm{col}_{w'}(i) = \mathrm{col}_{s_iw'}(i-1)$, it follows that $i-1 \notin \mathrm{D}(s_iw')$. Thus $s_iw' \rightarrow^{*1} w'$ with the index i, and since the same dual Knuth move takes (s_iu', s_it') to (u',t'), we have $(s_iu', s_it') \approx_k (s',t')$. It follows by Lemma 7.21 that $\mu(c_{\beta,s_iu'}, c_{\alpha,s_it'}) = \mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Since $s_it' < t'$, it follows from Corollary 6.14 that $s_it' <_{lex} t'$. But $(\alpha, s_it') \in \mathrm{Ini}_\lambda(\Gamma)$ and $s_it' \in \bigcup_{\alpha \in \mathcal{I}_\lambda} \mathrm{Ini}_{(\alpha,\lambda)}(\Gamma)$, this contradicts the assumption that $t' = t_\lambda$. Hence, $i = 1$, as claimed.

Let $v = j(u' \uparrow 1)$ and $x = j(t' \uparrow 1)$, and write $\zeta = \mathrm{Shape}(v)$ and $\xi = \mathrm{Shape}(x)$. We shall need the restriction of Γ to W_L constructed earlier.

By Remark 6.43, we can identify the vertex $c_{\beta,u'}$ of Γ_L with $c''_{\delta,v}$ for some $\delta \in \mathcal{I}_{L,\beta,\mu,\zeta}$, and the vertex $c_{\alpha,t'}$ of Γ_L with $c''_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{L,\alpha,\lambda,\xi}$. Note that since $\mu \neq \lambda$, and so $\beta \neq \alpha$, we have $\mathcal{I}_{L,\beta,\mu,\xi} \cap \mathcal{I}_{L,\alpha,\lambda,\zeta} = \varnothing$, and it follows that $\delta \neq \gamma$. Now since $l > 1$, we have $l \in \mathrm{D}(v) \smallsetminus \mathrm{D}(x)$, whence $\mathrm{D}(v) \nsubseteq \mathrm{D}(x)$, and it follows that $\underline{\mu}(c_{\delta,v}, c_{\gamma,x}) = \mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. Since Γ_L is ordered, and since $\delta \neq \gamma$, we obtain $v < x$; in particular, $\zeta \leqslant \xi$.

Let (g,p) and (h,q) be boxes vacated in $j(u' \uparrow 1)$ and $j(t' \uparrow 1)$ respectively. Since $t' \uparrow k = t \uparrow k$, we have $t'(2,2) = k+1$ and $\mathrm{col}_{t'}(n) = k-1$. Moreover, Corollary 8.12 shows that $t' \uparrow (k+1)$ is the minimal tableau of its shape, equivalently $\mathrm{col}_{t'}(k+2) \leqslant \cdots \leqslant \mathrm{col}_{t'}(n)$. It is therefore clear that $\mathrm{col}_{t'}(i) \leqslant \mathrm{col}_{t'}(n) = k-1$ for all $i \in [1,n]$, in particular, this shows that $q \leqslant k-1$. Since $u' \uparrow k = u \uparrow k$, we have $u'(1,k) = n$, and since $u'(1,k-1) = k$ while $(2,k-1) \notin [\mu]$, it follows that $\mathrm{col}_{u'}(i) \leqslant k-2$ for all $i \in [1,n] \smallsetminus \{k\} \cup \{n\}$. Note, moreover, that the box $(2,1)$ is in the slide path of $j((1,1), u' \uparrow 1))$, and so we have $g \geqslant 2$, and it follows that $p \leqslant k-2 < k-1$.

Hence, we obtain $\sum_{m=1}^{k-1}\xi_m = \sum_{m=1}^{k-1}\lambda_m - 1 = \sum_{m=1}^{k-1}\mu_m + 1 - 1 = \sum_{m=1}^{k-1}\zeta_m + 1 + 1 - 1$. Thus $\zeta \not\leqslant \xi$, a desired contradiction. $\square$

LEMMA 8.15. *Let $\lambda \in \Lambda$, let $\mu \in \Lambda \setminus \{\lambda\}$, and suppose that $(\alpha, t') \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u') \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ satisfy the condition that $\mu(c_{\beta,s'}, c_{\alpha,t'}) \neq 0$, where we write t' for t_λ. Then $\mu < \lambda$.*

Proof. It is clear that n is at least 2. Recall that $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$ implies that $\mathrm{D}(t') \subsetneqq \mathrm{D}(u')$ and $\mu(c_{\alpha,t'}, c_{\beta,u'}) = 0$, since vertices $c_{\beta,u'}$ and $c_{\alpha,t'}$ belong to different molecules. Let k be the restriction number of the pair (u', t') and note that $1 \leqslant k \leqslant n-1$. By Lemma 8.10, we have $A(u', t') \neq \varnothing$. Let (u, t) be an approximate of (u', t'). By Lemma 8.9, we have (u, t) is k-restricted. By Lemma 8.10, we have $(u, t) \approx (u', t')$, $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ with $k = \min(\mathrm{D}(u) \setminus \mathrm{D}(t))$, and $\mu(c_{\beta,u}, c_{\alpha,t}) = \mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$. By Proposition 8.11, $t \Uparrow k$ is k-critical. For later reference, let $\nu = \mathrm{Shape}(u \Downarrow k) = \mathrm{Shape}(t \Downarrow k)$.

If $k = 1$ then since t is 1-minimal with respect to u, it follows by Lemma 8.13 that $\mu < \lambda$, and if $k = n-1$ then since $\mathrm{D}(u) = \{n-1\} \cup \mathrm{D}(t)$, it follows by Lemma 7.24 that $\mu < \lambda$. We may therefore assume that $1 < k < n-1$.

Let $w = j(u \uparrow 1)$ and let $y = j(t \uparrow 1)$, and let $v = u \Downarrow (n-1)$ and let $x = t \Downarrow (n-1)$. Let $\zeta = \mathrm{Shape}(w)$ and $\xi = \mathrm{Shape}(y)$, and let $\eta = \mathrm{Shape}(v)$ and $\theta = \mathrm{Shape}(x)$. We shall need the restriction of Γ to W_K and W_L established earlier.

By Remark 6.43, the vertex $c_{\beta,u}$ of Γ_K coincides with the vertex $c'_{\delta,v}$ for some $\delta \in \mathcal{I}_{K,\eta}$ and the vertex $c_{\alpha,t}$ of Γ_K coincides with the vertex $c'_{\gamma,x}$ for some $\gamma \in \mathcal{I}_{K,\theta}$. Since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$ and $k < n-1$, we have $k \in \mathrm{D}(v) \setminus \mathrm{D}(x)$, and so, $\mu(c'_{\delta,v}, c_{\gamma,x}) = \underline{\mu}(c_{\beta,u}, c_{\alpha,t}) \neq 0$.

By Remark 6.43, the vertex $c_{\beta,u}$ of Γ_L coincides with the vertex $c''_{\pi,w}$ for some $\pi \in \mathcal{I}_{L,\zeta}$ and the vertex $c_{\alpha,t}$ of Γ_L coincides with the vertex $c''_{\varepsilon,y}$ for some $\varepsilon \in \mathcal{I}_{L,\xi}$. Since $k \in \mathrm{D}(u) \setminus \mathrm{D}(t)$ and $k > 1$, we have $k \in \mathrm{D}(w) \setminus \mathrm{D}(y)$, and so, $\mu(c''_{\pi,w}, c_{\varepsilon,y}) = \underline{\mu}(c_{\beta,u}, c_{\alpha,t}) \neq 0$.

Since $\alpha \neq \beta$ (since $\mu \neq \lambda$), we have $\gamma \neq \delta$ and $\varepsilon \neq \pi$. Since Γ_K and Γ_L are ordered, it follows that $v < x$ and $w < y$. In particular, this gives $\eta \leqslant \theta$ and $\zeta \leqslant \xi$.

Since $t \Uparrow k$ is k-critical, it follows from the minimality of $\mathrm{col}_{t \Uparrow k}(k)$ that $\mathrm{col}_t(n) \geqslant \mathrm{col}_t(k)$. We shall show that if $\mathrm{col}_t(n) > \mathrm{col}_t(k)$ then $\mathrm{col}_t(n) \geqslant \mathrm{col}_u(n)$. Suppose to the contrary that $\mathrm{col}_t(k) < \mathrm{col}_t(n) < \mathrm{col}_u(n)$. We aim to show that (u, t) satisfies the hypothesis of Lemma 8.14.

Let $l = \mathrm{col}_t(n)$. Since $\eta \leqslant \theta$, it follows that

$$\sum_{m=1}^{l} \theta_m \leqslant \sum_{m=1}^{l} \eta_m. \tag{9}$$

Moreover, since $\mathrm{col}_t(k+2) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(n)$, since $t \uparrow (k+1)$ is the minimal tableau of its shape, and since $\mathrm{col}_t(k+1) \leqslant \mathrm{col}_t(n)$, since $\mathrm{col}_t(k) + 1 = \mathrm{col}_t(k+1)$ and $\mathrm{col}_t(k) + 1 \leqslant \mathrm{col}_t(n)$ by assumption, we have

$$\mathrm{col}_t(i) \leqslant l \quad \text{if} \quad k < i < n, \tag{10}$$

and so Eq. (9) can be expressed in the form

$$\sum_{m=1}^{l} \nu_m + n - 1 - k \leqslant \sum_{m=1}^{l} \nu_m + \sum_{m=1}^{l} (\eta_m - \nu_m).$$

Thus $\sum_{m=1}^{l}(\eta_m - \nu_m) \geqslant n - 1 - k$. But for each $m \in [1, l]$, $\eta_m - \nu_m$ counts certain positive integers between k and n, and so, we have $\sum_{m=1}^{l}(\eta_m - \nu_m) \leqslant n - 1 - k$. It follows that $\sum_{m=1}^{l}(\eta_m - \nu_m) = n - 1 - k$. Equivalently, we have

$$\mathrm{col}_u(i) \leqslant l \quad \text{if} \quad k < i < n. \tag{11}$$

In particular, Eq.(11) shows that $\mathrm{col}_u(n-1) \leqslant l = \mathrm{col}_t(n)$. Since $\mathrm{col}_t(n) < \mathrm{col}_u(n)$ by our assumption, this implies that $\mathrm{col}_u(n-1) < \mathrm{col}_u(n)$. Thus $n - 1 \in \mathrm{A}(u)$. Since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, it follows that $n-1 \in \mathrm{A}(t)$. Since $\mathrm{col}_u(i) \leqslant l$ whenever $k < i < n$ by Eq.(11) and $\mathrm{col}_t(i) \leqslant l$ whenever $k < i < n$ by Eq.(10), and since $\mathrm{col}_u(n) > \mathrm{col}_t(n) = l$ by our assumption, we have

$$\sum_{m=1}^{l} \mu_m = \sum_{m=1}^{l} \nu_m + n - 1 - k = \sum_{m=1}^{l} \nu_m + (n-k) - 1 = \sum_{m=1}^{l} \lambda_m - 1. \tag{12}$$

Let (g,p) and (h,q) be the boxes vacated by $j((1,1),u\uparrow 1)$ and $j((1,1),t\uparrow 1)$, respectively. We claim that

$$q \leqslant l < p. \tag{13}$$

If $l < q$ then $\sum_{m=1}^{l} \xi_m = \sum_{m=1}^{l} \lambda_m$, and since $\sum_{m=1}^{l} \mu_m \geqslant \sum_{m=1}^{l} \zeta_m$, it follows by Eq.(12) that $\sum_{m=1}^{l} \xi_m > \sum_{m=1}^{l} \zeta_m$. If $p \leqslant l = \mathrm{col}_t(n)$ then $\sum_{m=1}^{l} \zeta_m = \sum_{m=1}^{l} \mu_m - 1 < \sum_{m=1}^{l} \mu_m$. Moreover, since $\sum_{m=1}^{l} \lambda_m - 1 \leqslant \sum_{m=1}^{l} \xi_m$, it follows by Eq.(12) that $\sum_{m=1}^{l} \zeta_m < \sum_{m=1}^{l} \xi_m$. Since $\zeta \leqslant \xi$, either case results in a contradiction, whence $q \leqslant l < p$, as claimed.

Let $u(g,p) = b$. We claim that $b = n$.

If $k+1 \leqslant b < n$ then $p = \mathrm{col}_u(b) \leqslant l$ by Eq. (11), contradicting Eq. (13). Thus $b \leqslant k$ or $b = n$. But $\mathrm{col}_u(k) = \mathrm{col}_t(k) < \mathrm{col}_t(n) = l$ by our assumption, and so the case $b = k$ is excluded by Eq. (13). Suppose that $b < k$. Since the box (g,p) in the diagram of $\mathrm{Shape}(u\uparrow 1)$ is vacated by $j((1,1),u\uparrow 1)$, and since $u \Downarrow k = t \Downarrow k$, the box (g,p) in the diagram of $\mathrm{Shape}(t\uparrow 1)$ is in the slide path of $j((1,1),t\uparrow 1)$, and it follows that $h \geqslant g$ and $q \geqslant p$ by Lemma 6.35. The latter inequality contradicts $q < p$ given by Eq.(13). Hence $b = n$, as claimed.

We claim that

$$\mathrm{col}_u(i) < p - 1 \quad \text{if} \quad k < i < n. \tag{14}$$

By Eq. (10), we have $\mathrm{col}_t(k+1) \leqslant \mathrm{col}_t(n)$.

Suppose first that $\mathrm{col}_t(k+1) = \mathrm{col}_t(n)$. Since $n-1 \in \mathrm{A}(t)$, as shown above, we have $\mathrm{col}_t(n-1) < \mathrm{col}_t(n)$, and since $\mathrm{col}_t(k+1) = \mathrm{col}_t(k)+1$, the assumption $\mathrm{col}_t(k+1) = \mathrm{col}_t(n)$ implies that $\mathrm{col}_t(n-1) \leqslant \mathrm{col}_t(k)$. But $\mathrm{col}_t(n-1) \geqslant \mathrm{col}_t(k)$ by the minimality of $\mathrm{col}_{t\Uparrow k}(k)$, we therefore have $\mathrm{col}_t(n-1) = \mathrm{col}_t(k)$. Since $t\uparrow(k+1)$ is the minimal tableau of its shape, this shows that $\mathrm{col}_t(k+2) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) = \mathrm{col}_t(k)$, and so it follows by the minimality of $\mathrm{col}_{t\Uparrow k}(k)$ that $\mathrm{col}_t(k) = \mathrm{col}_t(k+2) = \cdots = \mathrm{col}_t(n-1)$, whence $k+1, k+2, \ldots, n-2 \in \mathrm{D}(t)$. On the one hand, since $k \in \mathrm{D}(u)$, and since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$, we have $k, k+1, \ldots, n-2 \in \mathrm{D}(u)$, and it follows that $\mathrm{col}_u(n-1) \leqslant \mathrm{col}_u(n-2) \leqslant \cdots \leqslant \mathrm{col}_u(k+1) \leqslant \mathrm{col}_u(k)$. On the other hand, since $\mathrm{col}_u(k) = \mathrm{col}_t(k) = \mathrm{col}_t(k+1) - 1 = \mathrm{col}_t(n) - 1 = l - 1$, and it follows by Eq.(13) that $\mathrm{col}_u(k) < p-1$. Hence, if $k < i < n$ then $\mathrm{col}_u(i) < p-1$.

Suppose now that $\mathrm{col}_t(k+1) < \mathrm{col}_t(n) = l$. Since $t\uparrow(k+1)$ is the minimal tableau of its shape, and since $n-1 \in \mathrm{A}(t)$, as shown above, so that $\mathrm{col}_t(n-1) < \mathrm{col}_t(n)$, we have $\mathrm{col}_t(k+2) \leqslant \cdots \leqslant \mathrm{col}_t(n-1) < \mathrm{col}_t(n)$. Hence if $k < i < n$, we have $\mathrm{col}_t(i) < l$. Since $\eta \leqslant \theta$, we have $\sum_{m=1}^{l-1} \theta_m \leqslant \sum_{m=1}^{l-1} \eta_m$. This gives

$$\sum_{m=1}^{l-1} \nu_m + n - 1 - k \leqslant \sum_{m=1}^{l-1} \nu_m + \sum_{m=1}^{l-1} (\eta_m - \nu_m)$$

that is, $n-1-k \leqslant \sum_{m=1}^{l-1} (\eta_m - \nu_m)$. But since $\eta_m - \nu_m$ counts, for each $m \in [1, l-1]$, certain positive integers between k and n, it follows that $\sum_{m=1}^{l-1} (\eta_m - \nu_m) \leqslant n-1-k$. Therefore, we conclude that $\sum_{m=1}^{l-1} (\eta_m - \nu_m) = n-1-k$, that is, $\mathrm{col}_u(i) \leqslant l-1$ if $k < i < n$. Since $l < p$ by Eq. (13), we have $\mathrm{col}_u(i) < p-1$ if $k < i < n$. This completes the proof of our claim.

Obviously n slides from the box (g,p) of the diagram of $\mathrm{Shape}(u\uparrow 1)$ into either the box $(g-1,p)$ or the box $(g,p-1)$. Note that Eq. (14) gives $u(g,p-1) \leqslant k$ and $u(g-1,p) \leqslant k$, and so $t(g,p-1) = u(g,p-1)$ and $t(g-1,p) = u(g-1,p)$. Now if n slides into the box $(g-1,p)$, so that the box $(g-1,p)$ is in the slide path of $j((1,1),t\uparrow 1)$, then Lemma 6.35 gives $p \leqslant q$, contradicting Eq. (13). Thus n slides into the box $(g,p-1)$, so that the box

$(g, p-1)$ is in the slide path of $j((1,1),t\uparrow 1)$, and Lemma 6.35 gives $p-1 \leqslant q$. But since $q \leqslant l < p$ by Eq. (13), this shows that $\mathrm{col}_t(n) = l = q = p-1$.

Let $\kappa = (\kappa_1^{m_1}, \dots, \kappa_s^{m_s}) = \mathrm{Shape}(u \Downarrow k-1) = \mathrm{Shape}(t \Downarrow k-1)$.

We claim that $\mathrm{col}_u(k) = \mathrm{col}_t(k) = 1$.

Suppose to the contrary that $\mathrm{col}_t(k) > 1$. Choose (u,t) to be the minimal approximate of (u',t'). By Lemma 6.36, we have (κ_1, m_1) is vacated by $j((1,1),\tau_\kappa\uparrow 1)$. Since $(g,p-1)$ is vacated by $j((1,1),\tau_\kappa\uparrow 1)$, we have $\kappa_1 = g$ and $m_1 = p-1$. Since $\mathrm{col}_t(n) = p-1$, and since $t^{-1}(k)$ is a κ-addable box and $t^{-1}(k) \neq (\kappa_1, 1)$, we have $p-1 = m_1 < \mathrm{col}_t(k)$, the latter inequality shows that $\mathrm{col}_t(n) < \mathrm{col}_t(k)$, contradicting our assumption that $\mathrm{col}_t(k) < \mathrm{col}_t(n)$. Thus $\mathrm{col}_u(k) = \mathrm{col}_t(k) = 1$, as claimed.

Since $\kappa_1 = g$, as shown above, we have $\mathrm{row}_u(k) = \mathrm{row}_t(k) = \kappa_1 + 1 = g+1$. We claim that $g = 1$.

Suppose to the contrary that $g > 1$. We have $(g,p) \notin [\kappa]$ since $u(g,p) = n$ but $(g-1,p) \in [\kappa]$ because $g > 1$ and because of Eq. (14). It follows that $(g,p) = u^{-1}(n)$ is a κ-addable box, whence $s > 1$. Choose (u,t) to be the maximal approximate of (u',t'). Let $\kappa^* = (\kappa_1^{*n_1}, \dots, \kappa_r^{*n_r})$. It follows from Lemma 6.36 that (n_1, κ_1^*) is vacated by $j((1,1),\tau^\kappa\uparrow 1)$. Since $n_1 = m_s$ and $\kappa_1^* = m_1 + \cdots + m_s$, it follows that $(\kappa_1, m_1) \neq (n_1, \kappa_1^*)$, a clear contradiction. Thus $g = 1$, as claimed.

Since $t(2,1) = u(2,1) = k$, we deduce that κ consists of $(k-1)$ parts of length 1, that is, $m_1 = k-1$ and $\kappa_1 = 1$. Thus $\mathrm{col}_t(n) = k-1$ and $u(1,k) = n$, and since $\mathrm{col}_t(n) > \mathrm{col}_t(k)$ by our assumption, we have $\mathrm{col}_t(n) \geqslant 2$, and it follows that $k \geqslant 3$. Moreover, since $\mathrm{col}_u(i) < k-1$ for $k \leqslant i \leqslant n-1$, we have $(2,k-1) \notin [\mu]$. It is clear that (u,t) satisfies the hypothesis of Lemma 8.14. But Lemma 8.14 shows that $(u,t) \notin A(u',t')$, which completes our argument by contradiction.

We have shown that $\mathrm{col}_t(n) = \mathrm{col}_t(k)$ or if $\mathrm{col}_t(n) > \mathrm{col}_t(k)$ then $\mathrm{col}_t(n) \geqslant \mathrm{col}_u(n)$.

Suppose first that $\mathrm{col}_t(n) = \mathrm{col}_t(k)$. Since $t\uparrow(k+1)$ is the minimal tableau of its shape, we have $\mathrm{col}_t(n) \geqslant \mathrm{col}_t(n-1) \geqslant \cdots \geqslant \mathrm{col}_t(k+3) \geqslant \mathrm{col}_t(k+2)$. Since $\mathrm{col}_t(k) = \mathrm{col}_t(n)$, it follows from the minimality of $\mathrm{col}_{t\Uparrow k}(k)$ that $\mathrm{col}_t(k) = \mathrm{col}_t(k+2) = \cdots = \mathrm{col}_t(n)$. Moreover, since $\mathrm{col}_t(k+1) = \mathrm{col}_t(k) + 1 > \mathrm{col}_t(k)$, this shows that $\mathrm{col}_t(k+1) > \mathrm{col}_t(k+2)$. Thus, it follows that $k+1, k+2, \dots, n-1 \in \mathrm{D}(t)$. Now since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ and $k = \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$, in particular, $k \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, it follows that $k, k+1, k+2 \dots, n-1 \in \mathrm{D}(u)$. This shows that $\mathrm{col}_u(n) \leqslant \mathrm{col}_u(n-1) \leqslant \cdots \leqslant \mathrm{col}_u(k) = \mathrm{col}_t(k) = \mathrm{col}_t(n)$, whence $\mu < \lambda$ by Lemma 7.2.

Finally, suppose that $\mathrm{col}_u(n) \leqslant l = \mathrm{col}_t(n)$. Since $\eta \leqslant \theta$, we have $\mu \leqslant \lambda$ by Lemma 7.2, and since $\mu \neq \lambda$, we have $\mu < \lambda$. ☐

LEMMA 8.16. *Suppose further that* Γ *is a cell. Then* $\Lambda = \{\lambda\}$ *for some* $\lambda \in P(n)$.

Proof. Assume to the contrary that Λ consists of more than one partitions of n. Let $\lambda \in \Lambda$. Since Γ is strongly connected, the set $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$. Let $(\alpha, t_\lambda) \in \mathrm{Ini}_\lambda(\Gamma)$. Let $\mu \in \Lambda \smallsetminus \{\lambda\}$ be such that $\mu(c_{\beta,u}, c_{\alpha,t_\lambda}) \neq 0$, for some $(\beta,u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$. Then $\mu < \lambda$ by Lemma 8.15. Repeating the argument with μ in place of λ. Since Λ is a finite set and Γ is strongly connected, a finite chain $\lambda > \mu > \cdots > \gamma > \cdots > \nu > \gamma$ is eventually reached, a clear contradiction. ☐

Lemma 8.16 says that the set of molecule types for an admissible W_n- cell is a singleton set $\{\lambda\}$, where λ is a partition of n.

LEMMA 8.17. *Suppose that* $n \geqslant 2$. *Let* D *and* D' *be cells of* Γ, *and let* $\{\mu\}$ *and* $\{\lambda\}$ *be the sets of molecule types for* D *and* D', *respectively. Then* $D \leqslant_\Gamma D'$ *implies* $\mu \leqslant \lambda$. *In particular, if* $c_{\alpha,t} \in D'$ *and* $c_{\beta,u} \in D$ *satisfy the condition that* $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, *then* $\mu \leqslant \lambda$.

Proof. If $\mu = \lambda$ then the result holds trivially. So we can assume that $\mu \neq \lambda$. Let $(\mathcal{C}, \leqslant_\Gamma)$ be the poset of cells of Γ induced by the preorder $\leqslant_\Gamma$. It follows that $|\mathcal{C}| \geqslant 2$.

Suppose first that D and D' are the only cells of Γ. Since $D \leqslant_\Gamma D'$, the set $\mathrm{Ini}_\lambda(\Gamma) \neq \varnothing$. Let $(\alpha, t_\lambda) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u) \in \mathcal{I}_\mu \times \mathrm{Std}(\mu)$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t_\lambda}) \neq 0$. It follows readily from Lemma 8.15 that $\mu < \lambda$.

Suppose now that $|\mathcal{C}| > 2$ and the result holds for any admissible W_n-graph of less than $|\mathcal{C}|$ cells. Let C_0 and C_1 be a minimal and a maximal cell in $(\mathcal{C}, \leqslant_\Gamma)$. It is clear that C_0 and $C \setminus C_1$ are closed subsets of C, hence the full subgraphs $\Gamma(C \setminus C_0)$ and $\Gamma(C \setminus C_1)$ induced by $C \setminus C_0$ and $C \setminus C_1$ are themselves admissible W_n-graph with edge weights and vertex colours inherited from Γ. It follows that if both D and D' are cells of $\Gamma(C \setminus C_0)$ or $\Gamma(C \setminus C_1)$, then the result is given by the inductive hypothesis. Furthermore, since $D \leqslant_\Gamma D'$ by assumption, we can assume that $D = C_0$ and $D' = C_1$ are the (unique) minimal and maximal cells in $(\mathcal{C}, \leqslant_\Gamma)$.

Let $C' \neq C_0, C_1$ be a cell of Γ. By Lemma 8.16, the set of molecule types for C' is $\{\nu\}$ for some $\nu \in \Lambda$. Now since $C_0 \leqslant_\Gamma C'$ and C_0 and C' are cells of $\Gamma(C \setminus C_1)$, we have $\mu \leqslant \nu$ by the inductive hypothesis. Similarly, since $C' \leqslant_\Gamma C_1$ and C' and C_1 are cells of $\Gamma(C \setminus C_0)$, we have $\nu \leqslant \lambda$ by the inductive hypothesis. It follows that $\mu \leqslant \lambda$ as required.

Since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, we have $\mathrm{D}(u) \not\subseteq \mathrm{D}(t)$. It follows that $c_{\beta,u} \leqslant_\Gamma c_{\alpha,t}$, hence $D \leqslant_\Gamma D'$ by the definition of the preorder $\leqslant_\Gamma$. It follows from the result above that $\mu \leqslant \lambda$. □

THEOREM 8.18. *Γ is ordered.*

Proof. Suppose that $(\alpha, t) \in \mathcal{I}_\lambda \times \mathrm{Std}(\lambda)$ and $(\beta, u) \in \mathcal{I}_\mu$ satisfy $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$. It follows from Lemma 8.17 that $\mu \leqslant \lambda$. Now Proposition 8.4 says that $u < t$ unless $\alpha = \beta$ and $u = s_i t > t$ for some $i \in [1, n-1]$. That is, Γ is ordered. □

REMARK 8.19. Let $y, w \in W_n$, and let $RS(y) = (u, v)$ and $RS(w) = (t, x)$. It follows from Remark 8.2 that if$y \preceq_{\mathsf{LR}} w$ then $\mu \leqslant \lambda$, where $\mu = \mathrm{Shape}(x) = \mathrm{Shape}(u)$ and $\lambda = \mathrm{Shape}(y) = \mathrm{Shape}(v)$. This gives an alternative approach to the necessary part of the following well-known result. (See, for example, [6, Theorem 5.1].)

THEOREM 8.20. *Let $y, w \in W_n$ and $\mu, \lambda \in P(n)$, and suppose that $RS(y) \in \mathrm{Std}(\mu) \times \mathrm{Std}(\mu)$ and $RS(w) \in \mathrm{Std}(\lambda) \times \mathrm{Std}(\lambda)$. Then $y \preceq_{\mathsf{LR}} w$ if and only if $\mu \leqslant \lambda$. In particular, the sets $D(\lambda) := \{w \in W_n \mid RS(w) \in \mathrm{Std}(\lambda) \times \mathrm{Std}(\lambda)\}$, where $\lambda \in P(n)$, are precisely the Kazhdan–Lusztig two-sided cells.*

Let $\lambda \in P(n)$. For each $t \in \mathrm{Std}(\lambda)$, since $C(t) = \{w \in W_n \mid Q(w) = t\}$ gives rise to the left cell isomorphic to Γ_λ, we have $D(\lambda) = \bigsqcup_{t \in \mathrm{Std}(\lambda)} C(t)$ gives rise to the union of $|\mathrm{Std}(\lambda)|$ left cells whose molecule types are λ.

9. W-GRAPHS FOR ADMISSIBLE CELLS IN TYPE A

DEFINITION 9.1. Let $\lambda \in P(n)$. A pair of standard λ-tableaux (u, t) is said to be a *probable pair* if $u < t$ and $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$.

It can be seen that there is no probable pair unless $n \geqslant 5$.

LEMMA 9.2. *Let $\lambda \in P(n)$, and let $u, t \in \mathrm{Std}(\lambda)$. Let i be the restriction number of (u, t) and $j = \max(\mathrm{SD}(t))$. If (u, t) is favourable and satisfies $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ then $i < j$.*

Proof. Suppose to the contrary that $i \geqslant j$. Since (u, t) is favourable, and since $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$, we have $i \in \mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \setminus \mathrm{D}(t)$. Since $i \notin \mathrm{D}(t)$, we have $i \neq j$, and so $i > j$. Let $w = t \Downarrow i = u \Downarrow i \in \mathrm{Std}(\mu)$, where $\mu = \mathrm{Shape}(w)$. Since $j = \max(\mathrm{SD}(t))$ and since $i > j$, we have $\mathrm{D}(t \uparrow i) \cap [i+1, n-1] = \mathrm{WD}(t \uparrow i) \cap [i+1, n-1]$, and it follows by Remark 6.16 that $t \uparrow i$ is minimal, that is, $t \uparrow i = \tau_{\lambda/\mu}$. Moreover, since $i \notin \mathrm{D}(t)$, this shows that for all $k > i$, we have $\mathrm{col}_t(k) \geqslant \mathrm{col}_t(i+1) > \mathrm{col}_t(i)$, from which we have $\lambda_m = \mu_m$ for all $m \leqslant \mathrm{col}_t(i)$. Hence if $k > i$ then $\mathrm{col}_u(k) > \mathrm{col}_u(i)$, in particular, $\mathrm{col}_u(i+1) > \mathrm{col}_u(i)$, contradicting $i \in \mathrm{D}(u)$. □

LEMMA 9.3. *Let $\lambda \in P(n)$, and let $u,t \in \mathrm{Std}(\lambda)$. Let i be the restriction number of (u,t). Suppose that (u,t) is favourable and satisfies $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$. If, moreover, $i+1 = \max \mathrm{SD}(t)$, then $\mathrm{col}_t(i+2) \neq \mathrm{col}_t(i)$.*

Proof. Suppose to the contrary that $\mathrm{col}_t(i+2) = \mathrm{col}_t(i)$. Since (u,t) is i-restricted, we have $u{\Downarrow}i = t{\Downarrow}i$. Let $\mu = \mathrm{Shape}(t{\Downarrow}i) = \mathrm{Shape}(u{\Downarrow}i)$, and let $u(g,p) = t(g,p) = i$. Now since $i+1 = \max \mathrm{SD}(t)$, we have $t{\uparrow}(i+1)$ is minimal, hence $\mathrm{col}_t(i) = \mathrm{col}_t(i+2) \leqslant \mathrm{col}_t(k)$ for $k > i+2$. Furthermore, since (u,t) is favourable, we have $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, and so we have $\mathrm{col}_t(i) < \mathrm{col}_t(i+1)$. It follows that for $k > i$, we have $\mathrm{col}_t(k) \geqslant \mathrm{col}_t(i)$. Therefore, for each $j \in [1, p-1]$, we have $\lambda_j = \mu_j$. This shows that for $k > i$, we have $\mathrm{col}_u(k) \geqslant \mathrm{col}_u(i)$, in particular, we have $\mathrm{col}_u(i+1) \geqslant \mathrm{col}_u(i)$. Since $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, as (u,t) is favourable, we have $\mathrm{col}_u(i+1) \leqslant \mathrm{col}_u(i)$. Thus $\mathrm{col}_u(i+1) = \mathrm{col}_u(i)$, and we have $u(g+1,p) = i+1$. An easy induction on $l \in [1, \lambda_p - g]$ shows that if $t(g+l,p) = i+l+1$ then $u(g+1,p) = i+l$. However, this contradicts $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$, as desired. □

Let $\Gamma = \Gamma(C,\mu,\tau)$ be an admissible W_n-graph. Suppose that $\Lambda = \{\lambda\}$, where $\lambda \in P(n)$, is the set of molecule types for Γ, and let $\mathcal{I} = \mathcal{I}_\lambda$ index the molecules of Γ. By Remark 6.41, the vertex set of Γ is given by $C = \bigsqcup_{\alpha \in \mathcal{I}} C_{\alpha,\lambda}$, where for each $\alpha \in \mathcal{I}$, $C_{\alpha,\lambda} = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$, the simple edges of Γ are the pairs $\{c_{\beta,u}, c_{\alpha,t}\}$ such that $\alpha = \beta$ and u and t are related by a dual Knuth move, and $\tau(c_{\alpha,t}) = \mathrm{D}(t)$.

LEMMA 9.4. *Let $u,t \in \mathrm{Std}(\lambda)$, and suppose that the pair (u,t) is probable. Then for all $(v,x) \in F(u,t)$, the pair (v,x) is probable,* $\max(\mathrm{SD}(v)) = \max(\mathrm{SD}(t))$*, and $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$.*

Proof. We may assume that (u,t) is not favourable. Let $(v,x) \in F(u,t)$. Let i be the restriction number of (v,x) (which is also the restriction number of (u,t)), and let $j = \max(\mathrm{SD}(x))$. Since $(v,x) \in F(u,t)$, we have $(v,x) \approx_i (u,t)$, and so $(v,x) \approx (u,t)$ by Remark 7.7, and since (u,t) is probable, we have $u < t$, and it follows by Lemma 7.9 that $v < x$. Furthermore, $u < t$ implies that $u{\Downarrow}(i+1) \leqslant t{\Downarrow}(i+1)$, but since (u,t) is i-restricted. we have $u{\Downarrow}(i+1) \neq t{\Downarrow}(i+1)$, therefore $u{\Downarrow}(i+1) < t{\Downarrow}(i+1)$. It follows by Remark 6.9 that $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$. Moreover, as (u,t) is not favourable and $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$, so that $\mathrm{D}(u) \oplus \mathrm{D}(t) = \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$, we have $i < \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ by Remark 7.17. Thus (u,t) satisfies the hypothesis of Lemma 7.20. Since $\mathrm{col}_u(i+1) < \mathrm{col}_t(i+1)$ as shown above, it follows by Lemma 7.20 that (v,x) satisfies $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) = \{i\} \cup (\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ and $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \varnothing$. Since $\mathrm{D}(x) \smallsetminus \mathrm{D}(v) = \varnothing$ while $\mathrm{D}(v) \smallsetminus \mathrm{D}(x) \neq \varnothing$, we have $\mathrm{D}(x) \subsetneq \mathrm{D}(v)$. Hence (v,x) is probable.

Next, since $j = \max(\mathrm{SD}(x))$ and $j > i$ by Lemma 9.2, we have $j = \max(SD(x{\uparrow}i))$, and since $t{\uparrow}i = x{\uparrow}i$, we have $j = \max(\mathrm{SD}(t{\uparrow}i))$, and it follows that $j = \max(\mathrm{SD}(t))$, as required.

Finally, since $i < \min(\mathrm{D}(u) \smallsetminus \mathrm{D}(t))$ as shown above, we have $\mu(c_{\beta,v}, c_{\alpha,x}) = \mu(c_{\beta,u}, c_{\alpha,t})$ by Lemma 7.21. □

PROPOSITION 9.5. *Monomolecular admissible cells of type A_{n-1} are Kazhdan–Lusztig.*

Proof. Suppose that $\Gamma = \Gamma(C,\mu,\tau)$ is a monomolecular admissible W_n-cell. Then there is a partition λ of n such that $C = \{c_t \mid t \in \mathrm{Std}(\lambda)\}$, and $\{c_u, c_t\}$ is a simple edge of Γ if and only if $u,t \in \mathrm{Std}(\lambda)$ are related by a dual Knuth move. In view of Corollary 6.31, our task is to show that $\Gamma \cong \Gamma_\lambda = \Gamma(\mathrm{Std}(\lambda), \mu^{(\lambda)}, \tau^{(\lambda)})$. Recall from Remark 6.42 that Γ_λ is an admissible W_n-graph consisting of a single molecule of type λ. Since it follows from Remark 6.15 that $\tau(c_t) = \mathrm{D}(t) = \tau^{(\lambda)}(t)$ for all $t \in \mathrm{Std}(\lambda)$, it remains to show that $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t)$ for all $u,t \in \mathrm{Std}(\lambda)$. Note that, by Theorem 5.8, both Γ and Γ_λ satisfy the W_n-Compatibility Rule, the W_n-Simplicity Rule, the W_n-Bonding Rule and the W_n-Polygon Rule.

We have shown in Theorem 8.18 that Γ and Γ_λ are both ordered. Thus if $u,t \in \mathrm{Std}(\lambda)$ then $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t) = 0$ unless $u < t$ or $u = s_i t > t$ for some $i \in [1, n-1]$. If $u = s_i t > t$ for some $i \in [1, n-1]$ then we have $\mu(c_u, c_t) = \mu^{(\lambda)}(u,t) = 1$ by Corollary 7.22. Now suppose that $u < t$ and $\mathrm{D}(t) \nsubseteq \mathrm{D}(u)$. If one or other of $\mu(c_u,c_t)$ and $\mu^{(\lambda)}(u,t)$ is nonzero then, by the Simplicity Rule, one or other of $\{c_u,c_t\}$ and $\{u,t\}$ is a simple edge, whence u and t are related by a dual Knuth move (by Remark 6.41), and both $\{c_u,c_t\}$ and $\{u,t\}$ are simple edges. So $\mu(c_u,c_t) = \mu^{(\lambda)}(u,t) = 1$ in this case. Obviously there is nothing to show if $\mu(c_u,c_t)$ are $\mu^{(\lambda)}(u,t)$ both zero, and so all that remains is to show that $\mu(c_u,c_t) = \mu^{(\lambda)}(u,t)$ whenever $u < t$ and $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$. That is, it remains to show that $\mu(c_u,c_t) = \mu^{(\lambda)}(u,t)$ for all probable pairs of standard λ-tableaux.

Let (u',t') be a probable pair. If $t' = \tau_\lambda$ then there is nothing to prove. Proceeding inductively on the lexicographic order, let $\tau_\lambda \neq t' \in \mathrm{Std}(\lambda)$, and assume that the result holds for all $x \in \mathrm{Std}(\lambda)$ such that $x <_{\mathrm{lex}} t'$.

Let i be the restriction number of (u',t'), and let $j = \max(\mathrm{SD}(t'))$. Let $(u,t) \in F(u',t')$, and note that (u,t) is i-restricted and favourable, and satisfies $t\!\uparrow\! i = t'\!\uparrow\! i$ and $u\!\uparrow\! i = u'\!\uparrow\! i$. Moreover, Lemma 9.4 shows that (u,t) is probable, $\max(\mathrm{SD}(t)) = \max(\mathrm{SD}(t')) = j$, and $\mu(c_u,c_t) = \mu(c_{u'},c_{t'})$ and $\mu^{(\lambda)}(u,t) = \mu^{(\lambda)}(u',t')$. By the last result, it suffices to show that $\mu(c_u,c_t) = \mu^{(\lambda)}(u,t)$.

Since $j \in \mathrm{SD}(t)$, we have $s_jt \in \mathrm{Std}(\lambda)$ and $s_jt < t$. Let $v = s_jt$, and note that $v <_{\mathrm{lex}} t'$ by Lemma 7.29. Since $i < j$ by Lemma 9.2, we have either $j - i > 1$ or $j - i = 1$.

Case 1.
Suppose that $j - i > 1$, so that $m(i,j) = 2$. Since (u,t) is favourable, and since $\mathrm{D}(t) \subsetneqq \mathrm{D}(u)$ (since (u,t) is probable), we have $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ and $j \in D(u) \cap \mathrm{D}(t)$, that is, $i \notin \mathrm{D}(t)$ and $j \in \mathrm{D}(t)$, and $i,j \in \mathrm{D}(u)$. We also have $i,j \notin \mathrm{D}(v)$, by Lemma 7.28 (i).

If (c_v,c_{y_1},c_u) is any alternating directed path of type (j,i), then, since Γ is ordered, it follows that either $y_1 = s_jv = t > v$ or $y_1 < v$. Similarly, if (c_v,c_{x_1},c_u) is any alternating directed path of type (i,j), then it follows that either $x_1 = s_iv > v$ or $x_1 < v$. Note that if $x_1 = s_iv > v$, then since $x_1 \in \mathrm{Std}(\lambda)$, it follows that $i \in \mathrm{SA}(v)$. Thus, if $x_1 = s_iv > v$, then $i \in \mathrm{D}(s_iv)$ and $j \notin \mathrm{D}(s_iv)$ by Lemma 7.28 (i). Now since Γ satisfies the W_n-Polygon Rule, we have $N^2_{j,i}(\Gamma;c_v,c_u) = N^2_{i,j}(\Gamma;c_v,c_u)$, and it follows that

$$\begin{multline} \mu(c_t,c_v)\mu(c_u,c_t) + \sum_{y_1<v}\mu(c_{y_1},c_v)\mu(c_u,c_{y_1}) \\ = \mu(c_{s_iv},c_v)\mu(c_u,c_{s_iv}) + \sum_{x_1<v}\mu(c_{x_1},c_v)\mu(c_u,c_{x_1}), \end{multline} \tag{15}$$

where the term $\mu(c_{s_iv},c_v)\mu(c_u,c_{s_iv})$ on the right hand side of Eq. (15) should be omitted if $i \notin \mathrm{SA}(v)$. Note that if $i \in \mathrm{SA}(v)$ then (c_v,c_{s_iv},c_u) is not necessarily a directed path, since there need not be an arc from s_iv to u, but in this case $\mu(c_{s_iv},c_v)\mu(c_u,c_{s_iv}) = 0$ since $\mu(c_u,c_{s_iv}) = 0$. Similarly, (c_v,c_t,c_u) is not necessarily a directed path, since there need not be an arc from t to u, but $\mu(c_t,c_v)\mu(c_u,c_t) = 0$ in this case. So Eq. (15) still holds in these cases.

Since Corollary 7.22 gives $\mu(c_t,c_v) = 1$, and $\mu(c_{s_iv},c_v) = 1$ if $i \in \mathrm{SA}(v)$, Eq. (15) yields the following formula for $\mu(c_u,c_t)$:

$$\mu(c_u,c_t) = \mu(c_u,c_{s_iv}) + \sum_{x_1<v}\mu(c_{x_1},c_v)\mu(c_u,c_{x_1}) - \sum_{y_1<v}\mu(c_{y_1},c_v)\mu(c_u,c_{y_1}),$$

where $\mu(c_u,c_{s_iv})$ should be interpreted as 0 if $s_iv \notin \mathrm{Std}(\lambda)$.

Working similarly on Γ_λ yields the following formula for $\mu^{(\lambda)}(u,t)$:

$$\mu^{(\lambda)}(u,t) = \mu^{(\lambda)}(u,s_iv) + \sum_{x_1<v}\mu^{(\lambda)}(x_1,v)\mu^{(\lambda)}(u,x_1) - \sum_{y_1<v}\mu^{(\lambda)}(y_1,v)\mu^{(\lambda)}(u,y_1).$$

Since $v <_{\mathrm{lex}} t$ by Lemma 7.29, $s_iv <_{\mathrm{lex}} t$ (if $i \in \mathrm{SA}(t)$) by Lemma 7.29 (i), and $x_1 <_{\mathrm{lex}} t$ and $y_1 <_{\mathrm{lex}} t$ by Lemma 7.29 (ii), it follows by the inductive hypothesis that the corresponding edge weights that appear in the two formulae above are the same. Thus $\mu(c_u,c_t) = \mu^{(\lambda)}(u,t)$, as desired.

Case 2.
Suppose that $i = j-1$, so that $m(i,j) = 3$. By Lemma 9.3, $\mathrm{col}_t(j-1) \neq \mathrm{col}_t(j+1)$, and it follows that either one of the following situations occurs: $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$ or $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$.

If $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$, then the result follows by the same argument as above, with $j-1$ replacing i and Lemma 7.28(ii) replacing Lemma 7.28(i).

Suppose that $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$. Since $j-1 \in \mathrm{SD}(v)$ by Lemma 7.28 (iii), we have $s_{j-1}v \in \mathrm{Std}(\lambda)$ and $s_{j-1}v < v$. Let $w = s_{j-1}v$. It follows by Lemma 7.28 (iii) that $j-1,\, j \notin \mathrm{D}(w)$, but $j-1,\, j \in \mathrm{D}(u)$, since (u,t) is favourable and probable.

We consider length three alternating directed paths of type $(j-1, j)$ and $(j, j-1)$ from c_w to c_u. We have $j \in \mathrm{D}(t)$ and $j-1 \notin \mathrm{D}(t)$ (since (u,t) is favourable), while $j-1 \in \mathrm{D}(v)$ and $j \notin \mathrm{D}(v)$ by Lemma 7.28 (iii).

If $(c_w, c_{x_1}, c_{x_2}, c_u)$ is any alternating directed path of type $(j-1, j)$, then, since Γ is ordered, it follows that either $x_1 = s_{j-1}w = v > w$, or else $x_1 < w$. Moreover, since Γ satisfies the W_n-Simply Laced Bonding Rule, the fact that $j-1 \in \mathrm{D}(x_1)$ and $j \notin \mathrm{D}(x_1)$ shows that c_{x_2} is the unique vertex adjacent to c_{x_1} satisfying $j-1 \notin \mathrm{D}(x_2)$ and $j \in \mathrm{D}(x_2)$. That is, x_2 is the $(j-1)$-neighbour of x_1. Thus it follows that either $x_1 = v$ and $x_2 = s_j v = t$, or else $x_1 < w$ and either $x_2 = s_j x_1 > x_1$ or $x_2 = s_{j-1}x_1 < x_1$.

Similarly, if $(c_w, c_{y_1}, c_{y_2}, c_u)$ is any alternating directed path of type $(j, j-1)$, then it follows that either $y_1 = s_j w > w$ or $y_1 < w$, and y_2 is the $(j-1)$-neighbour of y_1. Note that if $y_1 = s_j w > w$, then since $y_1 \in \mathrm{Std}(\lambda)$, it follows that $j \in \mathrm{SA}(w)$. Thus, if $y_1 = s_j w > w$ then $y_2 = s_{j-1}y_1 = s_{j-1}s_j w > s_j w = y_1$, and $j \in \mathrm{D}(s_{jw})$ and $j-1 \notin \mathrm{D}(s_j w)$, and $j-1 \in \mathrm{D}(s_{j-1}s_j w)$ and $j \notin \mathrm{D}(s_{j-1}s_j w)$ by Lemma 7.28 (iii), while if $y_1 < w$ then either $y_2 = s_{j-1}y_1 > y_1$ or $y_2 = s_j y_1 < y_1$.

Now since Γ satisfies the W_n-Polygon Rule, we have $N^3_{j-1,j}(\Gamma; c_w, c_u) = N^3_{j,j-1}(\Gamma; c_w, c_u)$, and it follows that

$$\begin{aligned}(16)\quad \mu(c_v,c_w)\mu(c_t,c_v)\mu(c_u,c_t) + \sum_{\substack{x_1<w\\ x_2=(j-1)\text{-neb}(x_1)}} \mu(c_{x_1},c_w)\mu(c_{x_2},c_{x_1})\mu(c_u,c_{x_2})\\ = \mu(c_{s_jw},c_w)\mu(c_{s_{j-1}s_jw},c_{s_jw})\mu(c_u,c_{s_{j-1}s_jw}) +\\ \sum_{\substack{y_1<w\\ y_2=(j-1)\text{-neb}(y_1)}} \mu(c_{y_1},c_w)\mu(c_{y_2},c_{y_1})\mu(c_u,c_{y_2}),\end{aligned}$$

where the term $\mu(c_{s_jw},c_w)\mu(c_{s_{j-1}s_jw},c_{s_jw})\mu(c_u,c_{s_{j-1}s_jw})$ on the right hand side of Eq. (16 should be omitted if $j \notin \mathrm{SA}(w)$. Note that if $j \in \mathrm{SA}(w)$ then $(c_w, c_{s_jw}, c_{s_{j-1}s_jw}, c_u)$ is not necessarily a directed path, since there need not to be an arc from $c_{s_{j-1}s_jw}$ to c_u, but in this case $\mu(c_{s_jw},c_w)\mu(c_{s_{j-1}s_jw},c_{s_jw})\mu(c_u,c_{s_{j-1}s_jw}) = 0$ since $\mu(c_u,c_{s_{j-1}s_jw}) = 0$. Similarly, (c_w,c_v,c_t,c_u) is not necessarily a directed path, since there need not be an arc from c_t to c_u, but $\mu(c_v,c_w)\mu(c_t,c_v)\mu(c_u,c_t) = 0$ in this case. So Eq. (16) still holds in these cases.

Since $\mu(c_v,c_w) = \mu(c_{s_jw},c_w) = 1$ and $\mu(c_t,c_v) = \mu(c_{s_{j-1}s_jw},c_{s_jw}) = 1$, by Corollary 7.22, and since $\mu(c_{x_2},c_{x_1}) = \mu(c_{y_2},c_{y_1}) = 1$, since $\{c_{x_1},c_{x_2}\}$ and $\{c_{y_1},c_{y_2}\}$ are simple edges, Eq. (16) yields the following formula for $\mu(c_u,c_t)$:

$$\begin{aligned}\mu(c_u,c_t) = \mu(c_u,c_{s_{j-1}s_jw}) + \sum_{\substack{y_1<w\\ y_2=(j-1)\text{-neb}(y_1)}} \mu(c_{y_1},c_w)\mu(c_u,c_{y_2})\\ - \sum_{\substack{x_1<w\\ x_2=(j-1)\text{-neb}(x_1)}} \mu(c_{x_1},c_w)\mu(c_u,c_{x_2}),\end{aligned}$$

where $\mu(c_u,c_{s_{j-1}s_jw})$ should be interpreted as 0 if $s_jw \notin \mathrm{Std}(\lambda)$.

Working similarly on Γ_λ yields the following formula for $\mu^{(\lambda)}(u,t)$:

$$\mu^{(\lambda)}(u,t) = \mu^{(\lambda)}(u,s_{j-1}s_jw) + \sum_{\substack{y_1<w\\ y_2=(j-1)\text{-neb}(y_1)}} \mu^{(\lambda)}(y_1,w)\mu^{(\lambda)}(u,y_2) - \sum_{\substack{x_1<w\\ x_2=(j-1)\text{-neb}(x_1)}} \mu^{(\lambda)}(x_1,w)\mu^{(\lambda)}(u,x_2).$$

Since w, $s_{j-1}s_jw <_{\text{lex}} t'$ (if $j \in \mathrm{SA}(w)$) by Lemma 7.29 (iii), and $x_2, y_2 <_{\text{lex}} t'$ by Lemma 7.29 (iv), it follows by the inductive hypothesis that the corresponding edge weights that appear in the two formulae above are the same. Thus $\mu(c_{u'},c_{t'}) = \mu^{(\lambda)}(u',t')$, as desired. □

PROPOSITION 9.6. *Let $\Gamma = \Gamma(C,\mu,\tau)$ be an admissible W_n-graph. Suppose that $\Lambda = \{\lambda\}$, where $\lambda \in P(n)$, is the set of molecule types for Γ, and let $\mathcal{I} = \mathcal{I}_\lambda$ index the molecules of Γ. For each $\alpha \in \mathcal{I}$, let $C_\alpha = C_{\alpha,\lambda} = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ be the vertex set of a molecule of Γ. Then $\Gamma = \bigsqcup_{\alpha\in\mathcal{I}} \Gamma(C_\alpha)$, and $\Gamma(C_\alpha)$ is isomorphic to Γ_λ for each $\alpha \in \mathcal{I}$.*

Proof. If $|\mathcal{I}| = 1$ then $\Gamma = \Gamma(C_\alpha)$. Since $\Gamma(C_\alpha)$ is a monomolecular admissible W_n-cell of type λ, Proposition 9.5 says that $\Gamma(C_\alpha)$ is isomorphic to Γ_λ. So we assume that $|\mathcal{I}| > 1$.

If $\Gamma = \bigsqcup_{\alpha\in\mathcal{I}} \Gamma(C_\alpha)$ then for each $\alpha \in \mathcal{I}$ the set $C_\alpha = \{c_{\alpha,t} \mid t \in \mathrm{Std}(\lambda)\}$ is the vertex set of a monomolecular admissible W_n-cell of type λ. The result then follows immediately from Proposition 9.5, which says that each $\Gamma(C_\alpha)$ is isomorphic to Γ_λ. Thus it suffices to show that $\Gamma = \bigsqcup_{\alpha\in\mathcal{I}} \Gamma(C_\alpha)$.

Suppose otherwise. Then there exists $\alpha \in \mathcal{I}$ such that

$$\mathrm{Ini}_\alpha(\Gamma) = \{t \in \mathrm{Std}(\lambda) \mid \mu(c_{\beta,u},c_{\alpha,t}) \neq 0 \text{ for some } (\beta,u) \in (\mathcal{I} \smallsetminus \{\alpha\}) \times \mathrm{Std}(\lambda)\} \neq \varnothing,$$

and we let t' be the element of $\mathrm{Ini}_\alpha(\Gamma)$ that is minimal in the lexicographic order on $\mathrm{Std}(\lambda)$. Choose $(\beta,u') \in (\mathcal{I} \smallsetminus \{\alpha\}) \times \mathrm{Std}(\lambda)$ with $\mu(c_{\beta,u'},c_{\alpha,t'}) \neq 0$. Since Γ satisfies the W_n-Simplicity Rule (by Theorem 5.8), the assumption that $\alpha \neq \beta$ and $\mu(c_{\beta,u'},c_{\alpha,t'}) \neq 0$ implies that $\mathrm{D}(t') \subsetneq \mathrm{D}(u')$. Moreover, since Γ is ordered (by Theorem 8.18), $\alpha \neq \beta$ implies that $u' < t'$. Hence (u',t') is a probable pair.

Let i be the restriction number of (u',t') and $j = \max(\mathrm{SD}(t'))$. Let $(u,t) \in F(u',t')$. It is clear that (u,t) is i-restricted and favourable. Thus Lemma 9.4 shows that (u,t) is probable, $\max(\mathrm{SD}(t)) = \max(\mathrm{SD}(t')) = j$, and $\mu(c_{\beta,u},c_{\alpha,t}) = \mu(c_{\beta,u'},c_{\alpha,t'}) \neq 0$. Furthermore, since $i \in \mathrm{D}(u) \smallsetminus \mathrm{D}(t)$ (since (u,t') is favourable), and since $j \in \mathrm{D}(t)$ and $\mathrm{D}(t) \subsetneq \mathrm{D}(u)$ (since (u,t) is probable), it follows that $j \in \mathrm{D}(t)$ and $i \notin \mathrm{D}(t)$, and $i, j \in \mathrm{D}(u)$. Let $v = s_jt$, and note that $v \in \mathrm{Std}(\lambda)$ and $v < t$. Since $i < j$ by Lemma 9.2, either $i < j-1$ or $i = j-1$.

Suppose first that $i < j-1$. It follows by Lemma 7.28 (i) that $i, j \notin \mathrm{D}(v)$. Moreover, since $\mu(c_{\alpha,t},c_{\alpha,v}) = 1$ by Corollary 7.22, and since $\mu(c_{\beta,u},c_{\alpha,t}) \neq 0$, it follows that $(c_{\alpha,v},c_{\alpha,t},c_{\beta,u})$ is an alternating directed path of type (j,i).

Since Γ is admissible, if $\mu(c_{\beta,u},c_{\alpha,t}) \neq 0$ then $\mu(c_{\beta,u},c_{\alpha,t}) > 0$. So it follows that $N^2_{j,i}(\Gamma;v,u) > 0$, and so $N^2_{i,j}(\Gamma;v,u) > 0$, since Γ satisfies the W_n-Bonding Rule. Thus there exists at least one $(\delta,x_1) \in \mathcal{I} \times \mathrm{Std}(\lambda)$ such that $(c_{\alpha,v},c_{\delta,x_1},c_{\beta,u})$ is an alternating directed path of type (i,j). If $\delta \neq \alpha$ then $v \in \mathrm{Ini}_\alpha(\Gamma)$. Now since $t'{\uparrow}i = t{\uparrow}i$, we have $v <_{\text{lex}} t'$ by Lemma 7.29. This, however, contradicts the definition of t'. Hence $\delta = \alpha$, and $x_1 \in \mathrm{Ini}_\alpha(\Gamma)$. Now Theorem 8.18 shows that either $x_1 = s_iv$ and $i \in \mathrm{SA}(v)$, or else $x_1 < v$. But $x_1 <_{\text{lex}} t'$ by Lemma 7.29 (i) in the former case, and $x_1 <_{\text{lex}} t'$ by Lemma 7.29 (ii) in the latter case. Both alternatives contradict the definition of t', thus showing that $i < j-1$ is impossible.

Suppose now that $i = j-1$. By Lemma 9.3, we have $\mathrm{col}_t(j-1) \neq \mathrm{col}_t(j+1)$, and it follows that either $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$ or $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$.

Case 1.
Suppose that $\mathrm{col}_t(j-1) < \mathrm{col}_t(j+1)$. The result follows by the same argument as above, with $j-1$ replacing i and Lemma 7.28 (ii) replacing Lemma 7.28 (i).

Case 2.
Suppose that $\mathrm{col}_t(j-1) > \mathrm{col}_t(j+1)$. Then $j-1 \in \mathrm{SD}(v)$, and $j-1 \in \mathrm{D}(v)$, and $j \notin \mathrm{D}(v)$ by Lemma 7.28 (iii). Note that since $j-1 \in \mathrm{SD}(v)$, we have $s_{j-1}v \in \mathrm{Std}(\lambda)$ and $s_{j-1}v < v$. Let $w = s_{j-1}v$. It follows by Lemma 7.28 (iii) that $j-1, j \notin \mathrm{D}(w)$.

Next, since $\mu(c_{\alpha,c}, x_{\alpha,w}) = \mu(c_{\alpha,t}, c_{\alpha,v}) = 1$ by Corollary 7.22 and since $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$, it follows that $(c_{\alpha,w}, c_{\alpha,v}, c_{\alpha,t}, c_{\beta,u})$ is an alternating directed path of type $(j-1, j)$.

Since Γ is admissible, if $\mu(c_{\beta,u}, c_{\alpha,t}) \neq 0$ then $\mu(c_{\beta,u}, c_{\alpha,t}) > 0$. So it follows that $N^3_{j-1,j}(\Gamma; w, u) > 0$, and so $N^3_{j,j-1}(\Gamma; w, u) > 0$, since Γ satisfies the W_n-Bonding Rule. Thus there exists at least one $(\delta, x_1) \in \mathcal{I} \times \mathrm{Std}(\lambda)$ and one $(\gamma, x_2) \in \mathcal{I} \times \mathrm{Std}(\lambda)$ such that $(c_{\alpha,w}, c_{\delta,x_1}, c_{\gamma,x_2}, c_{\beta,u})$ is an alternating directed path of type $(j, j-1)$. If $\delta \neq \alpha$ then $w \in \mathrm{Ini}_\alpha(\Gamma)$. Now since $t\!\uparrow\!(j-1) = t'\!\uparrow\!(j-1)$, we have $w <_{\mathrm{lex}} t'$ by Lemma 7.29 (iii). This, however, contradicts the definition of t'. Therefore, $\delta = \alpha$.

Since $\mathrm{D}(x_1) \cap \{j-1, j\} = \{j\}$ and $\mathrm{D}(x_2) \cap \{j-1, j\} = \{j-1\}$, and $\mu(c_{\gamma,x_2}, c_{\delta,x_1}) \neq 0$, it follows from the W_n-Simplicity Rule that $\{c_{\delta,x_1}, c_{\gamma,x_2}\}$ is a simple edge. Thus $\gamma = \delta$, and x_1 and x_2 are related by a dual Knuth move. Thus x_2 is the $(j-1)$-neighbour of x_1. We see that $x_2 \in \mathrm{Ini}_\alpha(\Gamma)$, and it will suffice to show that $x_2 <_{\mathrm{lex}} t'$, contradicting the definition of t'.

By Theorem 8.18 either $x_1 = s_j w > w$ or $x_1 < w$. If $x_1 < w$ then since $t\!\uparrow\!(j-1) = t'\!\uparrow\!(j-1)$, the conclusion $x_2 <_{\mathrm{lex}} t'$ follows from Lemma 7.29 (iv). We are left with the case $x_1 = s_j w > w$. This gives $j \in \mathrm{SA}(w)$, and we see that the conditions of Lemma 7.28 (iii) are satisfied: we have $v = s_j t$ with $j \in \mathrm{SD}(t)$ and $\mathrm{col}_t(j+1) < \mathrm{col}_t(j-1)$, and $w = s_{j-1}v$. Since $j \in \mathrm{SA}(w)$ it follows that $j-1 \in \mathrm{SA}(x_1)$, and $s_{j-1}x_1$ is the $(j-1)$-neighbour of x_1. Thus $x_2 = s_{j-1}x_1 = s_{j-1}s_j w$, and since $t\!\uparrow\!(j-1) = t'\!\uparrow\!(j-1)$, we have $x_2 <_{\mathrm{lex}} t'$ by Lemma 7.29 (iii). □

REMARK 9.7. Since $\alpha \neq \beta$ and $\mu(c_{\beta,u'}, c_{\alpha,t'}) \neq 0$, it follows by Remark 8.8 that $A(u', t') \neq \varnothing$. Let $(u, t) \in A(u', t')$, noting that $(u, t) \in F(u', t')$. Then $t' \Uparrow i$ is i-critical. (The proof is very much the same as that for Proposition 8.11.) Since $i < j$ (as shown above) and $j < i+2$, since $t\!\uparrow\!(i+1)$ is minimal, we have $j = i+1$. Definition 6.17 says that $\mathrm{col}_t(i+2) = \mathrm{col}_t(i)$. But $\mathrm{col}_t(i+2) \neq \mathrm{col}_t(i)$ by Lemma 9.3. This contradiction provides an alternative proof for Proposition 9.6.

We are now in a position to state and prove the main result of the paper.

THEOREM 9.8. *Admissible cells of type A_{n-1} are Kazhdan–Lusztig.*

Proof. Let $\Gamma = \Gamma(C, \mu, \tau)$ be an admissible W_n-cell, and let Λ be the set of molecule types for Γ. By Lemma 8.16, $\Lambda = \{\lambda\}$ for some $\lambda \in P(n)$. Let $\mathcal{I} = \mathcal{I}_\lambda$ be the indexing set for the molecules of Γ, and let, for each $\gamma \in \mathcal{I}$, $C_\gamma = C_{\gamma,\lambda} = \{c_{\gamma,w} \mid w \in \mathrm{Std}(\lambda)\}$ be the vertex set of a molecule of Γ. By Proposition 9.6, $\Gamma = \bigsqcup_{\gamma \in \mathcal{I}} \Gamma(C_\gamma)$, where for each $\gamma \in \mathcal{I}$, $\Gamma(C_\gamma)$ is isomorphic to Γ_λ. Since Γ is an admissible W_n-cell by hypothesis, it follows that $\mathcal{I} = \{\gamma\}$, whence $\Gamma = \Gamma(C_\gamma)$ and Γ is isomorphic to Γ_λ. Since Γ_λ is isomorphic to $\Gamma(C(\tau_\lambda))$, it follows from Corollary 6.31 that Γ is isomorphic to a Kazhdan–Lusztig left cell. □

REMARK 9.9. Let $\lambda \in P(n)$ and let $D(\lambda) = \bigsqcup_{t \in \mathrm{Std}(\lambda)} C(t)$, the Kazhdan–Lusztig two-sided cell corresponding to λ. By Remark 8.19, the singleton set $\{\lambda\}$ is the set of molecule types of the admissible W_n-graph $\Gamma(D(\lambda))$. It follows from Proposition 9.6 that $\Gamma(D(\lambda))$ is a disjoint union of the Kazhdan–Lusztig left cells $\Gamma(C(t))$. This implies the following well known result (see, for example, [7, Theorem 5.3]).

THEOREM 9.10. *Let $\lambda \in P(n)$ and $y, w \in D(\lambda)$. If $y \preceq_{\mathrm{L}} w$ then $y, w \in C(t)$ for some $t \in \mathrm{Std}(\lambda)$.*

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