Dynamos driven by modified Beltrami flows

A.A. Bachtiar*[1] and R.W. James[2]

[1]Department of Mathematics, University of Indonesia
[2]School of Mathematics and Statistics, University of Sydney

February 8, 2019

Abstract

Bachtiar, Ivers and James (2006, BIJ), and Bachtiar & James (2010), showed that a planar fluid velocity $\mathbf{v}$ can support dynamo action in a conducting sphere. BIJ also tried to convert some historical flows to planar flows, via a process they termed 'planarizing'. In particular BIJ studied one of many flows considered by Pekeris, Accad and Shkoller (1973, PAS). The PAS flow was chosen because it produced a dynamo at low truncation levels and critical magnetic Reynolds number R_c, this possibly influenced by it being a helical Beltrami flow (i.e. $\nabla \times \mathbf{v} \parallel \mathbf{v}$). BIJ were able to planarize the poloidal part of the PAS flow, but not the toroidal part. The present paper considers more partly planarized PAS flows; and two modifications of PAS, labelled biPAS and quasiPAS, that can be fully planarized. BiPAS flows are just a sum of two PAS flows, and quasiPAS are PAS-like with a modified radial cell structure for the toroidal flow. We have studied 128 models using these flows, found 96 dynamos including 84 new dynamos, but found no dynamos with fully planar flows. The partly planarized PAS, the biPAS and quasiPAS flows are not Beltrami. But 20 (normalized using rms($\mathbf{v}$), or 22 using $\max|\mathbf{v}|$) R_c of the associated dynamos are lower than the R_c of the Beltrami dynamos from which they were derived, showing that the Beltrami property is not essential for low R_c.

*alhaji@sci.ui.ac.id

1 Introduction

It is generally believed that a magnetohydrodynamic dynamo process is responsible for the large-scale behavior of the geomagnetic field and other astroplanetary magnetic fields. This report has evolved from an attempt to find dynamos maintained by incompressible purely planar flows, i.e. where the conducting fluid's velocity

$$\mathbf{v} = \nabla \times (f\mathbf{e}_z) \tag{1}$$

is perpendicular to the cartesian unit vector $\mathbf{e}_z$, and has streamlines $f =$ const., $z =$ const. We will adopt the terminology of Bachtiar, Ivers & James (2006, BIJ). The existence of planar velocity dynamos is precluded in some circumstances by an antidynamo theorem, the Planar Velocity Theorem (PVT) (Zel'dovitch & Ruzmaikin 1980; BIJ; Bachtiar & James 2010, BJ). The PVT applies when $\mathbf{v}$ and the induced magnetic field $\mathbf{B}$ decrease sufficiently fast as radius $r \to \infty$; and when the conducting volume V is either all space V_∞, or has its boundary Σ everywhere normal to $\mathbf{e}_z$ (e.g. when V is a plane layer $|z| \leq$ const. or half-space $|z| \leq 0$). But the PVT proofs fail when V is finite (BIJ). In particular, for V being a sphere with an insulating exterior $\widehat{V}$, BIJ discovered a model, their p1Y22DM12, where $\mathbf{B}$ did not decay in time (corroborated by BJ; Li, Livermore & Jackson (2010, LLJ). [We note that there is some variability and misunderstanding in the literature concerning 2D flows, planar flows and the PVT, as discussed in the Appendix.]

The mathematical problem for a spherical conductor V is to find a non-decaying $\mathbf{B}$ that evolves according to the kinematic dynamo equations

$$\begin{gathered}\frac{\partial \mathbf{B}}{\partial t} = R\nabla \times (\mathbf{v} \times \mathbf{B}) + \nabla^2 \mathbf{B}\ , \quad \text{in } V, \\ \nabla \times \mathbf{B} = 0\ , \quad \text{in } \widehat{V}\ ,\end{gathered} \tag{2}$$

here non-dimensionalized on the diffusion time scale and radius of V as length scale. The exterior $\widehat{V}$ is assumed current-free. Since $\mathbf{v}$ is steady, solutions $\mathbf{B} = \mathbf{B}_\lambda(\mathbf{r})\, \mathrm{e}^{\lambda t}$ can be sought. The fundamental aim is to find, if it exists, the smallest critical magnetic Reynolds number R_c where $\Re(\lambda) = 0$.

Following BIJ, field $\mathbf{B}$ is expanded in poloidal-toroidal vectors $\mathbf{S}_n^m$, $\mathbf{T}_n^m$, and similarly for $\mathbf{v}$, $\mathbf{s}_n^m$, $\mathbf{t}_n^m$. Also expanding $f = \sum_{n,m} f_n^m(r) Y_n^m(\theta, \phi)$ in spherical harmonics, BIJ showed that the corresponding poloidal and toroidal velocity coefficients for planar velocity (1) were (with superscript m suppressed)

$$s_n = \frac{im}{n(n+1)} f_n \,, \tag{3}$$

$$t_{n-1} = \frac{\alpha_n}{n} \, d_{n+1} f_n \,, \tag{4}$$

$$t_{n+1} = -\frac{\alpha_{n+1}}{n+1} \, d_{-n} f_n \,. \tag{5}$$

Here $\alpha_n := \surd\,[(n^2 - m^2)/(4n^2 - 1)]$ and $d_n := \mathrm{d}/\mathrm{d}r + n/r$. Thus, for given f_n, the planar flow will generally have vector form $\mathbf{s}_n + \mathbf{t}_{n-1} + \mathbf{t}_{n+1}$. By prescribing f_n BIJ used (3)–(5) to find λ and $\mathbf{B}_\lambda$. In particular, the successful BIJ planar flow of model p1Y22DM12 used stream function

$$f = 2\Re\left\{ f_2^2 \, Y_2^2 \right\} \quad \text{where} \quad f_2^2 = r^2(1 - r^2) \,. \tag{6}$$

Variations with $f_2^2 \, Y_2^2$ replaced by $r^n(1 - r^q) Y_n^m$, with $q = 2, 4, 6, ...;\ n, m = 2, 4$; and including shell geometry, also exhibit dynamo action (Bachtiar 2009). But all these planar flow models suffer from slow convergence w.r.t. the discretization levels of the numerical solution.

Apart from investigating p1Y22DM12, BIJ also tried to transform several historical flows into planar flows consistent with (3)–(5). They coined the phrase ‘planarizing’ for this process, which comprised several steps.

To planarize a given s_n flow where $m \neq 0$, (3) can be solved for $f_n = -\mathrm{i}n(n+1)s_n/m$, and $t_{n\pm1}$ then found from (4),(5).

To planarize a given t_n flow two paths can be considered. (a) If $\alpha_{n+1} \neq 0$ then, assuming a fixed boundary at $r = 1$, (4) can be solved for

$$f_{n+1} = \frac{n+1}{\alpha_{n+1}} r^{-n-2} \int_1^r r^{n+2} \, t_n \, dr \ , \tag{7}$$

and (3),(5) used to create s_{n+1}, t_{n+2}. A necessary condition for $\mathbf{v}$ to be single-valued at $r = 0$ is $f_{n+1}(0) = 0$ which, given(7), requires

$$\int_0^1 r^{n+2} \, t_n \, dr = 0 \ . \tag{8}$$

If (8) is satisfied then a planar flow can be found with form $\mathbf{t}_n + \mathbf{s}_{n+1} + \mathbf{t}_{n+2}$. Alternatively (b) if $\alpha_n \neq 0$ then (5) can be solved for

$$f_{n-1} = -\frac{n}{\alpha_n} r^{n-1} \int_1^r r^{-n+1} \, t_n \, dr \ , \tag{9}$$

and (3),(4) used to create s_{n-1}, t_{n-2}. In this case a consistency condition like (8) is not required and the resulting planar flow has form $\mathbf{t}_n + \mathbf{s}_{n-1} + \mathbf{t}_{n-2}$.

BIJ referred to flows as 'partly planarized' when only the poloidal s, or toroidal t component of the flow was planarized. It is convenient to introduce parameters p_s, p_t and label flows as (p_s, p_t)-planarized. In this paper we consider only $p_s, p_t = 0$ or 1, so $(0,0)$-planarized means unplanarized flow, $(1,0)$ means the s component is planarized, $(0,1)$ means the t component is planarized, and $(1,1)$ means both s and t components are planarized.

Since the spherical harmonic order $m = 2$ for all flows herein, the magnetic field decouples into chains

$$\begin{aligned} &\mathrm{M02} : S_n^m\,, T_n^m\,;\; n = 1,2,3,\ldots;\; m = 0(\mathrm{mod}2)\;; \\ &\mathrm{M12} : S_n^m\,, T_n^m\,;\; n = 1,2,3,\ldots;\; m = 1(\mathrm{mod}2)\;; \end{aligned} \tag{10}$$

which can be considered separately as dynamo candidates. The first mode S_1^0 in M02 is an axial dipole, whereas S_1^1 in M12 is an equatorial dipole. We note that in the real formalism of PAS, flow (11) permits decoupling into 4 independent magnetic chains labelled A,B,C,D by PAS. But with any planarization, even only (1,0) or (0,1), such decoupling fails, whilst the decoupling in (10) remains valid.

One particular flow considered by BIJ was an s_2^2, t_2^2 flow of Pekeris, Accad and Shkoller (1973, PAS). Using their formulae, BIJ could only planarize s_2^2 since t_2^2 failed the consistency condition (8). However, the resulting $(1,0)$-planarized PAS flow produced an M02 dynamo with smaller R_c than the original PAS flow (BIJ). This result and the good convergence of PAS models prompted us to modify, as described in §2.1, the original PAS flows to forms that can be fully (1,1)-planarized by the BIJ procedure, the aim being to find planar flow dynamos with better convergence than p1Y22DM12 and related models.

2 PAS based flows

In the following subsection §2.1 we give the background theory for partly planarizing PAS flows. In §§2.2,2.3 we describe our 2 modifications to PAS, i.e. biPAS and quasiPAS, that allow full planarization.

2.1 PAS flows

PAS found a number of kinematic dynamos based on helical Beltrami flows (i.e. $\nabla \times \mathbf{v} \parallel \mathbf{v}$) in particular

$$\mathbf{v} = 2\Re\{\mathbf{s}_2^2 + \mathbf{t}_2^2\}, \tag{11}$$

where

$$s_2^2 = K\Lambda_i\, j_2(\Lambda_i r)\ , \qquad t_2^2 = \Lambda_i\, s_2^2\ . \tag{12}$$

Here Λ_i is the i^{th} positive root of the spherical Bessel function j_2, and $K = \sqrt{(6/5)}$ in the BIJ formalism. For geophysical and convergence reasons PAS initially paid special attention to the case $i = 3$. This is the flow considered by BIJ and Dudley & James (1989, DJ).

Whilst the PAS flow (11) cannot be fully planarized, its s_2^2 component can be planarized by using (3),(5) and the spherical Bessel property

$$d_{-n}\, j_n(\Lambda r) = -\Lambda r\, j_{n+1}(\Lambda r)\,. \tag{13}$$

One can thus construct an add-on component

$$t_3^2 = -\frac{\alpha_3^2}{3}\, d_{-2}\, f_2^2 = -\mathrm{i}\, \alpha_3 K \Lambda_i^2\, j_3(\Lambda_i r)\ . \tag{14}$$

[Equation (4) yields $t_1^2 = 0$ as usual for spherical harmonic coefficients.] The combination $2\Re\{\mathbf{s}_2^2 + \mathbf{t}_3^2\}$ is then planar, and (11) is (1,0)-planarized to

$$\mathbf{v} = 2\Re\{\mathbf{s}_2^2 + \mathbf{t}_3^2 + \mathbf{t}_2^2\}\,. \tag{15}$$

Our results for the (0,0)-PAS flow (11) and the (1,0)-planarized PAS flow (15) are given in §3.1.

2.2 BiPAS flows

Our first modified PAS flow, designated biPAS, is defined as the superposition of two flows of PAS type (12), i.e.

$$s_2^2 = K\Lambda_i\, j_2(\Lambda_i r) + CK\Lambda_k\, j_2(\Lambda_k r)\,, \tag{16}$$
$$t_2^2 = K\Lambda_i^2\, j_2(\Lambda_i r) + CK\Lambda_k^2\, j_2(\Lambda_k r)\,, \tag{17}$$

where $C = \text{const.} \neq 0$, Λ_i and Λ_k ($k \neq i$) are the i^{th} and k^{th} positive roots of $j_2(r)$, and $K = \surd(6/5)$ as before. Whilst this biPAS flow is the superposition of 2 Beltrami flows, it is itself not Beltrami. Use of the Bessel property

$$\int r^{n+1} j_{n-1}(\Lambda r)\, dr = \frac{1}{\Lambda}\, r^{n+1}\, j_n(\Lambda r) + \text{const.} \tag{18}$$

shows that consistency condition (8) is satisfied by choosing

$$C = -\frac{\Lambda_i\, j_3(\Lambda_i)}{\Lambda_k\, j_3(\Lambda_k)}. \tag{19}$$

Substituting (19) into (16),(17) yields

$$s_2^2 = \frac{K\Lambda_i}{j_3(\Lambda_k)}\left[j_3(\Lambda_k) j_2(\Lambda_i r) - j_3(\Lambda_i) j_2(\Lambda_k r)\right] , \tag{20}$$

$$t_2^2 = \frac{K\Lambda_i}{j_3(\Lambda_k)}\left[\Lambda_i\, j_3(\Lambda_k) j_2(\Lambda_i r) - \Lambda_k\, j_3(\Lambda_i) j_2(\Lambda_k r)\right] . \tag{21}$$

The biPAS flow defined by (16), (17) reverts to a PAS flow if $C = 0$, and yields free decay if $i = k$, whence $C = -1$ by (19). To planarize s_2^2 in (20) we used (3),(5) and (13) to construct the add-on

$$t_3^2 = \mathrm{i}\, \alpha_3\, \frac{K\Lambda_i}{j_3(\Lambda_k)}\Big(- \Lambda_i\, j_3(\Lambda_k)\, j_3(\Lambda_i r) + \Lambda_k\, j_3(\Lambda_i)\, j_3(\Lambda_k r)\Big) . \tag{22}$$

To planarize t_2^2 in (21), we used (7),(3),(5)(13) to construct the add-ons

$$s_3^2 = \frac{\mathrm{i}\sqrt{7}}{2}\frac{K\Lambda_i}{j_3(\Lambda_k)}\Big(j_3(\Lambda_k) j_3(\Lambda_i r) - j_3(\Lambda_i) j_3(\Lambda_k r)\Big) , \tag{23}$$

$$t_4^2 = \frac{\sqrt{3}}{2}\frac{K\Lambda_i}{j_3(\Lambda_k)}\Big(\Lambda_i\, j_3(\Lambda_k) j_4(\Lambda_i r) - \Lambda_k\, j_3(\Lambda_i) j_4(\Lambda_k r)\Big) . \tag{24}$$

The resulting combination

$$\mathbf{v} = 2\Re\{\mathbf{s}_2^2 + \mathbf{t}_3^2 + \mathbf{t}_2^2 + \mathbf{s}_3^2 + \mathbf{t}_4^2\} \tag{25}$$

is then a fully planarized biPAS flow, or (1,1)-biPAS for short. Our results for biPAS flows are given in §3.2.

2.3 QuasiPAS flows

Our second modification of PAS is the quasiPAS flow defined by changing the radial argument of the PAS t_2^2 flow component, i.e. replacing (12) by

$$s_2^2 = K\Lambda_i\, j_2(\Lambda_i r)\ , \qquad t_2^2 = K\Lambda_i^2\, j_2(\Gamma_k r), \tag{26}$$

where Γ_k is the k-th positive root of j_3 rather than j_2 as in PAS. Like biPAS, the quasiPAS flow is not Beltrami. Using (18), the quasiPAS flow (26) can be shown to satisfy the consistency condition (8).

The planarization of s_2^2 is achieved as in §2.1, by the addition of t_3^2 given by (14). To planarize t_2^2 in (26) we use (7),(3),(5),(13) to construct the add-ons

$$s_3^2 = \frac{\mathrm{i}K}{2\alpha_3}\frac{\Lambda_i^2}{\Gamma_k}\, j_3(\Gamma_k\, r), \tag{27}$$

$$t_4^2 = \frac{\sqrt{3}}{2}\, K\Lambda_i^2\, j_4(\Gamma_k\, r). \tag{28}$$

The resulting fully planarized (1,1)-quasiPAS flow has the same form as (25). Our results for quasiPAS flows are given in §3.3.

3 Results

In §§3.1–3.3 that follow we give results corresponding to the flows defined in §§2.1–2.3. But first we given some common background concerning notation and numerical method.

Including planarization parameters p_s, p_t to allow full, part or no planarization, yields PAS, biPAS and quasiPAS flows with general form

$$\mathbf{v} = 2\Re\{(\mathbf{s}_2^2 + p_s\mathbf{t}_3^2) + (\mathbf{t}_2^2 + p_t(\mathbf{s}_3^2 + \mathbf{t}_4^2))\}\ . \tag{29}$$

The poloidal and toroidal scalar functions for (29) are given by

(i) (12),(14) for PAS, with $p_t = 0$ since (8) can't be satisfied;

(ii) (20)–(24) for biPAS;

(iii) (26)–(28) for quasiPAS.

Since $m = 2$ for all flows in this paper, a rotation through $\Delta\phi = \pi/2$ simply reverses the flow. So any R_c occur in $\pm$ pairs and we may assume $R > 0$. Because some results depend on velocity normalization, we use 3 different magnetic Reynolds numbers:

(i) un-normalized R', notation as in PAS, calculated from (2) using $\mathbf{v}$ given by (12),(14);

(ii) $R = v_{\rm rms} R'$ as in PAS, normalized using $v_{\rm rms} = \sqrt{\left[3/(4\pi) \int_V |\mathbf{v}|^2 \, dV\right]}$;

(iii) $R^\uparrow = v_{\rm max} R'$ normalized using $v_{\rm max} = \max_V |\mathbf{v}|$. Results based on $R^\uparrow$ may differ markedly from those using R. So we use a $^\uparrow$ generally to indicate when differences occur. For results independent of normalization, the $^\uparrow$ will be omitted.

$v_{\rm rms}$ was calculated using the orthogonality of $\mathbf{s}_n^m$, $\mathbf{t}_n^m$ which implies

$$v_{\rm rms} = \left[\sum_{n,m} 3n(n+1) \int_0^1 \left(n(n+1)\,|s_n|^2 + \left| \frac{d(rs_n)}{dr} \right|^2 + |rt_n|^2 \right) dr \right]^{\frac{1}{2}} . \tag{30}$$

$v_{\rm max}$ was estimated by finding $\max_V |\mathbf{v}|$ over r, θ, ϕ grids using interval-halving with re-used $\mathbf{v}$ evaluations and the symmetry $\mathbf{v}(\phi) = -\mathbf{v}(\phi + \pi/2)$. The grids were chosen fine enough to obtain about 4 digit accuracy. Similar calculations using x, y, z grids gave good agreement.

The dynamo problem (2) was discretized using second-order finite differences with J radial subdivisions and Legendre degrees $n \le N$. The critical R_c was determined by considering convergence of λ w.r.t. J and N, this being shown by BJ to be a sensitive test. Generally we searched for non-decaying $\mathbf{B}$ over at least the interval $0 < R' < 4$, and for relatively small frequency $\omega = \Im(\lambda)$. If that failed we searched more widely over at least $0 < \omega < 100$. These search regions are much more extensive than the region needed for the original (0,0)-PAS flows (11) where $R'_c \lesssim 2.76$ with $\omega = 0$ suffices.

For brevity we will use labels like bPASps0pt1ΛiΛkM12 for the biPAS model with $p_s\!=\!0$, $p_t\!=\!1$, Λ_i, Λ_k, M12; and qPASps1pt0ΛiΓkM02 for the quasiPAS model with $p_s = 1$, $p_t = 0$, Λ_i, Γ_k, M02; and similarly for other models. We also use abbreviated substrings, e.g. ps0pt1, Λ1Γ2, where the context clearly identifies other parameters. For comparing R_c-values we will use the (0,0)-PAS results as a benchmark, and consider the effects of planarizing, and dependence on magnetic chains and velocity normalization.

3.1 PAS results

PAS reported results for the (0,0) flows (11) with Λ_i ($i = 1, \ldots, 8, 10, 15, 20$) for chain M02, and with Λ_i ($i = 1, 2, 3$) for M12. They found working dynamos in all those cases (PAS Tables 4,8).

Since PAS flows cannot be (0,1) or (1,1)-planarized, we here report only on (0,0) and (1,0) planarizations, the latter using (15). And since $p_t = 0$ for all these models we will suppress the pt0 sublabel in this section. We used Λ_i ($i = 1, \ldots, 5, 10, 15, 20$) for both M02 and M12, specifically

$$\begin{array}{l} \Lambda_1 = 5.7634\ldots,\ \Lambda_2 = 9.0950\ldots,\ \Lambda_3 = 12.322\ldots,\ \Lambda_4 = 15.514\ldots \\ \Lambda_5 = 18.689\ldots,\ \Lambda_{10} = 34.470\ldots,\ \Lambda_{15} = 50.205\ldots,\ \Lambda_{20} = 65.927\ldots \end{array} \quad (31)$$

and our results are shown in Table 1.

BIJ considered only $\Lambda3$ and chain M02, but found that the (1,0)-planarized PAS flow (15) produced a working dynamo with unnormalized R_c'(ps1Λ3M02) = 0.23 noticeably smaller than the original (0,0)-PAS R_c'(ps0Λ3M02) = 0.37. [R_c' was wrongly reported as 0.27 in BIJ but correctly depicted as 0.23 in BIJ's Figure 5(c).] BIJ did not consider velocity normalization, for which Table 1 shows that the difference is even larger, namely R_c(ps1Λ3M02) = 19.6 ($67.9^\uparrow$) compared with R_c(ps0Λ3M02) = 29.3 ($112^\uparrow$).

The roots (31) represent 8 flows for each of the (0,0),(1,0) planarizations, giving 16 flows for each magnetic chain M02,M12, and 32 PAS models in total. All 32 models work as stationary dynamos, with R_c as in Table 1. Convergent solutions were obtained at tractable truncation levels $J \leq 1600$, $N \leq 30$ with eigenmatrix sizes tending to increase with Λ_i but $\lesssim$ 50GB. Of these, 11 (i.e. all the ps0ΛiM02 ($i = 1, \ldots, 5, 10, 15, 20$) and ps0$\Lambda$iM12 ($i \leq 3$)) were reported in PAS Tables 4,8 which agree closely with Table 1 herein; and ps1Λ3M02 was reported by BIJ. [We also found that other i-values, e.g. $i = 6, \ldots, 9, 11, \ldots 14, 16, \ldots, 19$ produce working dynamos, but results are omitted here since they don't add substantially to the discussion.] So Table 1 contains 20 new dynamos and leads to the following comments.

For fixed Λ_i, the effects of planarizing are opposite for M02, M12. For M02, R_c(ps1Λi) < R_c(ps0Λi). So (1,0)-planarizing reduces R_c. Indeed all 8 (1,0) M02 models have significantly lower R_c than the corresponding (0,0) models. [Notably, for M02 the least (1,0)-PAS R_c, i.e. R_c(ps1Λ20) = 18.8 and $R_c^\uparrow$(ps1Λ2) = 54.5, are also significantly less than the least (0,0)-PAS,

R_c(ps0Λ20) = 26.3 and $R_c^{\uparrow}$(ps0Λ2) = 102.] On the other hand, for M12 R_c(ps1Λi) > R_c(ps0Λi). So (1,0)-planarizing increases R_c. All 8 (1,0) M12 models have higher R_c than the corresponding (0,0) models, but by relatively small differences compared to the M02 case. [For M12, the least (1,0) R_c, i.e. R_c(Λ20) = 19.0, is a bit greater than the least (0,0), R_c(Λ20) = 15.1, but the least (1,0) and least (0,0) $R_c^{\uparrow}$, i.e. $R_c^{\uparrow}$(Λ20) and $R_c^{\uparrow}$(Λ1), are both ≈ 32.]

R_c and $R_c^{\uparrow}$ exhibit contrasting trends w.r.t. their dependence on Λ_i, as depicted in Figure 1. (i) The $v_{\rm rms}$ R_c decrease monotonically as i, Λ_i increase, and approach asymptotic minima close to R_c(ps0Λ20M02) = 26.3 (this one known to PAS), R_c(ps0Λ20M12) = 15.1, R_c(ps1Λ20M02) = 18.8, and R_c(ps1Λ20M12) = 19.0. Also, for M12 there is very little dependence of R_c on i. (ii) By contrast, the $v_{\rm max}$ $R_c^{\uparrow}$ increase rapidly and linearly w.r.t. i for $i \gtrsim 3$.

For a given flow, i.e. for p_s, Λ_i fixed, R_c(ps0M12) < R_c(ps0M02) $\forall i$, and R_c(ps1ΛiM12) < R_c(ps1ΛiM02) for $i < 5$. For $p_s = 1$ and $i \geq 5$ there is little M dependence, indeed R_c(ps1M12) = R_c(ps1M02) to 2 digits. So where there is noticeable difference, M12 is advantageous in producing lower R_c.

Overall the least (0,0)-PAS R_c is just less than the least (1,0) R_c [i.e. R_c(ps0 Λ20M12) = 15.1 < R_c(ps1Λ20M02) = 18.8]. But the least (0,0) and least (1,0) $R_c^{\uparrow}$ are almost equal [$R_c^{\uparrow}$(ps0Λ1M12) = 32.3, $R_c^{\uparrow}$(ps1Λ1M12)= 32.4].

3.2 BiPAS results

Relations (20)–(24) for $s_2^2, t_2^2, t_3^2, s_3^2, t_4^2$ are all antisymmetric w.r.t. interchange of i, k except for a multiplicative factor $\tau_{ik} := \Lambda_i / j_3(\Lambda_k)$. So interchange of i, k corresponds to flow reversal (or reversing sign(R)) which can be counteracted by rotation of coordinate frame through $\Delta\phi = \pi/2$ as earlier indicated. And normalizing R_c removes dependence on the factor τ_{ik}, so if R_c exists

$$R_c(\Lambda\mathrm{i}\Lambda\mathrm{k}) = R_c(\Lambda\mathrm{k}\Lambda\mathrm{i})\,. \tag{32}$$

Also, if $i = k$ then $s_2^2, \ldots, t_4^2$ in (20)–(24) are all zero, corresponding to free decay. So, for determining R_c it suffices to consider $i < k$. [Aside: numerical tests on the ratio $\rho := |\tau_{ik}/\tau_{ki}|$ show that $|\rho - 1| < .05$. It follows by interchanging i, k in (20)–(24) that even without normalization $|R_c'(\Lambda\mathrm{i}\Lambda\mathrm{k}) - R_c'(\Lambda\mathrm{k}\Lambda\mathrm{i})| \lesssim 5\%$.]

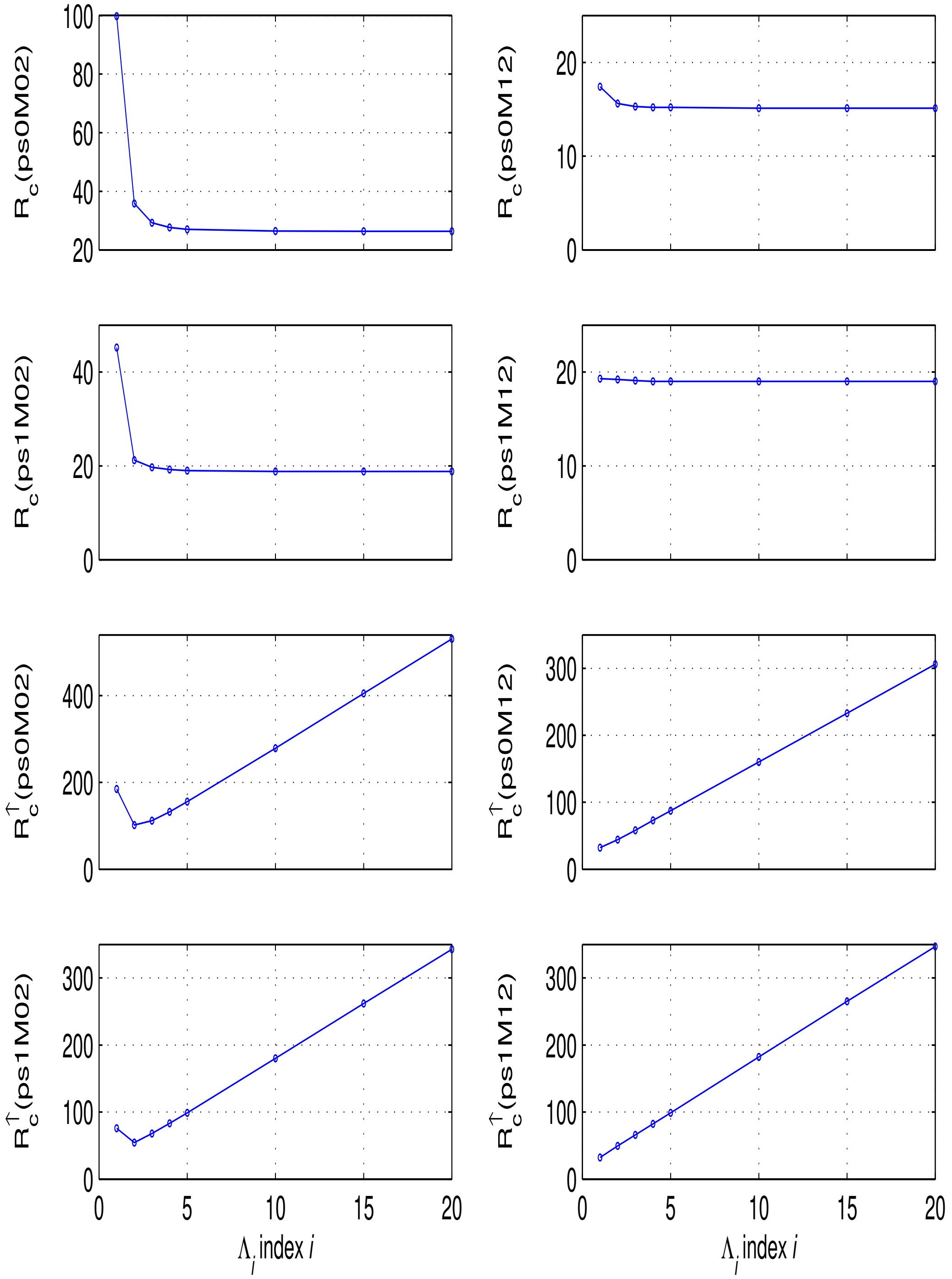

Figure 1: R_c(PAS) vs i showing monotonic decrease to asymptotic limits, and $R_c^{\uparrow}$(PAS) vs i showing linear increase, for $p_s = 0, 1$ and chains M01,M12.

					M02		M12	
p_s	p_t	i	$v_{\rm rms}$	$v_{\max}$	R_c	$R_c^\uparrow$	R_c	$R_c^\uparrow$
0	0	1	36.14	66.92	99.8	185.	17.4	32.3
		2	58.69	166.7	35.8	102.	15.6	44.3
		3	80.19	305.9	29.3	112.	15.3	58.5
		4	108.3	469.6	27.7	132.	15.2	72.8
		5	130.8	681.5	27.0	156.	15.2	87.3
		10	241.9	2318.	26.4	279.	15.1	160.
		15	352.6	4918.	26.3	405.	15.1	233.
		20	433.2	8756.	26.3	531.	15.1	306.
1	0	1	38.64	64.81	45.2	75.9	19.3	32.4
		2	62.74	161.4	21.2	54.5	19.2	49.3
		3	85.73	296.3	19.6	67.9	19.1	65.9
		4	108.3	469.6	19.2	83.1	19.0	82.3
		5	130.8	681.5	19.0	98.9	19.0	98.9
		10	241.9	2318.	18.8	180.	19.0	182.
		15	352.6	4918.	18.8	262.	19.0	265.
		20	463.1	8480.	18.8	343.	19.0	347.

Table 1: R_c for PAS models, i.e. flow (29) with parameters $p_s = 0, 1$ and $p_t = 0$; Bessel j_2 roots Λ_i; and magnetic chains M02, M12.

We used the first three positive roots of $j_2(r)$ given in (31). So the $\Lambda_i\Lambda_k$ combinations investigated were $\Lambda_1\Lambda_2$, $\Lambda_1\Lambda_3$, $\Lambda_2\Lambda_3$, giving 3 flows for each of the $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ planarizations, 12 flows for each of the magnetic chains M02, M12, and a total of 24 biPAS models.

All 18 of the non-(1,1) models worked as dynamos but we found no dynamos amongst the 6 (1,1) candidates. For the non-(1,1) models convergence w.r.t. truncation levels J, N was generally good, requiring $J \leq 500$, $N \leq 50$ and eigenmatrix sizes $\lesssim$ 100GB. One exception ps0pt1Λ2Λ3M12 required $[J, N] \sim [300, 70]$, and matrix size $\sim$250GB. All but 2 of the dynamos were stationary. The critical non-(1,1) R_c are shown in Table 2 leading to the following comments.

For fixed Λ_i, Λ_k the effects on R_c of planarizing biPAS are different for M02, M12. For M02, R_c(ps1pt0) $<$ R_c(ps0pt0) $<$ R_c(ps0pt1), i.e. (1,0)-planarizing reduces R_c whereas (0,1)-planarizing increases R_c. [So for M02 the least R_c are (1,0), namely R_c(ps1pt0Λ2Λ3M02) $=$ 16.9 (70.7$^\uparrow$).] But for M12, R_c(ps0pt0) $<$ R_c(ps1pt0) $<$ R_c(ps0pt1), i.e. any planarizing increases R_c. [So the least R_c are (0,0), namely R_c(Λ1Λ2M12)= 13.1 (42.1$^\uparrow$).] It is noticeable

						M02			M12	
p_s	p_t	i	k	$v_{\rm rms}$	$v_{\rm max}$	ω	R_c	$R_c^\uparrow$	R_c	$R_c^\uparrow$
0	0	1	2	67.52	216.3	6.42	40.8	131.	13.1	42.1
		1	3	85.31	257.2	33.0	107.	323.	18.9	56.9
		2	3	98.83	455.1		22.8	105.	11.9	54.7
0	1	1	2	135.0	385.8		189.	540.	161.	459.
		1	3	170.6	513.1		247.	744.	246.	739.
		2	3	197.7	818.5		91.5	379.	2.8e2	1.1e3
1	0	1	2	72.19	210.3		25.4	73.9	15.1	44.0
		1	3	91.20	277.1		24.4	74.0	25.9	78.6
		2	3	105.6	441.7		16.9	70.7	14.3	59.8

Table 2: R_c for model biPAS (29) with planarization parameters p_s, p_t; Bessel j_2 roots Λ_i, Λ_k; magnetic chains M02, M12; and frequency $\omega = \Im(\lambda)$, zero if not shown.

that the R_c(bPASps0pt1) are all large ($\gtrsim 91\,(379^\uparrow)$). So planarizing the toroidal flow alone, whilst it does produce dynamos, is disadvantageous.

For a given flow, i.e. p_s, p_t, Λ_i and Λ_k fixed, R_c(M12)<R_c(M02) for 7/9 of the successful flows. So, like the PAS case, M12 is advantageous.

Because of velocity normalization and the i, k symmetry in (20),(21), a biPAS flow with sub-label ΛiΛk can be derived from either the (0,0)-PASΛi or the (0,0)-PASΛk flow. Comparing Tables 1,2 shows that $8\,(3^\uparrow)/18$ biPAS have lower R_c than both the corresponding (0,0)-PAS dynamos. And overall the least biPAS R_c is less than the least (0,0)-PAS [i.e. R_c(bPASps0pt0Λ2Λ3M12) $= 11.9 < R_c$(PASps0pt0Λ20M12) $= 15.1$]. But for $v_{\rm max}$ normalization the reverse holds: the least biPAS exceeds the least (0,0)-PAS [$R_c^\uparrow$(bPASps0 pt0Λ1Λ2M12) $= 42.1 > R_c^\uparrow$(PASps0pt0Λ1M12) $= 32.3$].

3.3 QuasiPAS results

For quasiPAS we investigated flows using Λ_i ($i = 1, 2, 3$), and Γ_k ($k = 1, 2, 3$), the latter being the first 3 positive roots of j_3:

$$\Gamma_1 = 6.9879\ldots, \quad \Gamma_2 = 10.417\ldots, \quad \Gamma_3 = 13.698\ldots$$

So 9 Λ_i, Γ_k combinations were considered for each of the $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ planarizations; making 36 flows for each of the magnetic chains M02, M12; and a total of 72 quasiPAS models.

Amongst the 54 non-(1,1) models we found 46 dynamos, 23 M02 and 23 M12, 35 being stationary. But we found no successful (1,1) dynamos. The quasiPAS models were much more problematic than PAS or biPAS, often having higher R_c, needing higher truncation levels $J \leq 800$, $N \leq 70$ and eigenmatrix sizes > 200GB, or showing no convergence even for relatively small R. The The critical R_c for the non-(1,1) flows are shown in Table 3, with the following implications.

For fixed Λ_i, Γ_k the effects on R_c of planarizing are again different for M02, M12. For M02, R_c(ps1pt0) $<$ min(R_c(ps0pt0),R_c(ps0pt1)) for 6/9 of the $\Lambda_i\,\Gamma_k$ combinations (here counting "-" as larger than any R_c). So (1,0)-planarizing usually reduces R_c. [Indeed the least M02 R_c are R_c(ps1pt0Λ3Γ3) $= 28.2\,(105^\uparrow)$.] For M12, R_c(ps0pt0) $<$ min(R_c(ps1pt0),R_c(ps0pt1)) for 8 $(7^\uparrow)/9$ of the $\Lambda_i\,\Gamma_k$ combinations. So for M12 any planarizing usually increases R_c. [Indeed the least M12 R_c are R_c(ps0pt0Λ3Γ2) $= 17.1$ and $R_c^\uparrow$(ps0pt0Λ2Γ1) $= 44.0$.]

The 46 successful quasiPAS dynamos include 20 flows where both M02,M12 work (i.e. 40 dynamos), and 14/20 of these flows have R_c(M12)$<$$R_c$(M02). So M12 is advantageous, like PAS and biPAS but not as decisively. There were 3 flows successful for M12 but not for M02, and 3 other flows vice versa.

Comparing Tables 1,3 shows that only $3\,(5^\uparrow)/46$ of the quasiPAS dynamos have lower R_c than the corresponding (0,0)-PAS dynamos. And the overall least quasiPAS R_c, i.e. R_c(ps0pt0Λ3Γ2M12) $= 17.1\,(56.0^\uparrow)$, are a bit larger than the least (0,0)-PAS R_c(Λ3M12) $= 15.1$ and $R_c^\uparrow$(Λ1M12) $= 32.3$. Most of the quasiPAS R_c were large (> 100) or not found.

3.4 Conclusion

The original aim of this investigation was to find a dynamo using a planar flow defined by (1). The only known planar flow dynamos are model p1Y22DM12 of BIJ using stream function (6), and variations thereof (Bachtiar 2009), but all these exhibit high R_c and slow convergence w.r.t. truncation levels J, N. On the other hand, the PAS flows (26) produce dynamos with fast convergence at low R_c. Since the PAS flows can't be fully planarized by the BIJ algorithm (4)–(9), we have investigated partly planarized (1,0)-PAS

						M02			M12		
p_s	p_t	i	k	$v_{\rm rms}$	$v_{\rm max}$	ω	R_c	$R_c^{\uparrow}$	ω	R_c	$R_c^{\uparrow}$
0	0	1	1	32.86	72.84		68.4	152.	14.4	52.3	116.
		1	2	29.33	77.39		-	-		56.9	150.
		1	3	27.86	75.50		-	-		80.8	219.
		2	1	66.09	161.9		8.6e2	2.1e3		18.0	44.0
		2	2	54.84	177.6	36.9	101.	327.		19.2	62.2
		2	3	49.84	190.7	17.4	2.1e2	8.0e2		49.8	191.
		3	1	110.2	317.6		4.8e2	1.4e3		27.1	78.1
		3	2	86.88	284.3		3.6e2	1.2e3		17.1	56.0
		3	3	76.05	322.1		225.	953.		17.3	73.1
0	1	1	1	60.34	124.0		3.1e2	6.4e2	11e1	4.0e2	8.3e2
		1	2	45.88	127.5		-	-		4.6e2	1.3e3
		1	3	38.91	127.0		-	-		-	-
		2	1	142.3	304.7		2.7e2	5.8e2	7e1	4.6e2	9.8e3
		2	2	103.5	306.9		109.	322.	1.7e2	4.1e2	1.2e3
		2	3	84.01	315.2	10e1	4.5e2	1.7e3		-	-
		3	1	256.2	553.1		4.4e2	9.4e2		361.	7.8e2
		3	2	183.2	556.6	71.	282.	857.		-	-
		3	3	145.6	561.7		87.	3.4e2		-	-
1	0	1	1	35.59	67.81		70.8	135.	21.8	50.9	97.0
		1	2	32.36	74.52		222.	511.	19	3.1e2	7.1e2
		1	3	31.03	74.00		2.2e2	5.2e2		3.4e2	8.1e2
		2	1	69.72	194.4		5.9e2	1.6e3		20.1	56.1
		2	2	59.16	166.5		77.5	218.	12	36.0	101.
		2	3	54.56	180.9		32.0	106.		55.1	183.
		3	1	114.2	367.8		383.	1.23e3		31.6	102.
		3	2	92.02	336.3		2.9e2	1.0e3		19.9	72.6
		3	3	81.81	303.6		28.2	105.		22.0	81.7

Table 3: R_c for model quasiPAS (29) with planarization parameters p_s, p_t; Bessel j_2 roots Λ_i, j_3 roots Γ_k; magnetic chains M02, M12; and frequency $\omega = \Im(\lambda)$, zero if not shown. R_c not found is indicated by "-".

flows (15) and these exhibit equally good convergence and low R_c as shown in Table 1. So, seeking planar flow dynamos with better convergence than p1Y22DM12, we have investigated 2 modifications of PAS flows: (i) biPAS given by (20),(21), and (ii) quasiPAS given by (26). All investigated flows have the form (29) with planarization parameters p_s, $p_t = 0, 1$. Altogether we have studied 32 PAS, 24 biPAS and 72 quasiPAS models with flows of the form (29), and magnetic chains M01,M02 given by (10). We have confirmed 12 known PAS dynamos, and found 84 new dynamos comprising 20 PAS, 18 biPAS and 46 quasiPAS. But we have found no fully planar flow dynamos. Generally we have found (i) that the M12 magnetic chain has lower R_c than M02; (ii) that the (1,0) partly planarized dynamos have lower R_c for M02, and the (0,0) unplanarized dynamos have lower R_c for M12.

Whilst (0,0)-PAS flows are helical Beltrami flows, the (1,0)-PAS, and all the biPAS and quasiPAS flows are not. Nevertheless, of the non-Beltrami dynamos, $20\,(22^{\uparrow})$ possess lower R_c than the corresponding (0,0) PAS dynamos. So helicity is not necessary for an efficient dynamo in the sense of having lower R_c. Indeed for $v_{\rm rms}$ normalization the overall least R_c found was the non-Beltrami R_c(bPASps0pt0Λ2Λ3M12) ≈ 12, compared with the least Beltrami R_c(PASps0pt0ΛiM12) ≈ 15 ($i = 3, \ldots, 20$). But such comparisons depend on velocity normalization. For $v_{\max}$ normalization the overall least were the Beltrami (0,0)-PAS and non-Beltrami (1,0)-PAS $R_c^{\uparrow}$(Λ1M12), both ≈ 32, considerably less than the least biPAS $R_c^{\uparrow}$(ps0pt0Λ1Λ2M12) ≈ 42 and least quasiPAS $R_c^{\uparrow}$(ps0pt0Λ3Γ2M12) $\approx$ $R_c^{\uparrow}$(ps1pt0Λ2Γ1M12) ≈ 56.

Tables 1,2 show a low range for the (0,0)-PAS dynamos: $R_c \sim 15-100\,(32^{\uparrow}-531^{\uparrow})$, and even lower for the (1,0)-PAS: $R_c \sim 19-45\,(32^{\uparrow}-347^{\uparrow})$ and (1,0)-biPAS: $R_c \sim 14-26\,(44^{\uparrow}-79^{\uparrow})$. In contrast the BIJ planar flow dynamo p1Y22DM12 has high $R_c \sim 317\,(1150^{\uparrow})$. Larger R_c generally require larger truncation levels J and N, larger eigenmatrices and longer cpu solution times. The difficulties in finding planar flow dynamos are perhaps partly due to larger R_c, but the R_c of historical dynamos are quite variable. E.g. the simple roll dynamos s1t1, s2t1, s2t2 of DJ have low $R_c \sim 32-104\,(51^{\uparrow}-173^{\uparrow})$, whereas the Kumar-Roberts dynamos reported in DJ have higher $R_c \sim 398-875\,(808^{\uparrow}-1717^{\uparrow})$, whilst still having very good convergence. This suggests that the difficulties with planar flow dynamos are not due to high R_c alone. Whilst the (1,0)-PAS and (1,0)-biPAS dynamos stand out for having very low R_c, this work shows amongst other things that attempting to find planar flow dynamos by modifying successful flows with low R_c does not necessarily work. Finding a planar flow dynamo with good convergence remains an outstanding problem.

Appendix: 2D and planar flows, and the PVT

In his early work on a PVT Zel'dovich (1957, Z) defined his 2D motion via "$v_z = 0$, v_x and v_y depend only on x and y", with $\nabla \cdot \mathbf{v} = 0$. The subsequent Zel'dovich-Ruzmaǐken (1980, ZR) PVT removed the z-independence, describing their 2D motion by "$v_z = 0$, $v_x = v_x(x, y, z)$, $v_y = v_y(x, y, z)$", making their flow equivalent to (1). LLJ (p.8679) refer to "planar flows (flows independent of the vertical coordinate z)", and to "the z-independent flow P1Y22DM12" of BIJ. We emphasize here that whilst $\mathbf{v}$ defined by (1) has no $\mathbf{e}_z$ component, it does generally depend on z. Indeed the z-dependence of the flow for p1Y22DM12 is clearly present in Figure 2 of BIJ which includes v_ϕ meridional contours for the stream function (6).

For his PVT for finite V, Z assumed $B_z \equiv 0$ on the surface Σ of V, which is not true in general. Otherwise both Z and ZR assumed V was all space. So the PVTs of Z and ZR do not contradict the BIJ planar flow dynamo where V is a conducting finite sphere surrounded by free space.

Acknowledgement

Part of this work was completed whilst AAB was the recipient of an Australian Development Scholarship at the School of Mathematics and Statistics, University of Sydney, whilst on leave from the University of Indonesia.

References

Bachtiar, A.A., A study of planar velocity dynamos and related issues. Ph.D. Thesis, University of Sydney, 2009.

Bachtiar, A. A., Ivers D. J. & James, R. W., Planar velocity dynamos in a sphere. *Proc. R. Soc. Lond.* 2006, **A462**, 2439–2456.

Bachtiar, A. A. & James, R. W., Dynamo convergence tests, and application to a planar velocity dynamo. *Geophysical and Astrophysical Fluid Dynamics* 2010, **140**, 531–543

Dudley, M. L. & James, R. W., Time-dependent kinematic dynamos with stationary flows. *Proc. R. Soc. Lond.* 1989, **A425**, 407–429.

Li, K., Livermore, P. W. & Jackson, A., An optimal Galerkin scheme to solve the kinematic dynamo eigenvalue problem in a full sphere.*Journal of Comp. Physics* 2010, **229**, 8666–8683.

Pekeris, C. L., Accad, Y. & Shkoller, B., Kinematic dynamos and the Earth's magnetic field. *Phil. Trans. R. Soc. Lond.* 1973, **A275**, 425–461.

Zel'dovich, Ya. B., The magnetic field in the two-dimensional motion of a conducting turbulent liquid. *Soviet Phys. JETP* 1957, **4**, 460–462.

Zel'dovich, Ya. B. & Ruzmaǐkin, A. A., The magnetic field in a conducting fluid in two-dimensional motion. *Soviet Phys. JETP* 1980, **51**, 493–497.