THE STOCHASTIC LANDAU–LIFSHITZ–BARYAKHTAR EQUATION: GLOBAL SOLUTION AND INVARIANT MEASURE

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Abstract. The Landau–Lifshitz–Baryakhtar (LLBar) equation perturbed by a space-dependent noise is a system of fourth order stochastic PDEs which models the evolution of magnetic spin fields in ferromagnetic materials at elevated temperatures, taking into account longitudinal damping, long-range interactions, and noise-induced phenomena at high temperatures. In this paper, we show the existence of a martingale solution (which is analytically strong) to the stochastic LLBar equation posed in a bounded domain $\mathscr{D} \subset \mathbb{R}^d$, where $d = 1, 2, 3$. We also prove pathwise uniqueness of the solution, which implies the existence of a unique probabilistically strong solution. Finally, we show the Feller property of the Markov semigroup associated with the strong solution, which implies the existence of invariant measures.

Contents

1. Introduction

This paper aims to prove the existence and uniqueness of strong solutions and the existence of invariant measures to the stochastic Landau–Lifshitz–Baryakhtar (LLBar) equation.

Given a magnetic body $\mathscr{D} \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, with sufficiently smooth boundary $\partial\mathscr{D}$, its magnetisation vector $\boldsymbol{u}(t, \boldsymbol{x}) \in \mathbb{R}^3$ for any time $t > 0$ and at any point $\boldsymbol{x} \in \mathscr{D}$ (taking into account longitudinal damping and long-range interactions) at elevated temperatures can be described by the LLBar equation [3], which is a fourth-order vector-valued nonlinear PDE. For an isotropic polycrystalline ferromagnetic material at elevated temperatures, the LLBar equation [2, 14, 29] with spin-torque term can be written as:

$$\begin{cases} \partial_t \boldsymbol{u} = \lambda_r \boldsymbol{H} - \lambda_e \Delta \boldsymbol{H} - \gamma \boldsymbol{u} \times \boldsymbol{H} + R(\boldsymbol{u}) & \text{for } (t, \boldsymbol{x}) \in (0, T) \times \mathscr{D}, \\ \boldsymbol{H} = \Delta \boldsymbol{u} + \kappa_1 \boldsymbol{u} - \kappa_2 |\boldsymbol{u}|^2 \boldsymbol{u} & \text{for } (t, \boldsymbol{x}) \in (0, T) \times \mathscr{D}, \\ \boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u_0}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathscr{D}, \\ \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{0}, \ \dfrac{\partial \boldsymbol{H}}{\partial \boldsymbol{n}} = \boldsymbol{0} & \text{for } (t, \boldsymbol{x}) \in (0, T) \times \partial\mathscr{D}, \end{cases} \tag{1.1}$$

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where $\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{n}}$ stands for the derivative of $\boldsymbol{H}$ in the direction of the exterior normal vector $\boldsymbol{n}$ on $\partial\mathscr{D}$. The vector fields $\boldsymbol{H}(t) : \mathscr{D} \to \mathbb{R}^3$ and $\boldsymbol{u}(t) : \mathscr{D} \to \mathbb{R}^3$ denote the effective magnetic field and the magnetic spin field, respectively. The positive constants λ_r, λ_e and γ are, respectively, the relativistic damping constant, exchange damping constant, and electron gyromagnetic ratio. The constants κ_1 and κ_2 are related to the longitudinal magnetic susceptibility and equilibrium magnetisation magnitude of the material. For temperatures below the Curie temperature, these constants are positive, while above the Curie temperature κ_1 can be negative, similarly to the case of the Landau–Lifshitz–Bloch (LLB) equation. For simplicity, the constants κ_1 and κ_2 are taken to be positive, and the effective field is assumed to consist of only the exchange and internal exchange fields, although any additional zero-order terms (such as the external field and uniaxial/cubic anisotropy field) can be added without any difficulty.

The term $R(\boldsymbol{u})$ describes transport by the spin-polarised currents [4]. In recent years, various versions of the Landau–Lifshitz equation which include this term have been intensely studied in physics and mathematics, giving birth to spintronics, see [20, 30] for the Landau–Lifshitz–Gilbert (LLG) equation (see also [1, 25] for the LLB equation and [28] for the LLBar equation). This term consists of the adiabatic and non-adiabatic parts:

$$R(\boldsymbol{u}) := \beta_1(\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u} + \beta_2\boldsymbol{u}\times(\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}, \tag{1.2}$$

where $\boldsymbol{\nu}(t) : \mathscr{D} \to \mathbb{R}^d$ is the spin current, and β_1 and β_2 are constants. Convective terms of the form $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ can also be added to (1.2) without significantly affecting the analysis in this paper. Other phenomenological torque terms which are similar in form to (1.2) have been considered in the literature [18, 21] and can be treated in a similar manner.

The LLBar equation describes an averaged magnetisation trajectory. At high temperature, however, the dispersion of individual trajectories becomes important. To incorporate random fluctuations into the dynamics of the magnetisation and to describe noise-induced transitions between equilibrium states of a ferromagnet, many approaches are available in the physics literature. An approach due to Brown [6] is to perturb the effective field by a Gaussian noise ('thermal field'). Another suggestion is made in [15] to perturb the precessional term and add a random torque, leading to an equation with additive and multiplicative noise. Mathematically, these result in the stochastic LLG equation [7] and the stochastic LLB equation [8], respectively, where questions related to existence and uniqueness of solutions in the appropriate function spaces have been studied.

In this paper, we consider the stochastic LLBar equation with spin-torque terms. For simplicity of presentation, we set $\kappa_1 = \kappa_2 = 1$. The problem then reads:

$$\begin{cases} \mathrm{d}\boldsymbol{u} = \big(\lambda_r\boldsymbol{H} - \lambda_e\Delta\boldsymbol{H} - \gamma\boldsymbol{u}\times\boldsymbol{H} + S(\boldsymbol{u})\big)\,\mathrm{d}t + \displaystyle\sum_{k=1}^{\infty} G_k(\boldsymbol{u})\,\mathrm{d}W_k(t) & \text{for } (t,\boldsymbol{x})\in(0,T)\times\mathscr{D}, \\ \boldsymbol{H} = \Delta\boldsymbol{u} + \boldsymbol{u} - |\boldsymbol{u}|^2\boldsymbol{u} & \text{for } (t,\boldsymbol{x})\in(0,T)\times\mathscr{D}, \\ \boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u_0}(\boldsymbol{x}) & \text{for } \boldsymbol{x}\in\mathscr{D}, \\ \dfrac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} = \boldsymbol{0},\ \dfrac{\partial\boldsymbol{H}}{\partial\boldsymbol{n}} = \boldsymbol{0} & \text{for } (t,\boldsymbol{x})\in(0,T)\times\partial\mathscr{D}, \end{cases} \tag{1.3}$$

where $\{W_k\}_{k\in\mathbb{N}}$ is a family of independent real-valued Wiener processes. For concreteness, we take

$$G_k(\boldsymbol{u}) := \boldsymbol{g}_k + \gamma\boldsymbol{u}\times\boldsymbol{h}_k, \tag{1.4}$$

where $\boldsymbol{g}_k$ and $\boldsymbol{h}_k$ are given functions with certain regularity. In other words, randomness is incorporated into the system via a stochastic precession term and a stochastic torque term. This form of (additive and multiplicative) noise is similar to the one considered in [7, 8], and is necessary to capture important features of the physical system near equilibrium at elevated temperatures [15]. The argument presented here would also apply in a more general setting where $G_k(\boldsymbol{u})$ is prescribed to satisfy certain smoothness and linear growth assumptions.

In (1.3),

$$S(\boldsymbol{u}) := R(\boldsymbol{u}) + L(\boldsymbol{u}), \tag{1.5}$$

where $R(\boldsymbol{u})$ is the spin-torque terms defined in (1.2) and $L(\boldsymbol{u})$ is a Lipschitz function of $\boldsymbol{u}$. More precise assumptions on $G_k(\boldsymbol{u})$ and $L(\boldsymbol{u})$ are elaborated in Section 2.2. There are two physical reasons to include the term $L(\boldsymbol{u})$ in (1.5):

(1) The stochastic term is usually interpreted in the Stratonovich sense for the LLG or the LLB equation, and converting the Stratonovich differential to the Itô differential results in an additional drift term. For the function $G_k(\boldsymbol{u})$ defined in (1.4) (or other form which is linear in $\boldsymbol{u}$), this additional term satisfies our assumptions for $L(\boldsymbol{u})$ described in Section 2.2.
(2) There are other types of phenomenological torque terms which are linear in $\boldsymbol{u}$; see [21]. The simplest anisotropy field is linear as well. These terms can be included in $L(\boldsymbol{u})$ too.

More generally, very little is known about higher order systems of stochastic PDEs, and the deterministic systems are not well understood either. Recently, such equations appeared in the theory of micromagnetism to model the onset and stability of magnetic skyrmions in a frustrated ferromagnet [13].

It is interesting to note that systems of fourth-order equations similar to (1.1) also appear in certain areas of chemistry and biology to model long-range diffusion. In particular, if $\gamma = 0$ and $\beta_2 = 0$, then we have a generalised (multi-species) reaction-diffusion-advection model with long range effects, which is prominently used in mathematical biology [22, Chapter 11]. The deterministic model arises in population dynamics to model growth and dispersal in a population [11, 23], and has been analysed in 1-D [10]. Related fourth-order deterministic reaction-diffusion system is also suggested to model anomalous bi-flux diffusion process [5, 17]. To account for the effect of noise or random reaction terms in the model, it is physically reasonable to consider the stochastic counterpart of such problems, on which the analysis in the present paper would also apply.

The well-posedness of the deterministic LLBar equation has been studied in [26], while a numerical method to approximate its solution has been proposed in [27]. However, to the best of our knowledge, the stochastic LLBar equation (1.3) has not been studied before. This paper aims to bridge this gap and initiate a systematic investigation of fourth-order systems of SPDEs.

The main results of this paper are Theorems 2.4, 2.5, and 2.6. Theorems 2.4 and 2.5 state the existence and uniqueness of a pathwise solution that is strong in PDEs sense. This is achieved by going through the intermediate steps of proving, in Theorem 2.4, the existence of a martingale solution to (1.3) which is strong in the sense of PDEs. This is done by means of Faedo–Galerkin approximation and compactness argument. We then prove, in Theorem 2.5, further regularity properties of the solution and show pathwise uniqueness of the solution. By the Yamada–Watanabe theorem, this implies existence of a unique strong solution (in the sense of probability) and uniqueness in law of the martingale solution. In Theorem 2.6, we show the existence of a non-trivial invariant measure to (1.3) that is supported on $\mathbb{H}^4$. To this end, we use standard arguments: we show the Feller property of the transition semigroup associated to the strong solution and invoke the general theorem of Krylov–Bogoliubov. All the results we formulate hold for d-dimensional domains with $d \leq 3$.

2. Preliminaries

2.1. **Notations.** We begin by defining some notations used in this paper. The function space $\mathbb{L}^p := \mathbb{L}^p(\mathscr{D};\mathbb{R}^3)$ denotes the usual space of p-th integrable functions defined on $\mathscr{D}$ and taking values in $\mathbb{R}^3$, and $\mathbb{W}^{k,p} := \mathbb{W}^{k,p}(\mathscr{D};\mathbb{R}^3)$ denotes the usual Sobolev space of functions on $\mathscr{D} \subset \mathbb{R}^d$ taking values in $\mathbb{R}^3$. We write $\mathbb{H}^k := \mathbb{W}^{k,2}$. The partial derivative $\partial/\partial x_i$ will be written by ∂_i for short. The partial derivative of f with respect to time t will be denoted by ∂_t.

If X is a Banach space, the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ denote respectively the usual Lebesgue and Sobolev spaces of functions on $(0,T)$ taking values in X. The space $C([0,T];X)$ denotes the space of continuous functions on $[0,T]$ taking values in X, while $B_b(X)$ and $C_b(X)$ denotes the space of bounded Borel functions on X and the space of bounded continuous functions on X, respectively. The space $L^p(\Omega;X)$ denotes the space of X-valued random variable with finite p-th moment, where $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space.

Throughout this paper, we denote the scalar product in a Hilbert space H by $\langle\cdot,\cdot\rangle_H$ and its corresponding norm by $\|\cdot\|_H$. The expectation of a random variable Y will be denoted by $\mathbb{E}[Y]$. We will not distinguish between the scalar product of $\mathbb{L}^2$ vector-valued functions taking values in $\mathbb{R}^3$ and the scalar product of $\mathbb{L}^2$ matrix-valued functions taking values in $\mathbb{R}^{3\times 3}$, and still denote them by $\langle \cdot, \cdot \rangle_{\mathbb{L}^2}$.

In various estimates, the constant C in the estimate denotes a generic constant which takes different values at different occurrences. If the dependence of C on some variable, e.g. T, is highlighted, we often write $C(T)$. The notation $A \lesssim B$ means $A \leq CB$ where the specific form of the constant C is not important to clarify.

2.2. **Assumptions.** Assume that $\mathscr{D}$ is an open domain with C^2-smooth boundary. For equation (1.3), we assume the following:

(1) For each $k \in \mathbb{N}$,
$$G_k(\boldsymbol{u}) = \boldsymbol{g}_k + \gamma \boldsymbol{u} \times \boldsymbol{h}_k,$$
where $\boldsymbol{g}_k : \mathscr{D} \to \mathbb{R}^3$ and $\boldsymbol{h}_k : \mathscr{D} \to \mathbb{R}^3$ are functions such that
$$\sigma_g^2 := \sum_{k=1}^{\infty} \|\boldsymbol{g}_k\|_{\mathbb{H}^2}^2 < \infty \quad \text{and} \quad \sigma_h^2 := \sum_{k=1}^{\infty} \|\boldsymbol{h}_k\|_{\mathbb{H}^2}^2 < \infty$$
(2) The spin current vector field $\boldsymbol{\nu} \in L^\infty\big(\mathbb{R}^+; \mathbb{L}^\infty(\mathscr{D}; \mathbb{R}^d)\big)$ is given.
(3) $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz-continuous function. More precisely, there exists a constant $C > 0$ such that for all $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \mathbb{R}^3$,
$$|L(\boldsymbol{v}_1) - L(\boldsymbol{v}_2)| \leq C|\boldsymbol{v}_1 - \boldsymbol{v}_2|.$$

The operator Δ denotes the Neumann Laplacian acting on $\mathbb{R}^3$-valued functions with domain
$$\mathrm{D}(\Delta) := \left\{ \boldsymbol{v} \in \mathbb{H}^2 : \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\mathscr{D} \right\}.$$
We also define the operator
$$A := \lambda_e \Delta^2 - (\lambda_r - \lambda_e)\Delta + \beta I, \quad \text{with } \mathrm{D}(A) = \mathrm{D}(\Delta^2), \tag{2.1}$$
where $\beta > 0$ is sufficiently large so that $A \geq \lambda_0 I$ for some $\lambda_0 > 0$. Then A is a positive, self-adjoint operator in $\mathbb{L}^2$.

2.3. **Auxiliary facts.** In our analysis, we will frequently use the following results. For any vector-valued function $\boldsymbol{v} : \mathscr{D} \to \mathbb{R}^3$, we have
$$\nabla(|\boldsymbol{v}|^2 \boldsymbol{v}) = 2\boldsymbol{v}\ (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) + |\boldsymbol{v}|^2 \nabla \boldsymbol{v}, \tag{2.2}$$
$$\Delta(|\boldsymbol{v}|^2 \boldsymbol{v}) = 2|\nabla \boldsymbol{v}|^2 \boldsymbol{v} + 2(\boldsymbol{v} \cdot \Delta \boldsymbol{v})\boldsymbol{v} + 4\nabla \boldsymbol{v}\ (\boldsymbol{v} \cdot \nabla \boldsymbol{v})^\top + |\boldsymbol{v}|^2 \Delta \boldsymbol{v}, \tag{2.3}$$
provided that the partial derivatives are well defined.

Lemma 2.1. Let $\mathscr{D} \subset \mathbb{R}^d$ be an open bounded domain with smooth boundary and $\epsilon > 0$ be given. Then there exists a positive constant C such that the following inequalities hold:

(i) for any $\boldsymbol{v} \in \mathrm{D}(\Delta)$,
$$\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \leq \frac{1}{4\epsilon} \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2, \tag{2.4}$$
(ii) for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^s$, where $s > d/2$,
$$\|\boldsymbol{v} \odot \boldsymbol{w}\|_{\mathbb{H}^s} \leq C \|\boldsymbol{v}\|_{\mathbb{H}^s} \|\boldsymbol{w}\|_{\mathbb{H}^s}, \tag{2.5}$$
$$\|(\boldsymbol{u} \times \boldsymbol{v}) \odot \boldsymbol{w}\|_{\mathbb{H}^s} \leq C \|\boldsymbol{u}\|_{\mathbb{H}^s} \|\boldsymbol{v}\|_{\mathbb{H}^s} \|\boldsymbol{w}\|_{\mathbb{H}^s}. \tag{2.6}$$
Here $\odot$ denotes either the dot product or cross product.

Proof. This is shown in [27, Lemma 2.2]. ☐

Lemma 2.2. Let R be the map defined in (1.2) and $\boldsymbol{\nu}$ be given. Then for each $\epsilon > 0$, there exists a positive constant C such that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^1 \cap \mathbb{L}^\infty$,

$$\left| \langle R(\boldsymbol{v}), \boldsymbol{v} \rangle_{\mathbb{L}^2} \right| \leq C \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^2(\mathscr{D};\mathbb{R}^d)}^2 + \epsilon \left\| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \right\|_{\mathbb{L}^2}^2 , \tag{2.7}$$

$$\left| \langle R(\boldsymbol{v}), \boldsymbol{w} \rangle_{\mathbb{L}^2} \right| \leq C \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left(\left\| \nabla \boldsymbol{v} \right\|_{\mathbb{L}^2}^2 + \left\| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \right\|_{\mathbb{L}^2}^2 \right) + \epsilon \left\| \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 , \tag{2.8}$$

$$\left| \langle R(\boldsymbol{v}) - R(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w} \rangle_{\mathbb{L}^2} \right| \leq C \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left(1 + \left\| \boldsymbol{w} \right\|_{\mathbb{L}^\infty}^2 \right) \left\| \boldsymbol{v} - \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 + \epsilon \left\| \nabla \boldsymbol{v} - \nabla \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 . \tag{2.9}$$

Proof. Inequality (2.7) follows directly from Young's inequality. To show (2.8), we apply Hölder's and Young's inequalities to obtain

$$\begin{aligned} \left| \langle R(\boldsymbol{v}), \boldsymbol{w} \rangle_{\mathbb{L}^2} \right| &\leq C \Big(\left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)} \left\| \nabla \boldsymbol{v} \right\|_{\mathbb{L}^2} + \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)} \left\| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \right\|_{\mathbb{L}^2} \Big) \left\| \boldsymbol{w} \right\|_{\mathbb{L}^2} \\ &\leq C \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left(\left\| \nabla \boldsymbol{v} \right\|_{\mathbb{L}^2}^2 + \left\| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \right\|_{\mathbb{L}^2}^2 \right) + \epsilon \left\| \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 \end{aligned}$$

as required. Finally, writing

$$R(\boldsymbol{v}) - R(\boldsymbol{w}) = -(\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{v} - \boldsymbol{w}) + \beta(\boldsymbol{v} - \boldsymbol{w}) \times (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{v} + \beta \boldsymbol{w} \times (\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{v} - \boldsymbol{w}),$$

we then have by Young's inequality,

$$\begin{aligned} \left| \langle R(\boldsymbol{v}) - R(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w} \rangle_{\mathbb{L}^2} \right| &\leq C \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left\| \boldsymbol{v} - \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 + \epsilon \left\| \nabla \boldsymbol{v} - \nabla \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 \\ &\quad + C \left\| \boldsymbol{\nu} \right\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left\| \boldsymbol{w} \right\|_{\mathbb{L}^\infty}^2 \left\| \boldsymbol{v} - \boldsymbol{w} \right\|_{\mathbb{L}^2}^2 . \end{aligned}$$

This completes the proof of the lemma. □

2.4. **Formulations and main results.** We assume that we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions. We assume that on this probability space a sequence of independent real-valued $\mathbb{F}$-adapted Wiener processes $\{W_k(t)\}_{t\geq 0}$ is defined.

The notion of solution to (1.1) used in this paper can now be stated.

Definition 2.3. Given $T > 0$ and initial data $\boldsymbol{u_0} \in \mathbb{H}^2$, a martingale solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \boldsymbol{u})$ to problem (1.3) in $[0, T]$ consists of

(1) a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions,
(2) a sequence of real-valued $\mathbb{F}$-adapted Wiener processes $W_k = \{W_k(t)\}_{t\geq 0}$,
(3) a progressively measurable process $\boldsymbol{u} : [0, T] \times \Omega \to \mathbb{H}^2$ such that $\mathbb{P}$-a.s.

$$\boldsymbol{u} \in L^\infty(0, T; \mathbb{H}^2) \cap L^2(0, T; \mathbb{H}^4),$$

and for every $t \in [0, T]$, $\mathbb{P}$-a.s.

$$\begin{aligned} \boldsymbol{u}(t) = \boldsymbol{u_0} &+ \lambda_r \int_0^t \boldsymbol{H}(s) \, \mathrm{d}s - \lambda_e \int_0^t \Delta \boldsymbol{H}(s) \, \mathrm{d}s - \gamma \int_0^t \boldsymbol{u}(s) \times \boldsymbol{H}(s) \, \mathrm{d}s \\ &+ \int_0^t R\big(\boldsymbol{u}(s)\big) \, \mathrm{d}s + \int_0^t L\big(\boldsymbol{u}(s)\big) \, \mathrm{d}s + \sum_{k=1}^\infty \int_0^t (\boldsymbol{g}_k + \gamma \boldsymbol{u}(s) \times \boldsymbol{h}_k) \, \mathrm{d}W_k(s), \end{aligned} \tag{2.10}$$

where $\boldsymbol{H}(t) = \Delta \boldsymbol{u}(t) + \boldsymbol{u}(t) - |\boldsymbol{u}(t)|^2 \boldsymbol{u}(t)$ a.e.

The main results of this paper are stated below.

Theorem 2.4. Let $\mathscr{D} \subset \mathbb{R}^d$, where $d = 1, 2, 3$. Let $\boldsymbol{u_0} \in \mathbb{H}^2$ and $\boldsymbol{g}_k, \boldsymbol{h}_k \in \mathbb{H}^2$ be given. There exists a martingale solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \boldsymbol{u})$ of (1.3) such that

(1) for any $q \geq 1$,

$$\boldsymbol{u} \in L^q \Big(\Omega; L^\infty(0, T; \mathbb{H}^2) \cap L^2(0, T; \mathbb{H}^4) \Big),$$

(2) for every $\beta \in [0,1)$, $\delta \in (0, 1-\beta)$, $\mathbb{P}$-a.s.,
$$\boldsymbol{u} \in C^{\delta}\big((0,T); \mathrm{D}(A^{\beta})\big), \tag{2.11}$$
where A is defined in (2.1). In particular, $\boldsymbol{u} \in C([0,T];\mathbb{H}^2)$.

Theorem 2.5. Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \boldsymbol{u}_1)$ and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \boldsymbol{u}_2)$ are two martingale solutions to (1.3) in the sense of Definition 2.3. Then, for $\mathbb{P}$-a.s. $\omega \in \Omega$,
$$\boldsymbol{u}_1(\cdot,\omega) = \boldsymbol{u}_2(\cdot,\omega).$$
This implies the existence of a pathwise unique (probabilistically) strong solution of (1.3) and the uniqueness in law of the martingale solution of (1.3).

Theorem 2.6. There exists an invariant measure for (1.3) supported on $\mathbb{H}^4$.

3. Faedo–Galerkin Approximation

Let $\{\boldsymbol{e}_i\}_{i\in\mathbb{N}}$ denote an orthonormal basis of $\mathbb{L}^2$ consisting of smooth eigenfunctions $\boldsymbol{e}_i$ of $-\Delta$ such that
$$-\Delta \boldsymbol{e}_i = \mu_i \boldsymbol{e}_i \quad \text{in } \mathscr{D} \quad \text{and} \quad \frac{\partial \boldsymbol{e}_i}{\partial \boldsymbol{n}} = \boldsymbol{0} \quad \text{on } \partial\mathscr{D}, \quad \forall i \in \mathbb{N},$$
where $\mu_i \geq 0$ are the eigenvalues of $-\Delta$, associated with $\boldsymbol{e}_i$.

Let $\mathbb{V}_n := \operatorname{span}\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ and $\Pi_n : \mathbb{L}^2 \to \mathbb{V}_n$ be the orthogonal projection defined by
$$\langle \Pi_n \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2} = \langle \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2}, \quad \forall \boldsymbol{\phi} \in \mathbb{V}_n,\ \boldsymbol{v} \in \mathbb{L}^2. \tag{3.1}$$
Note that Π_n is self-adjoint and satisfies
$$\begin{aligned} \|\Pi_n \boldsymbol{v}\|_{\mathbb{L}^2} &\leq \|\boldsymbol{v}\|_{\mathbb{L}^2}, \quad \forall \boldsymbol{v} \in \mathbb{L}^2, \\ \|\nabla \Pi_n \boldsymbol{v}\|_{\mathbb{L}^2} &\leq \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}, \quad \forall \boldsymbol{v} \in \mathbb{H}^1. \end{aligned}$$
Moreover,
$$\langle \Pi_n \Delta \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2} = \langle \Delta \Pi_n \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2} = -\langle \nabla \Pi_n \boldsymbol{v}, \nabla \boldsymbol{\phi}\rangle_{\mathbb{L}^2}, \quad \forall \boldsymbol{\phi} \in \mathbb{V}_n,\ \boldsymbol{v} \in \mathbb{H}^2.$$
The Faedo–Galerkin method seeks to approximate the solution to (1.3) by $(\boldsymbol{u}_n(t), \boldsymbol{H}_n(t)) \in \mathbb{V}_n \times \mathbb{V}_n$ satisfying
$$\begin{cases} \mathrm{d}\boldsymbol{u}_n = \big(\lambda_r \boldsymbol{H}_n - \lambda_e \Delta \boldsymbol{H}_n - \gamma \Pi_n(\boldsymbol{u}_n \times \boldsymbol{H}_n) + \Pi_n S(\boldsymbol{u}_n)\big)\,\mathrm{d}t + \displaystyle\sum_{k=1}^{n} \Pi_n G_k(\boldsymbol{u}_n)\,\mathrm{d}W_k(t) & \text{in } (0,T)\times\mathscr{D}, \\ \boldsymbol{H}_n = \Delta \boldsymbol{u}_n + \boldsymbol{u}_n - \Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n) & \text{in } (0,T)\times\mathscr{D}, \\ \boldsymbol{u}_n(0) = \boldsymbol{u}_{0n} & \text{in } \mathscr{D}, \end{cases} \tag{3.2}$$
where $\boldsymbol{u}_{0n} = \Pi_n \boldsymbol{u_0} \in \mathbb{V}_n$.

Substituting the second equation in (3.2) into the first gives
$$\begin{aligned} \mathrm{d}\boldsymbol{u}_n = \big(&(\lambda_r - \lambda_e)\Delta \boldsymbol{u}_n - \lambda_e \Delta^2 \boldsymbol{u}_n + \lambda_r \boldsymbol{u}_n - \lambda_r \Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n) + \lambda_e \Delta \Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n) - \gamma \Pi_n(\boldsymbol{u}_n \times \Delta \boldsymbol{u}_n) \\ &+ \gamma \Pi_n(\boldsymbol{u}_n \times \Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n)) + \Pi_n S(\boldsymbol{u}_n)\big)\,\mathrm{d}t + \sum_{k=1}^{n} \Pi_n G_k(\boldsymbol{u}_n)\,\mathrm{d}W_k(t). \end{aligned} \tag{3.3}$$
The existence of a unique local strong solution to (3.3), hence to (3.2), follows from Lemmas 3.1 and 3.2 shown in the following and [9, Theorem 10.6].

Lemma 3.1. For each $n \in \mathbb{N}$ and $\boldsymbol{v} \in \mathbb{V}_n$, define
$$\begin{aligned} F_n^1(\boldsymbol{v}) &= \Delta \boldsymbol{v} + \boldsymbol{v}, \\ F_n^2(\boldsymbol{v}) &= -\Delta^2 \boldsymbol{v}, \\ F_n^3(\boldsymbol{v}) &= -\Pi_n(|\boldsymbol{v}|^2 \boldsymbol{v}), \\ F_n^4(\boldsymbol{v}) &= \Pi_n \Delta(|\boldsymbol{v}|^2 \boldsymbol{v}), \end{aligned}$$

$$\begin{aligned}
F_n^5(\boldsymbol{v}) &= \Pi_n(\boldsymbol{v} \times \Delta \boldsymbol{v}),\\
F_n^6(\boldsymbol{v}) &= \Pi_n(\boldsymbol{v} \times \Pi_n(|\boldsymbol{v}|^2 \boldsymbol{v})),\\
F_n^7(\boldsymbol{v}) &= \Pi_n S(\boldsymbol{v}),\\
G_n(\boldsymbol{v}) &= \Pi_n G_k(\boldsymbol{v}).
\end{aligned}$$

Then F_n^j, $j = 1, 2, \dots, 7$, are well-defined mappings from $\mathbb{V}_n$ into $\mathbb{V}_n$. Moreover, F_n^1, F_n^2, and G_n are globally Lipschitz while F_n^3, F_n^4, F_n^5, F_n^6, and F_n^7 are locally Lipschitz.

Proof. Global Lipschitz continuity for F_n^1 and F_n^2, and local Lipschitz continuity for F_n^3, F_n^4, and F_n^5 were shown in [26, Lemma 3.2]. Also, G_n is linear, so it is clearly globally Lipschitz. Next, for any $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{V}_n$,

$$\begin{aligned}
\left\| F_n^6(\boldsymbol{v}) - F_n^6(\boldsymbol{w}) \right\|_{\mathbb{L}^2} &\le \left\| \boldsymbol{v} \times F_n^3(\boldsymbol{v}) - \boldsymbol{w} \times F_n^3(\boldsymbol{w}) \right\|_{\mathbb{L}^2}\\
&\le \left\| \boldsymbol{v} \times \left(F_n^3(\boldsymbol{v}) - F_n^3(\boldsymbol{w}) \right) \right\|_{\mathbb{L}^2} + \left\| (\boldsymbol{v} - \boldsymbol{w}) \times F_n^3(\boldsymbol{w}) \right\|_{\mathbb{L}^2}\\
&\le \|\boldsymbol{v}\|_{\mathbb{L}^\infty} \left\| F_n^3(\boldsymbol{v}) - F_n^3(\boldsymbol{w}) \right\|_{\mathbb{L}^2} + \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^3 \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^2} .
\end{aligned}$$

Since all norms on $\mathbb{V}_n$ are equivalent and F_n^3 is locally Lipschitz, F_n^6 is locally Lipschitz.

Finally, $F_n^7(\boldsymbol{v}) = \Pi_n R(\boldsymbol{v}) + \Pi_n L(\boldsymbol{v})$. The map L is globally Lipschitz by assumption. Moreover, for any $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{V}_n$,

$$\begin{aligned}
&\|\Pi_n R(\boldsymbol{v}) - \Pi_n R(\boldsymbol{w})\|_{\mathbb{L}^2}\\
&\le \|(\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{v} - \boldsymbol{w})\|_{\mathbb{L}^2} + \beta \|(\boldsymbol{v} - \boldsymbol{w}) \times (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{v}\|_{\mathbb{L}^2} + \beta \|\boldsymbol{w} \times (\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{v} - \boldsymbol{w})\|_{\mathbb{L}^2}\\
&\le \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)} \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{H}^1} + \beta \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)} \|\boldsymbol{v}\|_{\mathbb{H}^1} \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^\infty} + \beta \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)} \|\boldsymbol{w}\|_{\mathbb{L}^\infty} \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{H}^1} .
\end{aligned}$$

Since all norms on $\mathbb{V}_n$ are equivalent, this implies F_n^7 is locally Lipschitz. This completes the proof. ☐

Lemma 3.2. For each $n \in \mathbb{N}$ and $\boldsymbol{v} \in \mathbb{V}_n$,

$$\left\langle \sum_{j=1}^{7} F_n^j(\boldsymbol{v}), \boldsymbol{v} \right\rangle_{\mathbb{L}^2} + \sum_{k=1}^{n} \|\Pi_n G_k(\boldsymbol{v})\|_{\mathbb{L}^2}^2 \le C_n \left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \right),$$

where C_n is a constant, possibly depending on n.

Proof. We will estimate each term on the left-hand side. Firstly,

$$\left\langle F_n^1(\boldsymbol{v}) + F_n^2(\boldsymbol{v}), \boldsymbol{v} \right\rangle_{\mathbb{L}^2} = - \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 .$$

Moreover,

$$\left\langle F_n^3(\boldsymbol{v}) + F_n^4(\boldsymbol{v}), \boldsymbol{v} \right\rangle_{\mathbb{L}^2} = - \|\boldsymbol{v}\|_{\mathbb{L}^4}^4 - \| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \|_{\mathbb{L}^2}^2 - \frac{1}{2} \left\| \nabla |\boldsymbol{v}|^2 \right\|_{\mathbb{L}^2}^2 \le - \| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \|_{\mathbb{L}^2}^2 ,$$

and

$$\left\langle F_n^5(\boldsymbol{v}) + F_n^6(\boldsymbol{v}), \boldsymbol{v} \right\rangle_{\mathbb{L}^2} = 0.$$

Also, by Young's inequality and the assumptions on $\boldsymbol{\nu}$ and L,

$$\left\langle F_n^7(\boldsymbol{v}), \boldsymbol{v} \right\rangle_{\mathbb{L}^2} = \langle R(\boldsymbol{v}) + L(\boldsymbol{v}), \boldsymbol{v} \rangle_{\mathbb{L}^2} \le C \|\boldsymbol{\nu}\|_{\mathbb{L}^2(\mathscr{D};\mathbb{R}^d)}^2 + \epsilon \| |\boldsymbol{v}| \, |\nabla \boldsymbol{v}| \|_{\mathbb{L}^2}^2 + C\big(1 + \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \big)$$

for any $\epsilon > 0$. Finally, by Hölder's inequality,

$$\sum_{k=1}^{n} \|\Pi_n G_k(\boldsymbol{v})\|_{\mathbb{L}^2}^2 \le C \sigma_g \sigma_h \left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \right).$$

Adding all the above inequalities yields the required result. ☐

Before proceeding to prove uniform bounds for the approximate solutions $\boldsymbol{u}_n$, to simplify notations we write (3.2) as

$$\begin{cases} \mathrm{d}\boldsymbol{u}_n = F_n(\boldsymbol{u}_n, \boldsymbol{H}_n)\,\mathrm{d}t + \displaystyle\sum_{k=1}^n G_k(\boldsymbol{u}_n)\,\mathrm{d}W_k(t), & \text{in } (0,T)\times\mathscr{D}, \\ \boldsymbol{H}_n = \Delta\boldsymbol{u}_n + \boldsymbol{u}_n - \Pi_n(|\boldsymbol{u}_n|^2\boldsymbol{u}_n) & \text{in } (0,T)\times\mathscr{D}, \\ \boldsymbol{u}_n(0) = \boldsymbol{u}_{0n} & \text{in } \mathscr{D}, \end{cases} \tag{3.4}$$

where

$$F_n(\boldsymbol{u}_n, \boldsymbol{H}_n) := \lambda_r \boldsymbol{H}_n - \lambda_e \Delta\boldsymbol{H}_n - \gamma\Pi_n(\boldsymbol{u}_n \times \boldsymbol{H}_n) - \Pi_n S(\boldsymbol{u}_n), \tag{3.5}$$

$$G_k(\boldsymbol{u}_n) := \Pi_n(\boldsymbol{g}_k + \gamma\boldsymbol{u}_n \times \boldsymbol{h}_k). \tag{3.6}$$

Now, we prove uniform bounds for the approximate solutions $\boldsymbol{u}_n$.

Proposition 3.3. For any $p \geq 1$, $n \in \mathbb{N}$, $t \in (0,\infty)$,

$$\mathbb{E}\left[\sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^{2p}\right] + \mathbb{E}\left[\left(\int_0^t \|\nabla\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\Delta\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]$$
$$+ \mathbb{E}\left[\left(\int_0^t \||\boldsymbol{u}_n(s)|\,|\nabla\boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s\right)^p\right] \leq C\,\|\boldsymbol{u}_0\|_{\mathbb{L}^2}^{2p} + C(1+e^{Ct}),$$

where C is a positive constant depending on p, σ_g, and σ_h (but is independent of n).

Proof. Let $\psi : \mathbb{V}_n \to \mathbb{R}$ be a function defined by $\boldsymbol{v} \mapsto \frac{1}{2}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2$. Then

$$\psi'(\boldsymbol{v})(\boldsymbol{h}) = \langle \boldsymbol{v}, \boldsymbol{h}\rangle_{\mathbb{L}^2} \quad \text{and} \quad \psi''(\boldsymbol{v})(\boldsymbol{h},\boldsymbol{k}) = \langle \boldsymbol{k}, \boldsymbol{h}\rangle_{\mathbb{L}^2}, \quad \forall \boldsymbol{h},\boldsymbol{k} \in \mathbb{V}_n.$$

By Itô's Lemma,

$$\mathrm{d}\psi(\boldsymbol{u}_n) = \langle \mathrm{d}\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \frac{1}{2}\langle \mathrm{d}\boldsymbol{u}_n, \mathrm{d}\boldsymbol{u}_n\rangle_{\mathbb{L}^2}. \tag{3.7}$$

Therefore, by (3.7) and (3.4), we deduce that

$$\frac{1}{2}\mathrm{d}\,\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 = \left(\langle F_n(\boldsymbol{u}_n,\boldsymbol{H}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \frac{1}{2}\sum_{k=1}^n \|G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\right)\mathrm{d}t + \sum_{k=1}^n \langle G_k(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(t). \tag{3.8}$$

Note that by (3.5) and (3.6), we have

$$\begin{aligned} \langle F_n(\boldsymbol{u}_n,\boldsymbol{H}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} &= \lambda_r \langle \boldsymbol{H}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \lambda_e \langle \Delta\boldsymbol{H}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle S(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &= \lambda_r \langle \boldsymbol{H}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \lambda_e \langle \Delta\boldsymbol{H}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \langle (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle L(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &= (\lambda_e - \lambda_r)\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \lambda_e\|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \lambda_r\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \lambda_r\|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 - \lambda_e\||\boldsymbol{u}_n||\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \\ &\quad - 2\lambda_e\|\boldsymbol{u}_n\cdot\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \langle (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle L(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \end{aligned} \tag{3.9}$$

and

$$\langle G_k(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} = \langle \boldsymbol{g}_k, \boldsymbol{u}_n\rangle_{\mathbb{L}^2}. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8) and applying Hölder's inequality yield

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 + \lambda_r\int_0^t \|\nabla\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_e\int_0^t \|\Delta\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_r\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s \\ &\quad + \lambda_e\int_0^t \||\boldsymbol{u}_n(s)|\,|\nabla\boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2\,\mathrm{d}s + 2\lambda_e\int_0^t \|\boldsymbol{u}_n(s)\cdot\nabla\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\ &= \frac{1}{2}\|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 + \lambda_r\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_e\int_0^t \|\nabla\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s - \int_0^t \langle (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_n(s), \boldsymbol{u}_n(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s \\ &\quad + \int_0^t \langle L(\boldsymbol{u}_n(s)), \boldsymbol{u}_n(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s + \frac{1}{2}\sum_{k=1}^n\int_0^t \|G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \sum_{k=1}^n\int_0^t \langle \boldsymbol{g}_k, \boldsymbol{u}_n(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left\|\boldsymbol{u}_0\right\|_{\mathbb{L}^2}^2+\lambda_r \int_0^t\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s+\lambda_e \int_0^t\left\|\nabla \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \\
&\quad +C \int_0^t\|\boldsymbol{\nu}\|_{\mathbb{L}^2(\mathscr{D};\mathbb{R}^d)}\left\||\boldsymbol{u}_n(s)|\,|\nabla \boldsymbol{u}_n(s)|\right\|_{\mathbb{L}^2} \mathrm{d}s+C \int_0^t\left(1+\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2\right) \mathrm{d}s \\
&\quad +\frac{1}{2} \sum_{k=1}^n \int_0^t\left(\left\|\boldsymbol{g}_k\right\|_{\mathbb{L}^2}^2+\left\|\boldsymbol{h}_k\right\|_{\mathbb{L}^\infty}^2\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2\right) \mathrm{d}s+\sum_{k=1}^n \int_0^t\left\langle\boldsymbol{g}_k, \boldsymbol{u}_n(s)\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s) \\
&\leq C+C\left\|\boldsymbol{u}_{\mathbf{0}}\right\|_{\mathbb{L}^2}^2+C \int_0^t\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s+\epsilon \int_0^t\left\|\Delta \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s+\epsilon \int_0^t\left\||\boldsymbol{u}_n(s)|\,|\nabla \boldsymbol{u}_n(s)|\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \\
&\quad +\sum_{k=1}^n \int_0^t\left\langle\boldsymbol{g}_k, \boldsymbol{u}_n(s)\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s),
\end{aligned}
\tag{3.11}
$$

for any $\epsilon>0$, where in the penultimate step we used the assumptions on L and $\boldsymbol{\nu}$, while in the final step we used the assumptions on $\boldsymbol{g}_k$ and $\boldsymbol{h}_k$, Young's inequality, and interpolation inequality (2.4).

It follows from (3.11) and Jensen's inequality that, after rearranging the terms, for any $p\geq 1$,

$$
\begin{aligned}
&\sup_{\tau\in[0,t]}\left\|\boldsymbol{u}_n(\tau)\right\|_{\mathbb{L}^2}^{2p}+\left(\int_0^t\left\|\nabla \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p+\left(\int_0^t\left\|\Delta \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p+\left(\int_0^t\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^4}^4 \mathrm{d}s\right)^p \\
&\lesssim 1+\left\|\boldsymbol{u}_{\mathbf{0}}\right\|_{\mathbb{L}^2}^{2p}+\int_0^t\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^{2p} \mathrm{d}s+\sup_{\tau\in[0,t]}\left|\sum_{k=1}^n \int_0^\tau\left\langle\boldsymbol{g}_k, \boldsymbol{u}_n(s)\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s)\right|^p.
\end{aligned}
\tag{3.12}
$$

It remains to estimate the last term in (3.12). To this end, by the Burkholder–Davis–Gundy inequality and the Hölder inequality (noting the assumption on $\boldsymbol{g}_k$), we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{\tau\in[0,t]}\left|\sum_{k=1}^n \int_0^\tau\left\langle\boldsymbol{g}_k, \boldsymbol{u}_n(s)\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s)\right|^p\right]
&\leq C_p\,\mathbb{E}\left[\left(\sum_{k=1}^n \int_0^t\left|\left\langle\boldsymbol{g}_k, \boldsymbol{u}_n(s)\right\rangle_{\mathbb{L}^2}\right|^2 \mathrm{d}s\right)^{p/2}\right] \\
&\leq C\,\mathbb{E}\left[\left(\sum_{k=1}^n \int_0^t\left\|\boldsymbol{g}_k\right\|_{\mathbb{L}^2}^2\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{p/2}\right] \\
&\leq C\left(\sum_{k=1}^n\left\|\boldsymbol{g}_k\right\|_{\mathbb{L}^2}^2\right)^{p/2} \mathbb{E}\left[\left(\int_0^t\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{p/2}\right] \\
&\lesssim 1+\int_0^t \mathbb{E}\left[\sup_{\tau\in[0,s]}\left\|\boldsymbol{u}_n(\tau)\right\|_{\mathbb{L}^2}^{2p}\right] \mathrm{d}s.
\end{aligned}
$$

Therefore, taking expectation on both sides of (3.12), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\in[0,t]}\left\|\boldsymbol{u}_n(\tau)\right\|_{\mathbb{L}^2}^{2p}\right]+\mathbb{E}\left[\left(\int_0^t\left\|\nabla \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right]+\mathbb{E}\left[\left(\int_0^t\left\|\Delta \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right] \\
&\quad+\mathbb{E}\left[\left(\int_0^t\left\||\boldsymbol{u}_n(s)|\,|\nabla \boldsymbol{u}_n(s)|\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right]+\mathbb{E}\left[\left(\int_0^t\left\|\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^4}^4 \mathrm{d}s\right)^p\right] \lesssim 1+\int_0^t \mathbb{E}\left[\sup_{\tau\in[0,s]}\left\|\boldsymbol{u}_n(\tau)\right\|_{\mathbb{L}^2}^{2p}\right] \mathrm{d}s.
\end{aligned}
$$

The required estimate then follows by the Gronwall inequality. □

Proposition 3.4. For any $p\geq 1$, $n\in\mathbb{N}$, $t\in(0,\infty)$,

$$
\mathbb{E}\left[\sup_{\tau\in[0,t]}\left(\left\|\nabla \boldsymbol{u}_n(\tau)\right\|_{\mathbb{L}^2}^{2p}+\left\|\boldsymbol{u}_n(\tau)\right\|_{\mathbb{L}^4}^{4p}\right)\right]+\mathbb{E}\left[\left(\int_0^t\left\|\boldsymbol{H}_n(s)\right\|_{\mathbb{H}^1}^2 \mathrm{d}s\right)^p\right] \leq C\left\|\boldsymbol{u}_{\mathbf{0}}\right\|_{\mathbb{H}^1}^{2p}+C(1+e^{Ct}),
$$

where C is a positive constant depending on p, σ_g, and σ_h (but is independent of n).

Proof. Let $\psi : \mathbb{V}_n \to \mathbb{R}$ be a function defined by $\boldsymbol{v} \mapsto \frac{1}{2}\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{1}{4}\|\boldsymbol{v}\|_{\mathbb{L}^4}^4 - \frac{1}{2}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2$. Note that by Itô's formula [19], we have

$$\begin{aligned}
\frac{1}{2}\mathrm{d}\,\|\nabla \boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 &= \left(-\langle F(\boldsymbol{u}_n,\boldsymbol{H}_n), \Delta\boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \frac{1}{2}\sum_{k=1}^n \|\nabla G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\right)\mathrm{d}t - \sum_{k=1}^n \langle G_k(\boldsymbol{u}_n), \Delta \boldsymbol{u}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(t),\\
\frac{1}{4}\mathrm{d}\,\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^4}^4 &= \left(\left\langle F(\boldsymbol{u}_n,\boldsymbol{H}_n), |\boldsymbol{u}_n|^2\boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \frac{3}{2}\sum_{k=1}^n \||\boldsymbol{u}_n||G_k(\boldsymbol{u}_n)|\|_{\mathbb{L}^2}^2\right)\mathrm{d}t\\
&\quad + \sum_{k=1}^n \left\langle G_k(\boldsymbol{u}_n), |\boldsymbol{u}_n|^2\boldsymbol{u}_n\right\rangle_{\mathbb{L}^2}\mathrm{d}W_k(t),\\
-\frac{1}{2}\mathrm{d}\,\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 &= \left(-\langle F(\boldsymbol{u}_n,\boldsymbol{H}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \frac{1}{2}\sum_{k=1}^n \|G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\right)\mathrm{d}t - \sum_{k=1}^n \langle G_k(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(t).
\end{aligned}$$

Adding the above equations and noting (3.4), we obtain

$$\begin{aligned}
\mathrm{d}\psi(\boldsymbol{u}_n) &= \left(-\langle F_n(\boldsymbol{u}_n,\boldsymbol{H}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \sum_{k=1}^n \left[\frac{1}{2}\|\nabla G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2 + \frac{3}{2}\||\boldsymbol{u}_n|\,|G_k(\boldsymbol{u}_n)|\|_{\mathbb{L}^2}^2 - \frac{1}{2}\|G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\right]\right)\mathrm{d}t\\
&\quad - \sum_{k=1}^n \langle G_k(\boldsymbol{u}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(t).
\end{aligned} \tag{3.13}$$

By (3.5) and (3.6), we have

$$\langle F_n(\boldsymbol{u}_n,\boldsymbol{H}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2} = \lambda_r\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \lambda_e\|\nabla\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \langle S(\boldsymbol{u}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2}, \tag{3.14}$$

and

$$\langle G_k(\boldsymbol{u}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2} = \langle \boldsymbol{g}_k + \gamma\boldsymbol{u}_n\times\boldsymbol{h}_k, \boldsymbol{H}_n\rangle_{\mathbb{L}^2}. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), then applying Hölder's inequality and inequality (2.8) (noting the assumptions on R and L) yield

$$\begin{aligned}
&\psi(\boldsymbol{u}_n(t)) + \lambda_r\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_e\int_0^t\|\nabla\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{1}{2}\sum_{k=1}^n\int_0^t \|G_k(\boldsymbol{u}_n(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\\
&= \psi(\boldsymbol{u}_n(0)) - \int_0^t \langle R(\boldsymbol{u}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}s - \int_0^t \langle L(\boldsymbol{u}_n),\boldsymbol{H}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}s + \frac{1}{2}\sum_{k=1}^n\int_0^t \|\nabla G_k(\boldsymbol{u}_n(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s\\
&\quad + \frac{3}{2}\sum_{k=1}^n\int_0^t \||\boldsymbol{u}_n(s)|\,|G_k(\boldsymbol{u}_n(s))|\|_{\mathbb{L}^2}^2\,\mathrm{d}s - \sum_{k=1}^n\int_0^t \langle\boldsymbol{g}_k+\gamma\boldsymbol{u}_n(s)\times\boldsymbol{h}_k,\boldsymbol{H}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s)\\
&\le \psi(\boldsymbol{u}_n(0)) + C\int_0^t \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2\left(\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right)\mathrm{d}s + \epsilon\int_0^t\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C\int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2\,\mathrm{d}s\\
&\quad + C\sum_{k=1}^n\int_0^t\left(\|\nabla\boldsymbol{g}_k\|_{\mathbb{L}^2}^2 + \|\boldsymbol{h}_k\|_{\mathbb{L}^2}^2\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2\|\nabla\boldsymbol{h}_k\|_{\mathbb{L}^2}^2\right)\mathrm{d}s\\
&\quad + C\sum_{k=1}^n\int_0^t\left(\|\boldsymbol{g}_k\|_{\mathbb{L}^\infty}^2\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{h}_k\|_{\mathbb{L}^\infty}^2\|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4\right)\mathrm{d}s - \sum_{k=1}^n\int_0^t\langle\boldsymbol{g}_k+\gamma\boldsymbol{u}_n\times\boldsymbol{h}_k,\boldsymbol{H}_n\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s)\\
&\lesssim 1 + \epsilon\int_0^t\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \int_0^t \||\boldsymbol{u}_n|\,|\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\,\mathrm{d}s\\
&\quad + \left[\sum_{k=1}^n\left(\|\boldsymbol{g}_k\|_{\mathbb{H}^1}^2 + \|\boldsymbol{h}_k\|_{\mathbb{H}^1}^2\right)\right]\left(\int_0^t 1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left[\sum_{k=1}^n \left(\|\boldsymbol{g}_k\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{h}_k\|_{\mathbb{L}^\infty}^2\right)\right] \left(\int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right) - \sum_{k=1}^n \int_0^t \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \boldsymbol{H}_n \rangle_{\mathbb{L}^2} \,\mathrm{d}W_k(s) \\
&\lesssim 1 + \epsilon \int_0^t \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \int_0^t \| |\boldsymbol{u}_n| \, |\nabla \boldsymbol{u}_n| \|_{\mathbb{L}^2}^2 \,\mathrm{d}s \\
&\quad + \int_0^t \left(\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2\right) \mathrm{d}s - \sum_{k=1}^n \int_0^t \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \boldsymbol{H}_n \rangle_{\mathbb{L}^2} \,\mathrm{d}W_k(s)
\end{aligned} \tag{3.16}
$$

for any $\epsilon > 0$, where in the last step we used the assumptions on $\boldsymbol{g}_k$ and $\boldsymbol{h}_k$. By Jensen's inequality, after rearranging the terms and noting Proposition 3.3, it follows that

$$
\begin{aligned}
&\sup_{\tau\in[0,t]} \left(\|\nabla \boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^{2p} + \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^4}^{4p}\right) + \left(\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p + \left(\int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p \\
&\lesssim 1 + \left(\int_0^t \| |\boldsymbol{u}_n| \, |\nabla \boldsymbol{u}_n| \|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p + \left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 + \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p \\
&\quad + \sup_{\tau\in[0,t]} \left| \sum_{k=1}^n \int_0^\tau \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \boldsymbol{H}_n \rangle_{\mathbb{L}^2} \,\mathrm{d}W_k(s) \right|^p .
\end{aligned} \tag{3.17}
$$

Now, we estimate the last term on the right-hand side of (3.17). By the Burkholder–Davis–Gundy inequality and the Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\in[0,t]} \left| \sum_{k=1}^n \int_0^\tau \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \boldsymbol{H}_n \rangle_{\mathbb{L}^2} \,\mathrm{d}W_k(s) \right|^p\right] \\
&\le C_p\, \mathbb{E}\left[\left(\sum_{k=1}^n \int_0^t \left| \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \boldsymbol{H}_n \rangle_{\mathbb{L}^2} \right|^2 \mathrm{d}s\right)^{p/2}\right] \\
&\le C \left(\sum_{k=1}^n \left(\|\boldsymbol{g}_k\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{h}_k\|_{\mathbb{L}^\infty}^2\right)\right)^{p/2} \mathbb{E}\left[\left(\int_0^t \left(1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2\right) \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{p/2}\right] \\
&\le C\, \mathbb{E}\left[\left(1 + \sup_{\tau\in(0,t)} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^2\right)^{p/2} \left(\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{p/2}\right] \\
&\lesssim \mathbb{E}\left[\left(1 + \sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^2\right)^p\right] + \epsilon \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p\right] \\
&\lesssim 1 + \epsilon \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p\right]
\end{aligned} \tag{3.18}
$$

for any $\epsilon > 0$, where in the penultimate step we used Young's inequality, while in the final step we used Proposition 3.3. Taking expectation on both sides of (3.17), choosing $\epsilon > 0$ sufficiently small, and substituting (3.18) into the resulting expression, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\in[0,t]} \left(\|\nabla \boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^{2p} + \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^4}^{4p}\right)\right] + \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p\right] \\
&\qquad \lesssim 1 + \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^1}^2 + \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^p\right].
\end{aligned}
$$

Applying Proposition 3.3 yields the required estimate. □

Proposition 3.5. For any $p \ge 1$, $n \in \mathbb{N}$, $t \in (0, \infty)$,

$$
\mathbb{E}\left[\left(\int_0^t \|\nabla \Delta \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^p\right] \le C \|\boldsymbol{u_0}\|_{\mathbb{H}^1}^{2p} + C(1 + e^{Ct}),
$$

where C is a positive constant depending on p, σ_g, and σ_h (but is independent of n).

Proof. Using the second equation in (3.2) and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \|\nabla\Delta \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \\
&\leq \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \left\|\nabla\left(|\boldsymbol{u}_n(s)|^2\boldsymbol{u}_n(s)\right)\right\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \\
&\leq \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^6}^4\,\|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^6}^2\,\mathrm{d}s\right)^p\right] \\
&\leq \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \\
&\quad + C\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\|\boldsymbol{u}_n(\tau)\|_{\mathbb{H}^1}^4\right)^p\left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right)^p\right] \\
&\leq \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \\
&\quad + C\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\|\boldsymbol{u}_n(\tau)\|_{\mathbb{H}^1}^4\right)^{2p}\right] + C\mathbb{E}\left[\left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right)^{2p}\right] \leq C,
\end{aligned}$$

where in the fourth line we also used the Sobolev embedding $\mathbb{H}^1 \subset \mathbb{L}^6$, and in the last step we used Young's inequality, Proposition 3.3, and 3.4. This shows the required estimate. ☐

Proposition 3.6. For any $p \geq 1$, $n \in \mathbb{N}$, $t \in (0, \infty)$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{\tau\in[0,t]}\|\Delta \boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^{2p}\right] &+ \mathbb{E}\left[\left(\int_0^t \|\nabla\Delta \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \\
&+ \mathbb{E}\left[\left(\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right] \leq C\,\|\boldsymbol{u_0}\|_{\mathbb{H}^2}^{2p} + C(1+e^{Ct}). \qquad (3.19)
\end{aligned}$$

Furthermore,

$$\mathbb{E}\left[\sup_{\tau\in[0,t]}\|\boldsymbol{H}_n(\tau)\|_{\mathbb{L}^2}^{2p}\right] \leq C\,\|\boldsymbol{u_0}\|_{\mathbb{H}^2}^{2p} + C(1+e^{Ct}), \qquad (3.20)$$

where C is a positive constant depending on p, σ_g, and σ_h (but is independent of n).

Proof. Let $\psi : \mathbb{V}_n \to \mathbb{R}$ be a function defined by $\boldsymbol{v} \mapsto \frac{1}{2}\|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2$. Similarly to the proof of Proposition 3.3, by Itô's lemma,

$$\frac{1}{2}\mathrm{d}\,\|\Delta \boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 = \left(\left\langle F_n(\boldsymbol{u}_n, \boldsymbol{H}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \frac{1}{2}\sum_{k=1}^n \|\Delta G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\right)\mathrm{d}t + \sum_{k=1}^n \left\langle G_k(\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2}\mathrm{d}W_k(t). \qquad (3.21)$$

Note that by (3.5) and (3.6), integrating by parts as necessary, we have

$$\begin{aligned}
&\left\langle F_n(\boldsymbol{u}_n, \boldsymbol{H}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\
&= \lambda_r\left\langle \boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} - \lambda_e\left\langle \Delta\boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} - \gamma\left\langle \boldsymbol{u}_n\times\boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \langle S(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \\
&= \lambda_r\left\langle \boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} - \lambda_e\left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 + \lambda_e\|\nabla\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \lambda_e\left\langle \Delta(|\boldsymbol{u}_n|^2\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\
&\quad - \gamma\left\langle \boldsymbol{u}_n\times\boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \left\langle R(\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \left\langle L(\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \qquad (3.22)
\end{aligned}$$

and

$$\left\langle G_k(\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} = \left\langle \boldsymbol{g}_k + \gamma\boldsymbol{u}_n\times\boldsymbol{h}_k, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2}. \qquad (3.23)$$

Substituting (3.22) and (3.23) into (3.21) yields

$$\begin{aligned}
&\frac{1}{2}\|\Delta \boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 + \lambda_e \int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \\
&= \frac{1}{2}\|\Delta \boldsymbol{u_0}\|_{\mathbb{L}^2}^2 + \frac{1}{2}\sum_{k=1}^n \int_0^t \|\Delta G_k(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2 \mathrm{d}s + \lambda_r \int_0^t \left\langle \boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \mathrm{d}s + \lambda_e \int_0^t \|\nabla \Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \mathrm{d}s \\
&\quad + \lambda_e \int_0^t \left\langle \Delta(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \mathrm{d}s - \gamma \int_0^t \left\langle \boldsymbol{u}_n \times \boldsymbol{H}_n, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \mathrm{d}s + \int_0^t \left\langle R(\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \mathrm{d}s \\
&\quad + \int_0^t \left\langle L(\boldsymbol{u}_n), \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \mathrm{d}s + \sum_{k=1}^n \int_0^t \left\langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \Delta^2 \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s) \\
&=: \frac{1}{2}\|\Delta \boldsymbol{u_0}\|_{\mathbb{L}^2}^2 + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t) + I_8(t) + I_9(t).
\end{aligned} \tag{3.24}$$

We shall bound each term appearing on the right-hand side of (3.24). Firstly, for the term $I_2(t)$, by (2.5),

$$|I_2(t)| \le C \sum_{k=1}^n \int_0^t \left(\|\boldsymbol{g}_k\|_{\mathbb{H}^2}^2 + \gamma \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 \|\boldsymbol{h}_k\|_{\mathbb{H}^2}^2\right) \mathrm{d}s \le C\left(1 + \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 \mathrm{d}s\right).$$

Secondly, for the term $I_3(t)$, by Young's inequality,

$$|I_3(t)| \le C \int_0^t \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s$$

for any $\epsilon > 0$. The term $I_4(t)$ will be left as is. For the term $I_5(t)$, noting (2.3) and applying Hölder's and Young's inequalities, we have

$$\begin{aligned}
|I_5(t)| &\le \lambda_e \int_0^t \left(\|\boldsymbol{u}_n\|_{\mathbb{L}^\infty}^2 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2} \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2} + \|\boldsymbol{u}_n\|_{\mathbb{L}^\infty} \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^4}^2 \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}\right) \mathrm{d}s \\
&\le C \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^\infty}^4 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s + C \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^\infty}^2 \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \mathrm{d}s \\
&\le C \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s
\end{aligned}$$

for any $\epsilon > 0$, where in the last step we assumed $d = 3$ for brevity and used the Gagliardo–Nirenberg inequalities

$$\|\boldsymbol{u}_n\|_{\mathbb{L}^\infty}^2 \le C \|\boldsymbol{u}_n\|_{\mathbb{H}^1} \|\boldsymbol{u}_n\|_{\mathbb{H}^2} \quad \text{and} \quad \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \le C \|\boldsymbol{u}_n\|_{\mathbb{H}^1} \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^3,$$

and the interpolation inequality

$$\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 \le C \|\boldsymbol{u}_n\|_{\mathbb{H}^1} \|\boldsymbol{u}_n\|_{\mathbb{H}^3}.$$

The case $d = 1$ and $d = 2$ can be handled similarly. Next, for the term $I_6(t)$, we applied Young's inequality and Sobolev embedding to obtain

$$|I_6(t)| \le C \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^2 \|\boldsymbol{H}_n\|_{\mathbb{L}^4}^2 \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \le C \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^2 \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s.$$

For the term $I_7(t)$, we used (2.8) to obtain

$$|I_7(t)| \le C \int_0^t \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right) \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s.$$

For the term $I_8(t)$, by the assumption on L and Young's inequality,

$$|I_8(t)| \le C \int_0^t \left(1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2\right) \mathrm{d}s + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s.$$

For the moment, we keep the term $I_9(t)$ as is. Altogether, substituting these bounds into (3.24) implies

$$\begin{aligned}
&\frac{1}{2}\|\Delta \boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 + \lambda_e \int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \\
&\lesssim \|\Delta \boldsymbol{u_0}\|_{\mathbb{L}^2}^2 + \epsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}_n\right\|_{\mathbb{L}^2}^2 \mathrm{d}s + \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 \mathrm{d}s + \int_0^t \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \mathrm{d}s + \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 \mathrm{d}s \\
&\quad + \int_0^t \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^2 \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 \mathrm{d}s + \int_0^t \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \mathrm{d}s + \sum_{k=1}^n \int_0^t \left\langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \Delta^2 \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s)
\end{aligned}$$

for any $\epsilon > 0$. Choosing $\epsilon > 0$ sufficiently small, rearranging the terms, and applying Jensen's inequality, we obtain for $p \geq 1$,

$$\begin{aligned}
&\sup_{\tau\in[0,t]} \|\Delta \boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^{2p} + \left(\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p \\
&\lesssim 1 + \left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2 \mathrm{d}s\right)^p + \left(\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p + \left(\sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{H}^1}^4\right)^p \left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2 \mathrm{d}s\right)^p \\
&\quad + \left(\sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^4}^2\right)^p \left(\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^1}^2 \mathrm{d}s\right)^p + \left(\int_0^t \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p \\
&\quad + \sup_{\tau\in[0,t]} \left|\sum_{k=1}^n \int_0^t \left\langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \Delta^2 \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s)\right|^p. \qquad (3.25)
\end{aligned}$$

We will estimate the last term on the right-hand side of (3.25). By the Burkholder–Davis–Gundy inequality and the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\in[0,t]} \left|\sum_{k=1}^n \int_0^\tau \left\langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \Delta^2 \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s)\right|^p\right] \\
&\leq C_p\, \mathbb{E}\left[\left(\sum_{k=1}^n \int_0^t \left| \left\langle \boldsymbol{g}_k + \gamma \boldsymbol{u}_n \times \boldsymbol{h}_k, \Delta^2 \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \right|^2 \mathrm{d}s\right)^{p/2}\right] \\
&\leq C \left(\sum_{k=1}^n \left(\|\boldsymbol{g}_k\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{h}_k\|_{\mathbb{L}^\infty}^2\right)\right)^{p/2} \mathbb{E}\left[\left(\int_0^t \left(1 + \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\right) \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{p/2}\right] \\
&\leq C\mathbb{E}\left[\left(1 + \sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^2\right)^{p/2} \left(\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{p/2}\right] \\
&\lesssim \mathbb{E}\left[\left(1 + \sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^2\right)^p\right] + \epsilon\mathbb{E}\left[\left(\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right] \\
&\lesssim 1 + \epsilon\mathbb{E}\left[\left(\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right] \qquad (3.26)
\end{aligned}$$

for any $\epsilon > 0$, where in the penultimate step we used Young's inequality, while in the last step we used Proposition 3.3. Taking expectation on both sides of (3.25), substituting (3.26) into the resulting expression, and choosing $\epsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\in[0,t]} \|\Delta \boldsymbol{u}_n(\tau)\|_{\mathbb{L}^2}^{2p}\right] + \mathbb{E}\left[\left(\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right] \\
&\lesssim 1 + \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 \mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\int_0^t \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^p\right] + \mathbb{E}\left[\left(\sup_{\tau\in[0,t]} \|\boldsymbol{u}_n(\tau)\|_{\mathbb{H}^1}^4\right)^p \left(\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2 \mathrm{d}s\right)^p\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^4}^2\right)^p\left(\int_0^t\|\boldsymbol{H}_n(s)\|_{\mathbb{H}^1}^2\,\mathrm{d}s\right)^p\right]+\mathbb{E}\left[\left(\int_0^t\||\boldsymbol{u}_n|\,|\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\\
&\lesssim 1+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2\,\mathrm{d}s\right)^p\right]+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\\
&\quad+\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\|\boldsymbol{u}_n(\tau)\|_{\mathbb{H}^1}^4\right)^{2p}\right]+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2\,\mathrm{d}s\right)^{2p}\right]\\
&\quad+\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\|\boldsymbol{u}_n(\tau)\|_{\mathbb{L}^4}^2\right)^{2p}\right]+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{H}_n(s)\|_{\mathbb{H}^1}^2\,\mathrm{d}s\right)^{2p}\right]+\mathbb{E}\left[\left(\int_0^t\||\boldsymbol{u}_n|\,|\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\lesssim 1,
\end{aligned}
$$

where in the penultimate step we used Young's inequality and in the final step we used Proposition 3.3, 3.4, and 3.5. This proves (3.19).

Finally, since $H_n=\Delta\boldsymbol{u}_n+\boldsymbol{u}_n-\Pi_n(|\boldsymbol{u}_n|^2\boldsymbol{u}_n)$, we obtain (3.20) by using (3.19), Proposition 3.4, and the triangle inequality. This completes the proof of the proposition. □

Proposition 3.7. For any $p\geq 1$, $n\in\mathbb{N}$, $t\in(0,\infty)$,

$$
\mathbb{E}\left[\left(\int_0^t\|\Delta\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\leq C, \tag{3.27}
$$

$$
\mathbb{E}\left[\left(\int_0^t\left\|\Pi_n\big(\boldsymbol{u}_n(s)\times\boldsymbol{H}_n(s)\big)\right\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right)^p\right]\leq C, \tag{3.28}
$$

$$
\mathbb{E}\left[\left(\int_0^t\|\Pi_nS(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\leq C, \tag{3.29}
$$

where C is a positive constant depending on p, t, σ_g, σ_h, and $\|\boldsymbol{u_0}\|_{\mathbb{H}^2}$.

Proof. By (3.2) and (2.6),

$$
\begin{aligned}
\|\Delta\boldsymbol{H}_n\|_{\mathbb{L}^2}&\leq\left\|\Delta^2\boldsymbol{u}_n\right\|_{\mathbb{L}^2}+\|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}+\left\|\Delta(|\boldsymbol{u}_n|^2\boldsymbol{u}_n)\right\|_{\mathbb{L}^2}\\
&\leq\left\|\Delta^2\boldsymbol{u}_n\right\|_{\mathbb{L}^2}+\|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}+\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^3.
\end{aligned}
$$

Similarly, by Hölder's and Young's inequalities,

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^t\|\Delta\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]&\leq\mathbb{E}\left[\left(\int_0^t\left\|\Delta^2\boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]+\mathbb{E}\left[\left(\int_0^t\|\Delta\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\\
&\quad+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^6\,\mathrm{d}s\right)^p\right]\leq C,
\end{aligned}
$$

where in the last step we used Proposition 3.3, 3.4, and 3.5. This proves (3.27).

Next, by (2.5) and Young's inequality,

$$
\begin{aligned}
\left\|\Pi_n\big(\boldsymbol{u}_n\times\boldsymbol{H}_n\big)\right\|_{\mathbb{L}^2}&\leq\|\boldsymbol{u}_n\times\Delta\boldsymbol{u}_n\|_{\mathbb{H}^2}+\left\|\boldsymbol{u}_n\times\Pi_n(|\boldsymbol{u}_n|^2\boldsymbol{u}_n)\right\|_{\mathbb{H}^2}\\
&\leq C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}\|\Delta\boldsymbol{u}_n\|_{\mathbb{H}^2}+C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^4\\
&\leq C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2+C\|\Delta\boldsymbol{u}_n\|_{\mathbb{H}^2}^2+C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^4.
\end{aligned}
$$

Similar argument as before (noting (3.19)) then yields (3.28).

Finally, by the definition of R and L, and Hölder's inequality,

$$
\|\Pi_nS(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\leq C\|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2+C\|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2\|\boldsymbol{u}_n\|_{\mathbb{L}^\infty}^2\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2+C\big(1+\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2\big).
$$

Therefore, taking expectation, then applying Young's inequality and Sobolev embedding,

$$
\mathbb{E}\left[\left(\int_0^t\|\Pi_nS(\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^p\right]\lesssim 1+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^1}^2\right)^p\right]+\mathbb{E}\left[\left(\int_0^t\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^4\right)^p\right]\leq C,
$$

where we also used Proposition 3.3, 3.4, and 3.5. This shows (3.29), thus completing the proof of the proposition. ☐

4. Proof of Theorem 2.4: Existence of a Martingale Solution

In this section, we prove the existence of martingale solution to (1.1) as stated in Theorem 2.4. First, note that equation (3.4) can be written as

$$\begin{aligned}\boldsymbol{u}_n(t) &= \boldsymbol{u}_n(0) + \int_0^t F_n(\boldsymbol{u}_n, \boldsymbol{H}_n)\,\mathrm{d}s + \sum_{k=1}^n \int_0^t G_k(\boldsymbol{u}_n)\,\mathrm{d}W_k(s) \\ &=: \boldsymbol{u}_n(0) + \int_0^t F_n(\boldsymbol{u}_n, \boldsymbol{H}_n)\,\mathrm{d}s + B_n(\boldsymbol{u}_n, W)(t),\end{aligned} \tag{4.1}$$

where F_n and G_k were defined in (3.5) and (3.6), and $\boldsymbol{H}_n = \Delta \boldsymbol{u}_n + \boldsymbol{u}_n - \Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n)$. Moreover, uniform bounds for $\boldsymbol{u}_n$ established in Section 3 imply the following proposition.

Proposition 4.1. Let $p \geq 2$, $q \geq 1$, and $\alpha \in \left(0, \frac{1}{2}\right)$ with $\alpha p > 1$. There exists a constant C such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[\left\|\int_0^t F_n(\boldsymbol{u}_n, \boldsymbol{H}_n)\,\mathrm{d}s\right\|_{\mathbb{H}^1(0,T;\mathbb{L}^2)}^q\right] \leq C, \tag{4.2}$$

$$\mathbb{E}\left[\|B_n(\boldsymbol{u}_n, W)(t)\|_{\mathbb{W}^{\alpha,p}(0,T;\mathbb{L}^2)}^q\right] \leq C, \tag{4.3}$$

where C is a constant depending on p, q, α, T, σ_g, σ_h, and $\|\boldsymbol{u_0}\|_{\mathbb{H}^2}$ (but is independent of n).

Proof. Inequality (4.2) follows immediately from Proposition 3.4 and 3.7, while inequality (4.3) is a consequence of Proposition 3.4. ☐

Proposition 4.2. For any $p \in [1, \infty)$ and $\beta > 0$, the set of laws $\{\mathcal{L}(\boldsymbol{u}_n) : n \in \mathbb{N}\}$ on the space

$$\mathbb{Y} := L^p(0,T;\mathbb{W}^{1,4}) \cap L^2(0,T;\mathbb{H}^3) \cap C([0,T];\mathbb{X}^{-\beta}) \tag{4.4}$$

is tight.

Proof. Proposition 3.3, 3.4, 3.6, and 4.1 show that for any $q \geq 1$,

$$\mathbb{E}\left[\|\boldsymbol{u}_n\|_{L^p(0,T;\mathbb{H}^2)\cap L^2(0,T;\mathbb{H}^4)\cap W^{\alpha,p}(0,T;\mathbb{L}^2)}^q\right] \leq C.$$

This and the following compact embeddings

$$\begin{aligned}L^p(0,T;\mathbb{H}^2) \cap W^{\alpha,p}(0,T;\mathbb{L}^2) &\hookrightarrow L^p(0,T;\mathbb{W}^{1,4}) \cap C([0,T];\mathbb{X}^{-\beta}), \\ L^2(0,T;\mathbb{H}^4) \cap W^{\alpha,p}(0,T;\mathbb{L}^2) &\hookrightarrow L^2(0,T;\mathbb{H}^3) \cap C([0,T];\mathbb{X}^{-\beta})\end{aligned}$$

implies the required result. ☐

By the above proposition, we can find a subsequence of $\{\boldsymbol{u}_n\}$, which is not relabelled, such that the laws $\mathcal{L}(\boldsymbol{u}_n, W)$ converge weakly to a probability measure μ on $\mathbb{Y} \times C([0,T];\mathbb{R}^\infty)$. Noting that the space $\mathbb{Y} \times C([0,T];\mathbb{R}^\infty)$ is separable, we then have the following proposition by the Skorohod theorem.

Proposition 4.3. Let $\mathbb{Y}$ be the space defined in (4.4). Then there exist

(1) a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$,
(2) a sequence of random variables $\{(\boldsymbol{u}_n', W_n')\}$ defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ and taking values in the space $\mathbb{Y} \times C([0,T];\mathbb{R}^\infty)$.
(3) a random variable $(\boldsymbol{u}', W')$ defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ and taking values in the space $\mathbb{Y} \times C([0,T];\mathbb{R}^\infty)$,

such that in the space $\mathbb{Y} \times C([0,T];\mathbb{R}^\infty)$,

(1) the laws $\mathcal{L}(\boldsymbol{u}_n, W) = \mathcal{L}(\boldsymbol{u}_n', W_n')$ for all $n \in \mathbb{N}$,
(2) $(\boldsymbol{u}_n', W_n') \to (\boldsymbol{u}', W')$ strongly, $\mathbb{P}'$-a.s.

Remark 4.4. By the Kuratowski–Suslin theorem, the Borel subsets of $C([0,T];\mathbb{V}_n)$ are Borel subsets of $\mathbb{Y}$. Moreover, $\mathbb{P}$-a.s., $\boldsymbol{u}_n \in C([0,T];\mathbb{V}_n)$. Therefore, we may assume that $\boldsymbol{u}_n'$ takes values in $\mathbb{V}_n$ and that the laws on $C([0,T];\mathbb{V}_n)$ of $\boldsymbol{u}_n$ and $\boldsymbol{u}_n'$ are equal. It is then straightforward to show that the sequence $\{\boldsymbol{u}_n'\}$ satisfies the same estimates as the original sequence $\{\boldsymbol{u}_n\}$, namely for any $q \geq 1$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\sup_{t\in[0,T]} \left\|\boldsymbol{u}_n'(t)\right\|_{\mathbb{H}^2}^{2q}\right] < \infty, \tag{4.5}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\left(\int_0^T \left\|\boldsymbol{u}_n'(s)\right\|_{\mathbb{H}^4}^2 \mathrm{d}s\right)^q\right] < \infty, \tag{4.6}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\left(\int_0^T \left\|\boldsymbol{H}_n'(s)\right\|_{\mathbb{H}^2}^2 \mathrm{d}s\right)^q\right] < \infty, \tag{4.7}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\left(\int_0^t \left\|\Pi_n S(\boldsymbol{u}_n')\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^q\right] < \infty, \tag{4.8}$$

where $\boldsymbol{H}_n' = \Delta \boldsymbol{u}_n' + \boldsymbol{u}_n' - \Pi_n(|\boldsymbol{u}_n'|^2 \boldsymbol{u}_n')$.

Subsequently, we will work solely with processes defined on the probability space $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$. To simplify notations, we will write $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ instead. The new processes W_n', W', and $\boldsymbol{u}_n'$ will also be written as W_n, W, and $\boldsymbol{u}_n$, respectively. We remark that, as in [7], the processes W_n' and W' are Wiener processes on $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$.

Now, define a sequence of $\mathbb{L}^2$-valued processes

$$\boldsymbol{M}_n(t) := \boldsymbol{u}_n(t) - \boldsymbol{u}_n(0) - \int_0^t F_n(\boldsymbol{u}_n, \boldsymbol{H}_n)\,\mathrm{d}s. \tag{4.9}$$

Then for each $t \in [0,T]$, we have

$$\boldsymbol{M}_n(t) = B_n(\boldsymbol{u}_n, W_n)(t), \quad \mathbb{P}\text{-a.s.},$$

where B_n and F_n are as in (4.1).

Lemma 4.5. Let $H_n := \Delta \boldsymbol{u}_n + \boldsymbol{u}_n - \Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n)$ and $H := \Delta \boldsymbol{u} + \boldsymbol{u} - |\boldsymbol{u}|^2 \boldsymbol{u}$. Then $\mathbb{P}$-a.s.,

$$\boldsymbol{H}_n \to \boldsymbol{H} \text{ strongly in } L^2(0,T;\mathbb{H}^1).$$

Proof. Since $\boldsymbol{u}_n \to \boldsymbol{u}$ strongly in $L^2(0,T;\mathbb{H}^3)$, it is clear that $\Delta \boldsymbol{u}_n \to \Delta \boldsymbol{u}$ strongly in $L^2(0,T;\mathbb{H}^1)$. Moreover, by Hölder's inequality,

$$\begin{aligned}
\left\||\boldsymbol{u}_n|^2 \boldsymbol{u}_n - |\boldsymbol{u}|^2 \boldsymbol{u}\right\|_{L^2(0,T;\mathbb{H}^1)} &\leq \left\||\boldsymbol{u}_n|^2 (\boldsymbol{u}_n - \boldsymbol{u})\right\|_{L^2(0,T;\mathbb{H}^1)} + \left\|(\boldsymbol{u}_n - \boldsymbol{u}) \cdot (\boldsymbol{u}_n + \boldsymbol{u}) \boldsymbol{u}\right\|_{L^2(0,T;\mathbb{H}^1)} \\
&\leq \|\boldsymbol{u}_n\|_{L^6(0,T;\mathbb{L}^6)}^2 \|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^6(0,T;\mathbb{L}^6)} \\
&\quad + 2\|\boldsymbol{u}_n\|_{L^8(0,T;\mathbb{L}^8)} \|\nabla \boldsymbol{u}_n\|_{L^4(0,T;\mathbb{L}^4)} \|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^6(0,T;\mathbb{L}^6)} \|\boldsymbol{u}_n + \boldsymbol{u}\|_{L^6(0,T;\mathbb{L}^6)} \|\boldsymbol{u}\|_{L^6(0,T;\mathbb{L}^6)} \\
&\quad + \|\nabla \boldsymbol{u}_n - \nabla \boldsymbol{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\boldsymbol{u}_n + \boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \|\boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \|\nabla \boldsymbol{u}_n + \nabla \boldsymbol{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \|\boldsymbol{u}_n + \boldsymbol{u}\|_{L^8(0,T;\mathbb{L}^8)} \|\nabla \boldsymbol{u}\|_{L^4(0,T;\mathbb{L}^4)} .
\end{aligned}$$

Noting the embedding $\mathbb{W}^{1,4} \hookrightarrow \mathbb{L}^8$ and using Proposition 4.3, we have that $\Pi_n(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n) \to |\boldsymbol{u}|^2 \boldsymbol{u}$ strongly, $\mathbb{P}$-a.s. in $L^2(0,T;\mathbb{H}^1)$. This implies $\boldsymbol{H}_n \to \boldsymbol{H}$ strongly in $L^2(0,T;\mathbb{H}^1)$, as required. □

Lemma 4.6. Let $\boldsymbol{u}_n$ and $\boldsymbol{u}$ be processes defined in Proposition 4.3. Then for any $q \geq 1$,

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \text{ weakly in } L^{2q}\big(\Omega; L^\infty(0,T;\mathbb{H}^2) \cap L^2(0,T;\mathbb{H}^4)\big).$$

Furthermore, $\boldsymbol{u} \in L^{2q}\big(\Omega; C([0,T];\mathbb{H}^2_{\mathrm{w}})\big)$.

Proof. By Proposition 4.3 and Vitali's convergence theorem, we have for any $\boldsymbol{\phi} \in L^4\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$,

$$\mathbb{E}\left[\int_0^T \langle \boldsymbol{u}_n(t), \boldsymbol{\phi}(t)\rangle_{\mathbb{L}^2} \,\mathrm{d}t\right] \to \mathbb{E}\left[\int_0^T \langle \boldsymbol{u}(t), \boldsymbol{\phi}(t)\rangle_{\mathbb{L}^2} \,\mathrm{d}t\right]. \tag{4.10}$$

On the other hand, the Banach–Alaoglu theorem (noting (4.5) and (4.6)) confers a subsequence of $\{\boldsymbol{u}_n\}$, which we do not relabel, and $\boldsymbol{v} \in L^{2q}\big(\Omega; L^\infty(0,T;\mathbb{H}^2) \cap L^2(0,T;\mathbb{H}^4)\big)$ such that

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } L^{2q}\big(\Omega; L^\infty(0,T;\mathbb{H}^2) \cap L^2(0,T;\mathbb{H}^4)\big).$$

Let $\mathbb{U} := L^{\frac{2q}{2q-1}}\big(\Omega; L^1(0,T;\mathbb{X}^{-1}) \cap L^2(0,T;\mathbb{X}^{-2})\big)$. Noting that $L^{2q}\big(\Omega; L^\infty(0,T;\mathbb{H}^2) \cap L^2(0,T;\mathbb{H}^4)\big) \cong \mathbb{U}^*$, we have

$$\mathbb{E}\left[\int_0^T \int_{\mathscr{D}} \boldsymbol{u}_n(t,\boldsymbol{x}) \cdot \boldsymbol{\psi}(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t\right] \to \mathbb{E}\left[\int_0^T \int_{\mathscr{D}} \boldsymbol{v}(t,\boldsymbol{x}) \cdot \boldsymbol{\psi}(t,\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}t\right] \tag{4.11}$$

for any $\boldsymbol{\psi} \in \mathbb{U}$. By the density of $L^4\big(\Omega; L^2(0,T;\mathbb{L}^2)\big)$ in $\mathbb{U}$, we conclude from (4.10) and (4.11) that $\boldsymbol{u} = \boldsymbol{v}$ in $L^{2q}\big(\Omega; L^\infty(0,T;\mathbb{H}^2) \cap L^2(0,T;\mathbb{H}^4)\big)$.

Finally, Proposition 4.3 implies $\boldsymbol{u}_n \in C\big([0,T];\mathbb{H}^2_{\mathrm{w}}\big)$. Since $\boldsymbol{u}_n \to \boldsymbol{u}$ weakly in $L^{2q}\big(\Omega; L^\infty(0,T;\mathbb{H}^2)\big)$ and the space $C\big([0,T];\mathbb{H}^2_{\mathrm{w}}\big)$ is complete, we deduce $\boldsymbol{u} \in L^{2q}\big(\Omega; C([0,T];\mathbb{H}^2_{\mathrm{w}})\big)$. This completes the proof of the lemma. ☐

We now show the convergence of each term in (4.1).

Lemma 4.7. For any $\boldsymbol{\chi} \in \mathbb{H}^1$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\int_0^t \langle \boldsymbol{H}_n(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right] &= \mathbb{E}\left[\int_0^t \langle \boldsymbol{H}(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right],\\
\lim_{n\to\infty} \mathbb{E}\left[\int_0^t \langle \nabla\boldsymbol{H}_n(s), \nabla\boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right] &= \mathbb{E}\left[\int_0^t \langle \nabla\boldsymbol{H}(s), \nabla\boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right],\\
\lim_{n\to\infty} \mathbb{E}\left[\int_0^t \big\langle \Pi_n\big(\boldsymbol{u}_n(s) \times \boldsymbol{H}_n(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right] &= \mathbb{E}\left[\int_0^t \langle \boldsymbol{u}(s) \times \boldsymbol{H}(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right],\\
\lim_{n\to\infty} \mathbb{E}\left[\int_0^t \big\langle \Pi_n S\big(\boldsymbol{u}_n(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right] &= \mathbb{E}\left[\int_0^t \big\langle S\big(\boldsymbol{u}(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right].
\end{aligned}$$

Proof. By the same argument as in [26, Proposition 4.1], using Proposition 4.3 and Lemma 4.5, we have $\mathbb{P}$-a.s.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^t \langle \boldsymbol{H}_n(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s &= \int_0^t \langle \boldsymbol{H}(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s,\\
\lim_{n\to\infty} \int_0^t \langle \nabla\boldsymbol{H}_n(s), \nabla\boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s &= \int_0^t \langle \nabla\boldsymbol{H}(s), \nabla\boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s,\\
\lim_{n\to\infty} \int_0^t \big\langle \Pi_n\big(\boldsymbol{u}_n(s) \times \boldsymbol{H}_n(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s &= \int_0^t \langle \boldsymbol{u}(s) \times \boldsymbol{H}(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s,\\
\lim_{n\to\infty} \int_0^t \big\langle \Pi_n S\big(\boldsymbol{u}_n(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s &= \int_0^t \big\langle S\big(\boldsymbol{u}(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s.
\end{aligned}$$

Furthermore, by Hölder's inequality (noting (4.5), (4.6), (4.7), and (4.8)), we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|\int_0^t \langle \boldsymbol{H}_n(s), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right|^2\right] &\le \|\boldsymbol{H}_n\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} \|\boldsymbol{\chi}\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} < \infty,\\
\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|\int_0^t \langle \nabla\boldsymbol{H}_n(s), \nabla\boldsymbol{\chi}\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right|^2\right] &\le \|\nabla\boldsymbol{H}_n\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} \|\nabla\boldsymbol{\chi}\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} < \infty,\\
\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|\int_0^t \big\langle \Pi_n\big(\boldsymbol{u}_n(s) \times \boldsymbol{H}_n(s)\big), \boldsymbol{\chi}\big\rangle_{\mathbb{L}^2} \,\mathrm{d}s\right|^2\right] &\le \|\boldsymbol{u}_n\|^2_{L^8(\Omega;L^4(0,T;\mathbb{L}^4))} \|\boldsymbol{H}_n\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} \|\boldsymbol{\chi}\|^2_{L^8(\Omega;L^4(0,T;\mathbb{L}^4))}
\end{aligned}$$

$$< \infty,$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|\int_0^t \langle \Pi_n S(\boldsymbol{u}_n(s)), \boldsymbol{\chi}\rangle_{\mathbb{L}^2} \, \mathrm{d}s\right|^2\right] \le \|S(\boldsymbol{u}_n(s))\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} \|\boldsymbol{\chi}\|^2_{L^4(\Omega;L^2(0,T;\mathbb{L}^2))} < \infty.$$

The results then follow from Vitali's convergence theorem. □

Lemma 4.8. Let $\beta > 0$. For every $t \in [0,T]$, the sequence of random variables $\boldsymbol{M}_n(t)$ defined in (4.9) converges weakly in $L^2(\Omega;\mathbb{X}^{-\beta})$ to a limit $\boldsymbol{M}(t)$ given by

$$\begin{aligned}\boldsymbol{M}(t) = \boldsymbol{u}(t) - \boldsymbol{u}(0) - \lambda_r \int_0^t \boldsymbol{H}(s)\,\mathrm{d}s + \lambda_e \int_0^t \Delta \boldsymbol{H}(s)\,\mathrm{d}s + \gamma \int_0^t \boldsymbol{u}(s)\times \boldsymbol{H}(s)\,\mathrm{d}s \\ - \int_0^t R(\boldsymbol{u}(s))\,\mathrm{d}s - \int_0^t L(\boldsymbol{u}(s))\,\mathrm{d}s,\end{aligned}$$

where $\boldsymbol{H}(s) = \Delta\boldsymbol{u}(s) + \boldsymbol{u}(s) - |\boldsymbol{u}(s)|^2\boldsymbol{u}(s)$.

Proof. Let $t\in[0,T]$ and $\boldsymbol{\phi} \in L^2(\Omega;\mathbb{X}^{\beta})$. Since $\boldsymbol{u}_n$ converges to $\boldsymbol{u}$ in $C([0,T];\mathbb{X}^{-\beta})$, $\mathbb{P}$-a.s., we infer that

$$\lim_{n\to\infty} {}_{\mathbb{X}^{-\beta}}\langle \boldsymbol{u}_n(t), \boldsymbol{\phi}\rangle_{\mathbb{X}^\beta} = {}_{\mathbb{X}^{-\beta}}\langle \boldsymbol{u}(t), \boldsymbol{\phi}\rangle_{\mathbb{X}^\beta}, \quad \mathbb{P}\text{-a.s.}.$$

Furthermore, by the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{X}^{-\beta}$ and (4.5), we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|{}_{\mathbb{X}^{-\beta}}\langle \boldsymbol{u}_n(t), \boldsymbol{\phi}\rangle_{\mathbb{X}^\beta}\right|^2\right] \le \sup_{n\in\mathbb{N}} \left(\mathbb{E}\left[\|\boldsymbol{u}_n(t)\|^4_{\mathbb{X}^{-\beta}}\right]\right)^{\frac12} \left(\mathbb{E}\left[\|\boldsymbol{\phi}\|^4_{\mathbb{X}^\beta}\right]\right)^{\frac12} < \infty.$$

Therefore, by Vitali's convergence theorem,

$$\lim_{n\to\infty} \mathbb{E}\big[{}_{\mathbb{X}^{-\beta}}\langle \boldsymbol{u}_n(t), \boldsymbol{\phi}\rangle_{\mathbb{X}^\beta}\big] = \mathbb{E}\big[{}_{\mathbb{X}^{-\beta}}\langle \boldsymbol{u}(t), \boldsymbol{\phi}\rangle_{\mathbb{X}^\beta}\big].$$

This, together with Lemma 4.7, imply the required result. □

Lemma 4.9. Let $\beta > 0$, and let $\boldsymbol{u}_n$ and $\boldsymbol{u}$ be processes defined in Proposition 4.3. Then for every $t\in[0,T]$,

$$\lim_{n\to\infty} \left\|\sum_{k=1}^n \left(\int_0^t G_k(\boldsymbol{u}_n(s))\,\mathrm{d}W_{k,n}(s) - \int_0^t G_k(\boldsymbol{u}(s))\,\mathrm{d}W_k(s)\right)\right\|_{\mathbb{X}^{-\beta}} = 0.$$

Proof. The proof of this lemma is similar to that of [7, Lemma 5.2] and is omitted. □

We can now prove the first main theorem of the paper (Theorem 2.4).

Proof of Theorem 2.4. By equation (4.9), Lemma 4.8, and Lemma 4.9, we deduce that for every $t\in[0,T]$,

$$\boldsymbol{M}(t) = \sum_{k=1}^\infty \int_0^t G_k(\boldsymbol{u})\,\mathrm{d}W_k(s) \quad \text{in } L^2(\Omega;\mathbb{X}^{-\beta}).$$

This implies that $\mathbb{P}$-a.s., $\boldsymbol{u}\in C([0,T];\mathbb{H}^2_{\mathrm{w}})\cap L^2(0,T;\mathbb{H}^3)$, and $(\boldsymbol{u},\boldsymbol{W})$ satisfies equation (2.10). The proof that $\boldsymbol{u}\in C([0,T];\mathbb{H}^2)$ and (2.11) is deferred to Lemma 4.10 below. □

The proof of Theorem 2.4 will be complete once we show the following parabolic regularisation estimate. Let e^{-tA} denote the analytic semigroup generated by A, where A was defined in (2.1). Given $\boldsymbol{u_0}\in \mathrm{D}(A^{1/2})$, we can write the solution (see [12]) as:

$$\begin{aligned}\boldsymbol{u}(t) &= e^{-tA}\boldsymbol{u_0} + (\lambda_r+\beta)\int_0^t e^{-(t-s)A}\boldsymbol{u}(s)\,\mathrm{d}s - \lambda_r \int_0^t e^{-(t-s)A}\big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\,\mathrm{d}s \\ &\quad + \lambda_e \int_0^t e^{-(t-s)A}\Delta\big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\,\mathrm{d}s - \gamma\int_0^t e^{-(t-s)A}\big(\boldsymbol{u}(s)\times\Delta\boldsymbol{u}(s)\big)\,\mathrm{d}s + \int_0^t e^{-(t-s)A} S\big(\boldsymbol{u}(s)\big)\,\mathrm{d}s \\ &\quad + \sum_{k=1}^\infty \int_0^t e^{-(t-s)A} G_k(\boldsymbol{u}(s))\,\mathrm{d}W_k(s) \\ &=: I_0(t) + I_1(t) + \ldots + I_6(t). \end{aligned} \tag{4.12}$$

Lemma 4.10. Let $\boldsymbol{u_0} \in \mathrm{D}(A^{1/2})$, $\beta \in [0,1)$, and $\delta \in (0, 1-\beta)$. Then $\mathbb{P}$-a.s., we have $\boldsymbol{v} \in C^{\delta}([0,T]; \mathrm{D}(A^{\beta}))$. In particular, $\boldsymbol{u} \in C([0,T];\mathbb{H}^2)$.

Proof. Let $p > 2$ and $\delta > 0$ be numbers to be specified later. We aim to estimate $\mathbb{E}\big[\, \|\boldsymbol{v}(t+h) - \boldsymbol{v}(t)\|_{\mathrm{D}(A^\beta)}^p \,\big]$ for any $h > 0$. To this end, it suffices to bound

$$\mathbb{E}\left[\left\|A^\beta I_j(t+h) - A^\beta I_j(t)\right\|_{\mathbb{L}^2}^p\right]$$

for $j = 1, 2, \dots, 6$ in the following, where I_j are defined in (4.12). For the first term, noting Theorem 2.4, $\mathbb{P}$-a.s. we have

$$\begin{aligned}
\mathbb{E}\left[\left\|A^\beta I_1(t+h) - A^\beta I_1(t)\right\|_{\mathbb{L}^2}^p\right] &\le \mathbb{E}\left[\left\|\int_0^t A^{\beta-\frac{1}{2}+\delta} e^{-(t-s)A} A^{-\delta}(e^{-hA} - I) A^{\frac{1}{2}} \boldsymbol{u}(s)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\quad + \mathbb{E}\left[\left\|\int_t^{t+h} A^{\beta-\frac{1}{2}} e^{-(t-s)A} e^{-hA} A^{\frac{1}{2}} \boldsymbol{u}(s)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\le Ch^{\delta p}\, \mathbb{E}\left[\left(\int_0^t (t-s)^{-(\beta-\frac{1}{2})-\delta} \left\|A^{\frac{1}{2}}\boldsymbol{u}(s)\right\|_{\mathbb{L}^2} \mathrm{d}s\right)^p\right] \\
&\quad + C\,\mathbb{E}\left[\left(\int_t^{t+h} (t+h-s)^{-(\beta-\frac{1}{2})} \left\|A^{\frac{1}{2}}\boldsymbol{u}(s)\right\|_{\mathbb{L}^2} \mathrm{d}s\right)^p\right] \\
&\le C\,\mathbb{E}\left[\sup_{\tau\in[0,T]} \|\boldsymbol{u}(\tau)\|_{\mathbb{H}^2}^p\right] \left(h^{\delta p} t^{p(\frac{3}{2}-\beta-\delta)} + h^{p(\frac{3}{2}-\beta)}\right).
\end{aligned} \tag{4.13}$$

Similarly for the second term, using (2.3) and (2.5), $\mathbb{P}$-a.s. we have

$$\begin{aligned}
\mathbb{E}\left[\left\|A^\beta I_2(t+h) - A^\beta I_2(t)\right\|_{\mathbb{L}^2}^p\right] &\le \mathbb{E}\left[\left\|\int_0^t A^{\beta-\frac{1}{2}+\delta} e^{-(t-s)A} A^{-\delta}(e^{-hA} - I) A^{\frac{1}{2}} \big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\quad + \mathbb{E}\left[\left\|\int_t^{t+h} A^{\beta-\frac{1}{2}} e^{-(t-s)A} e^{-hA} A^{\frac{1}{2}} \big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\le Ch^{\delta p}\, \mathbb{E}\left[\left(\int_0^t (t-s)^{-(\beta-\frac{1}{2})-\delta} \left\||\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\right\|_{\mathbb{H}^2} \mathrm{d}s\right)^p\right] \\
&\quad + C\,\mathbb{E}\left[\left(\int_t^{t+h} (t+h-s)^{-(\beta-\frac{1}{2})} \left\||\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\right\|_{\mathbb{H}^2} \mathrm{d}s\right)^p\right] \\
&\le C\,\mathbb{E}\left[\sup_{\tau\in[0,T]} \|\boldsymbol{u}(\tau)\|_{\mathbb{H}^2}^{3p}\right] \left(h^{\delta p} t^{p(\frac{3}{2}-\beta-\delta)} + h^{p(\frac{3}{2}-\beta)}\right).
\end{aligned} \tag{4.14}$$

For the third term, using (2.3), (2.5) and interpolation inequalities, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|A^\beta I_3(t+h) - A^\beta I_3(t)\right\|_{\mathbb{L}^2}^p\right] \\
&\le \mathbb{E}\left[\left\|\int_0^t A^{\beta-\frac{1}{2}+\delta} e^{-(t-s)A} A^{-\delta}(e^{-hA} - I) A^{\frac{1}{2}} \Delta\big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\quad + \mathbb{E}\left[\left\|\int_t^{t+h} A^{\beta-\frac{1}{2}} e^{-(t-s)A} e^{-hA} A^{\frac{1}{2}} \Delta\big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\le Ch^{\delta p}\, \mathbb{E}\left[\left(\int_0^t (t-s)^{-(\beta-\frac{1}{2})-\delta} \left\|\Delta\big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\right\|_{\mathbb{H}^2} \mathrm{d}s\right)^p\right] \\
&\quad + C\,\mathbb{E}\left[\left(\int_t^{t+h} (t+h-s)^{-(\beta-\frac{1}{2})} \left\|\Delta\big(|\boldsymbol{u}(s)|^2\boldsymbol{u}(s)\big)\right\|_{\mathbb{H}^2} \mathrm{d}s\right)^p\right]
\end{aligned}$$

$$
\begin{aligned}
&\le Ch^{\delta p}\,\mathbb{E}\left[\left(\int_0^t (t-s)^{-(\beta-\frac{1}{2})-\delta}\,\|\boldsymbol{u}(s)\|_{\mathbb{H}^2}^2\,\|\boldsymbol{u}(s)\|_{\mathbb{H}^4}\,\mathrm{d}s\right)^p\right] \\
&\quad + C\,\mathbb{E}\left[\left(\int_t^{t+h} (t+h-s)^{-(\beta-\frac{1}{2})}\,\|\boldsymbol{u}(s)\|_{\mathbb{H}^2}^2\,\|\boldsymbol{u}(s)\|_{\mathbb{H}^4}\,\mathrm{d}s\right)^p\right] \\
&\le Ch^{\delta p}\,\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\|\boldsymbol{u}(\tau)\|_{\mathbb{H}^2}\right)^{2p}\left(\int_0^t (t-s)^{-(2\beta-1)-2\delta}\mathrm{d}s\right)^{\frac{p}{2}}\left(\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{H}^4}^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\quad + C\,\mathbb{E}\left[\left(\sup_{\tau\in[0,T]}\|\boldsymbol{u}(\tau)\|_{\mathbb{H}^2}\right)^{2p}\left(\int_t^{t+h} (t+h-s)^{-(2\beta-1)}\mathrm{d}s\right)^{\frac{p}{2}}\left(\int_t^{t+h} \|\boldsymbol{u}(s)\|_{\mathbb{H}^4}^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\le C\left(\mathbb{E}\left[\sup_{\tau\in[0,T]}\|\boldsymbol{u}(\tau)\|_{\mathbb{H}^2}^{4p}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T \|\boldsymbol{u}(s)\|_{\mathbb{H}^4}^2\,\mathrm{d}s\right]\right)^{\frac{1}{2}}\left(h^{\delta p}t^{p(1-\beta-\delta)}+h^{p(1-\beta)}\right). \qquad (4.15)
\end{aligned}
$$

The term in I_4 can be estimated in a similar manner, giving

$$
\mathbb{E}\left[\left\|A^\beta I_4(t+h)-A^\beta I_4(t)\right\|_{\mathbb{L}^2}^p\right] \le C\left(h^{\delta p}t^{p(1-\beta-\delta)}+h^{p(1-\beta)}\right). \qquad (4.16)
$$

Next, for the term I_5, recall that $S(\boldsymbol{u}) = R(\boldsymbol{u}) + L(\boldsymbol{u})$ as defined in (1.5) and (1.2). Noting assumptions on L in Section 2.2, we have

$$
\begin{aligned}
\mathbb{E}\left[\left\|A^\beta I_5(t+h)-A^\beta I_5(t)\right\|_{\mathbb{L}^2}^p\right] &\le \mathbb{E}\left[\left\|\int_0^t A^{\beta-\frac{1}{4}+\delta}e^{-(t-s)A}A^{-\delta}(e^{-hA}-I)A^{\frac{1}{4}}S\boldsymbol{u}(s)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\quad + \mathbb{E}\left[\left\|\int_t^{t+h} A^{\beta-\frac{1}{4}+\delta}e^{-(t-s)A}e^{-hA}A^{\frac{1}{4}}S\boldsymbol{u}(s)\,\mathrm{d}s\right\|_{\mathbb{L}^2}^p\right] \\
&\le Ch^{\delta p}\,\mathbb{E}\left[\left(\int_0^t (t-s)^{-(\beta-\frac{1}{4})-\delta}\left\|A^{\frac{1}{4}}S(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}\mathrm{d}s\right)^p\right] \\
&\quad + C\,\mathbb{E}\left[\left(\int_t^{t+h} (t+h-s)^{-(\beta-\frac{1}{4})}\left\|A^{\frac{1}{4}}S(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}\mathrm{d}s\right)^p\right] \\
&\le C\,\mathbb{E}\left[\sup_{\tau\in[0,T]}\|\boldsymbol{u}(\tau)\|_{\mathbb{H}^1}^p\right]\left(h^{\delta p}t^{p(\frac{5}{4}-\beta-\delta)}+h^{p(\frac{5}{4}-\beta)}\right). \qquad (4.17)
\end{aligned}
$$

Finally, we need to consider I_6. To this end, we first note that the process

$$
W(t) = \sum_{k=1}^\infty \boldsymbol{g}_k W_k(t)
$$

is by assumption a Wiener process taking values in $\mathbb{H}^2$, hence the process $B(t) = \Delta W(t)$ is a Wiener process taking values in $\mathbb{L}^2$. Also, since $\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|_{\mathbb{H}^2} < \infty$, we find that the process

$$
\Delta\left[\sum_{k=1}^\infty \big(\boldsymbol{h}_k \times \boldsymbol{u}(t)\big)W_k(t)\right]
$$

is well defined in $\mathbb{L}^2$, and moreover

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\sum_{k=1}^\infty \|G_k(\boldsymbol{u}(t))\|_{\mathbb{H}^2}^2\right] \le \sum_{k=1}^\infty \|\boldsymbol{g}_k\|_{\mathbb{H}^2}^2 + \mathbb{E}\left[\sup_{t\in[0,T]}\sum_{k=1}^\infty \|\boldsymbol{h}_k\times\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2\right] < \infty.
$$

We then have

$$
\mathbb{E}\left[\left\|A^\beta I_6(t+h)-A^\beta I_6(t)\right\|_{\mathbb{L}^2}^p\right] \le 2^{p-1}\mathbb{E}\left[\left\|\sum_{k=1}^\infty\int_0^t A^{\beta-\frac{1}{2}}e^{-(t-s)A}(e^{-hA}-I)A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\,\mathrm{d}W_k(s)\right\|_{\mathbb{L}^2}^p\right]
$$

$$
\begin{aligned}
&+ 2^{p-1}\mathbb{E}\left[\left\|\sum_{k=1}^{\infty}\int_t^{t+h} A^{\beta-\frac{1}{2}}e^{-(t-s)A}e^{-hA}A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\,\mathrm{d}W_k(s)\right\|_{\mathbb{L}^2}^p\right]\\
&= 2^{p-1}(J_1+J_2).
\end{aligned}
$$

Invoking the Burkholder–Davis–Gundy inequality, we obtain for J_1:

$$
\begin{aligned}
J_1 &\le C\,\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t \left\|A^{\beta-\frac{1}{2}}e^{-(t-s)A}(e^{-hA}-I)A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{p/2}\right]\\
&= C\,\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t \left\|A^{\beta-\frac{1}{2}+\delta}e^{-(t-s)A}A^{-\delta}(e^{-hA}-I)A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{p/2}\right]\\
&\le Ch^{\delta p}\left(\int_0^t \frac{\mathrm{d}s}{(t-s)^{2\beta-1+2\delta}}\right)^{p/2}\mathbb{E}\left[\left(\sup_{\tau\in[0,t]}\sum_{k=1}^{\infty}\left\|A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}^2\right)^{p/2}\right]\\
&\le Ch^{\delta p}t^{p(1-\beta-\delta)}. \qquad (4.18)
\end{aligned}
$$

Similar arguments for J_2 give

$$
\begin{aligned}
J_2 &\le C\,\mathbb{E}\left[\sum_{k=1}^{\infty}\int_t^{t+h}\left\|A^{\beta-\frac{1}{2}}e^{-(t-s)A}e^{-hA}A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}^2\mathrm{d}s\right]^{p/2}\\
&\le C\,\mathbb{E}\left[\sum_{k=1}^{\infty}\int_t^{t+h}\frac{1}{(t+h-s)^{2\beta-1}}\left\|A^{\frac{1}{2}}G_k(\boldsymbol{u}(s))\right\|_{\mathbb{L}^2}^2\mathrm{d}s\right]^{p/2}\\
&\le \left(\int_0^h \frac{\mathrm{d}s}{s^{2\beta-1}}\right)^{p/2}\mathbb{E}\left[\left(\sup_{\tau\in[0,T]}\sum_{k=1}^{\infty}\left\|A^{1/2}G_k(u(s))\right\|_{\mathbb{L}^2}^2\right)^{p/2}\right]\\
&\le Ch^{p(1-\beta)}. \qquad (4.19)
\end{aligned}
$$

Combining (4.18) and (4.19), we obtain

$$
\mathbb{E}\left[\left\|A^\beta I_6(t+h)-A^\beta I_6(t)\right\|_{\mathbb{L}^2}^p\right] \le C\left(h^{\delta p}t^{p(1-\beta-\delta)}+h^{p(1-\beta)}\right). \qquad (4.20)
$$

Altogether, estimates (4.13), (4.14), (4.15), (4.16), (4.17), and (4.20) imply that for any $t>0$,

$$
\mathbb{E}\left[\left\|A^\beta \boldsymbol{v}(t+h)-A^\beta \boldsymbol{v}(t)\right\|_{\mathbb{L}^2}^p\right] \le Ch^{\delta p}t^{p(1-\beta-\delta)}+h^{p(1-\beta)}. \qquad (4.21)
$$

Since p can be taken arbitrarily large, we find that for every $\delta\in(0,1-\beta)$, where $\beta\in(0,1]$, $\mathbb{P}$-a.s. we have $\boldsymbol{v}\in C^\delta([0,T];\mathrm{D}(A^\beta))$ by the Kolmogorov continuity theorem.

Finally, since the map $(t,\boldsymbol{u_0})\mapsto e^{-tA}\boldsymbol{u_0}$ is continuous in $[0,\infty)\times\mathrm{D}(A^{1/2})$, noting (4.21) for $\beta=\frac{1}{2}$, we have $\boldsymbol{u}\in C([0,T];\mathbb{H}^2)$. This completes the proof of the lemma. □

5. Proof of Theorem 2.5: Pathwise Uniqueness

We prove Theorem 2.5 in this section.

Proof of Theorem 2.5. Let $\boldsymbol{v}:=\boldsymbol{u}_1-\boldsymbol{u}_2$ and $\boldsymbol{B}=\boldsymbol{H}_1-\boldsymbol{H}_2$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are martingale solutions of (1.1) (on the same probability space and with the same Wiener process). Then $\boldsymbol{v}$ satisfies

$$
\begin{aligned}
\mathrm{d}\boldsymbol{v} &= \big(\lambda_r\boldsymbol{B}-\lambda_e\Delta\boldsymbol{B}-\gamma(\boldsymbol{v}\times\boldsymbol{H}_1+\boldsymbol{u}_2\times\boldsymbol{B})+R(\boldsymbol{u}_1)-R(\boldsymbol{u}_2)+L(\boldsymbol{u}_1)-L(\boldsymbol{u}_2)\big)\,\mathrm{d}t\\
&\quad+\left(\sum_{k=1}^{\infty}G_k(\boldsymbol{u}_1)-G_k(\boldsymbol{u}_2)\right)\mathrm{d}W_k(t)
\end{aligned}
$$

with $\boldsymbol{v}(0)=\mathbf{0}$. Moreover, for any $R>0$, define the stopping time

$$\tau_R := \inf\{t\geq 0 : \|\boldsymbol{u}_1(t)\|_{\mathbb{H}^2} \vee \|\boldsymbol{u}_2(t)\|_{\mathbb{H}^2} > R\}. \tag{5.1}$$

Note that $\tau_R\to\infty$ as $R\to\infty$. Now, by Itô's lemma,

$$\begin{aligned}\frac{1}{2}\mathrm{d}\,\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 &= \Big(\lambda_r \langle \boldsymbol{B},\boldsymbol{v}\rangle_{\mathbb{L}^2} + \lambda_e \langle \nabla\boldsymbol{B},\nabla\boldsymbol{v}\rangle_{\mathbb{L}^2} - \gamma\langle \boldsymbol{u}_2\times\boldsymbol{B},\boldsymbol{v}\rangle_{\mathbb{L}^2} \\ &\quad + \langle R(\boldsymbol{u}_1)-R(\boldsymbol{u}_2),\boldsymbol{v}\rangle_{\mathbb{L}^2} + \langle L(\boldsymbol{u}_1)-L(\boldsymbol{u}_2),\boldsymbol{v}\rangle_{\mathbb{L}^2} + \frac{1}{2}\sum_{k=1}^\infty \|G_k(\boldsymbol{u}_1)-G_k(\boldsymbol{u}_2)\|_{\mathbb{L}^2}^2\Big)\,\mathrm{d}t. \end{aligned}\tag{5.2}$$

We have

$$\lambda_r\langle \boldsymbol{B},\boldsymbol{v}\rangle_{\mathbb{L}^2} = -\lambda_r\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \lambda_r\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \lambda_r \||\boldsymbol{u}_1|\,|\boldsymbol{v}|\|_{\mathbb{L}^2}^2 - \lambda_r\|\boldsymbol{u}_2\cdot\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \lambda_r\langle (\boldsymbol{u}_1\cdot\boldsymbol{v})\boldsymbol{u}_2,\boldsymbol{v}\rangle_{\mathbb{L}^2}, \tag{5.3}$$

$$\lambda_e\langle \nabla\boldsymbol{B},\nabla\boldsymbol{v}\rangle_{\mathbb{L}^2} = -\lambda_e\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \lambda_e\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \lambda_e\big\langle |\boldsymbol{u}_1|^2\boldsymbol{v},\Delta\boldsymbol{v}\big\rangle_{\mathbb{L}^2} + \lambda_e\big\langle \big((\boldsymbol{u}_1+\boldsymbol{u}_2)\cdot\boldsymbol{v}\big)\boldsymbol{u}_2,\Delta\boldsymbol{v}\big\rangle_{\mathbb{L}^2}, \tag{5.4}$$

and

$$-\gamma\langle \boldsymbol{u}_2\times\boldsymbol{B},\boldsymbol{v}\rangle_{\mathbb{L}^2} = -\gamma\langle \boldsymbol{u}_2\times\Delta\boldsymbol{v},\boldsymbol{v}\rangle_{\mathbb{L}^2}. \tag{5.5}$$

Substituting (5.3), (5.4), and (5.5) into (5.2), and integrating with respect to t yield

$$\begin{aligned}&\frac{1}{2}\|\boldsymbol{v}(t)\|_{\mathbb{L}^2}^2 + \lambda_e\int_0^t \|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + (\lambda_r-\lambda_e)\int_0^t \|\nabla\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\ &\leq \lambda_r\int_0^t \|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_r\int_0^t \langle (\boldsymbol{u}_1(s)\cdot\boldsymbol{v}(s))\boldsymbol{u}_2(s),\boldsymbol{v}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s + \lambda_e\int_0^t \big\langle |\boldsymbol{u}_1(s)|^2\boldsymbol{v}(s),\Delta\boldsymbol{v}(s)\big\rangle_{\mathbb{L}^2}\,\mathrm{d}s \\ &\quad + \lambda_e\int_0^t \big\langle \big((\boldsymbol{u}_1(s)+\boldsymbol{u}_2(s))\cdot\boldsymbol{v}(s)\big)\boldsymbol{u}_2(s),\Delta\boldsymbol{v}(s)\big\rangle_{\mathbb{L}^2}\,\mathrm{d}s - \gamma\int_0^t \langle \boldsymbol{u}_2(s)\times\Delta\boldsymbol{v}(s),\boldsymbol{v}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}s \\ &\quad + \int_0^t \langle R(\boldsymbol{u}_1)-R(\boldsymbol{u}_2),\boldsymbol{v}\rangle_{\mathbb{L}^2}\,\mathrm{d}s + \int_0^t \langle L(\boldsymbol{u}_1)-L(\boldsymbol{u}_2),\boldsymbol{v}\rangle_{\mathbb{L}^2}\,\mathrm{d}s + \frac{1}{2}\sum_{k=1}^\infty\int_0^t \|G_k(\boldsymbol{u}_1)-G_k(\boldsymbol{u}_2)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\ &=: I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t)+I_7(t)+I_8(t). \end{aligned}\tag{5.6}$$

With τ_R as defined in (5.1), we will derive bounds for $I_j(t\wedge\tau_R)$, for $j=2,\dots,8$, as follows. For the term $I_2(t\wedge\tau_R)$, by Hölder's inequality and Sobolev embedding,

$$I_2(t\wedge\tau_R) \leq \lambda_r\int_0^{t\wedge\tau_R} \|\boldsymbol{u}_1(s)\|_{\mathbb{L}^\infty}\|\boldsymbol{u}_2(s)\|_{\mathbb{L}^\infty}\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \leq C\lambda_r R^2\int_0^{t\wedge\tau_R}\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s.$$

For the term $I_3(t\wedge\tau_R)$, by Hölder's and Young's inequalities, and Sobolev embedding, we have

$$\begin{aligned}I_3(t\wedge\tau_R) &\leq \lambda_e\int_0^{t\wedge\tau_R} \|\boldsymbol{u}_1(s)\|_{\mathbb{L}^\infty}^2\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}\|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}\,\mathrm{d}s \\ &\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R}\|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{1}{2\lambda_e}\int_0^{t\wedge\tau_R}\|\boldsymbol{u}_1(s)\|_{\mathbb{L}^\infty}^4\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\ &\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R}\|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{CR^4}{\lambda_e}\int_0^{t\wedge\tau_R}\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \end{aligned}$$

Similarly,

$$\begin{aligned}I_4(t\wedge\tau_R) &\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R}\|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{1}{2\lambda_e}\int_0^{t\wedge\tau_R}\|\boldsymbol{u}_1(s)+\boldsymbol{u}_2(s)\|_{\mathbb{L}^\infty}^2\|\boldsymbol{u}_2(s)\|_{\mathbb{L}^\infty}^2\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\ &\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R}\|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{CR^4}{\lambda_e}\int_0^{t\wedge\tau_R}\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \end{aligned}$$

and

$$I_5(t\wedge\tau_R) \leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R}\|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{\gamma^2}{2\lambda_e}\int_0^{t\wedge\tau_R}\|\boldsymbol{u}_2(s)\|_{\mathbb{L}^\infty}^2\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s$$

$$\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R} \|\Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{C\gamma^2 R^2}{\lambda_e}\int_0^{t\wedge\tau_R} \|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s.$$

Next, for the term $I_6(t\wedge\tau_R)$, by (2.9),

$$\begin{aligned} I_6(t\wedge\tau_R) &\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R} \|\Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C\int_0^{t\wedge\tau_R} \|\boldsymbol{\nu}\|_{L^\infty(0,T;\mathbb{L}^\infty)}^2 \left(1+\|\boldsymbol{u}_2(s)\|_{\mathbb{L}^\infty}^2\right)\|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\ &\leq \frac{\lambda_e}{8}\int_0^{t\wedge\tau_R} \|\Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C(1+R^2)\int_0^{t\wedge\tau_R} \|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \end{aligned}$$

For the term $I_7(t\wedge\tau_R)$, by the assumptions on L,

$$I_7(t\wedge\tau_R) \leq C\int_0^{t\wedge\tau_R} \|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s.$$

Finally, for the term $I_8(t\wedge\tau_R)$, we have

$$I_8(t\wedge\tau_R) \leq \sigma_h\gamma^2\int_0^{t\wedge\tau_R} \|\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s.$$

For brevity, we will assume $\lambda_e - \lambda_r > 0$ (otherwise interpolation inequality (2.4) can be used). Therefore, replacing t by $t\wedge\tau_R$ in (5.6), rearranging the terms, and taking expectations, we obtain

$$\mathbb{E}\left[\sup_{\tau\in[0,t\wedge\tau_R]} \|\boldsymbol{v}(\tau)\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_R} \|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right] \leq C(1+R^4)\int_0^t \mathbb{E}\left[\sup_{\tau\in[0,s\wedge\tau_R]} \|\boldsymbol{v}(\tau)\|_{\mathbb{L}^2}^2\right]\mathrm{d}s. \tag{5.7}$$

By Gronwall's inequality, we infer that for any $t\geq 0$,

$$\mathbb{E}\left[\sup_{\tau\in[0,t\wedge\tau_R]} \|\boldsymbol{v}(\tau)\|_{\mathbb{L}^2}^2\right] = 0.$$

Letting $R\uparrow\infty$, and applying the monotone convergence theorem, we obtain $\boldsymbol{v}(t)\equiv \mathbf{0}$ for $\mathbb{P}$-a.s. $\omega\in\Omega$, as required. The existence of a pathwise unique strong solution (in the sense of probability) and the uniqueness in law of the martingale solution are consequences of [24, Theorem 2.2 and Theorem 12.1]. ☐

6. Proof of Theorem 2.6: Existence of Invariant Measures

We will show the existence of invariant measures for the stochastic LLBar equation (1.3). Recall that by Theorem 2.4 and Theorem 2.5, for any initial data $\boldsymbol{u_0}\in\mathbb{H}^2$, the stochastic LLBar equation has a unique pathwise strong solution which defines an $\mathbb{H}^2$-valued Markov process $\boldsymbol{u}$. Let $\boldsymbol{u}(t;\boldsymbol{u_0})$ denote the process $\boldsymbol{u}$ starting at $\boldsymbol{u}(0)=\boldsymbol{u_0}$. For a Borel set B and any $t\geq 0$, we define the transition functions

$$P_t(\boldsymbol{u_0}, B) := \mathbb{P}(\boldsymbol{u}(t;\boldsymbol{u_0})\in B).$$

We can study the Markov semigroup $\{P_t\}_{t\geq 0}$, defined for $\phi\in B_b(\mathbb{H}^2)$ by

$$P_t\phi(\boldsymbol{u_0}) := \mathbb{E}\left[\phi(\boldsymbol{u}(t;\boldsymbol{u_0}))\right] = \int_{\mathbb{H}^2}\phi(\boldsymbol{u})P_t(\boldsymbol{u_0},\mathrm{d}\boldsymbol{u}),\quad \forall \boldsymbol{u_0}\in\mathbb{H}^2. \tag{6.1}$$

A Borel probability measure μ on $\mathbb{H}^2$ is called an invariant measure for the Markov semigroup associated to the problem (1.3) if for every $t\geq 0$,

$$\int_{\mathbb{H}^2}\phi(\boldsymbol{u_0})\,\mathrm{d}\mu(\boldsymbol{u_0}) = \int_{\mathbb{H}^2} P_t\phi(\boldsymbol{u_0})\,\mathrm{d}\mu(\boldsymbol{u_0}).$$

First, we prove the following bounds which will be used later.

Lemma 6.1. Let $\boldsymbol{u}$ be the solution of (1.3) given by Theorem 2.4. Then there exists a positive constant C such that for all $t\geq 0$,

$$\mathbb{E}\left[\|\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2\right] + \int_0^t \mathbb{E}\left[\|\boldsymbol{u}(s)\|_{\mathbb{H}^2}^2\right]\mathrm{d}s + \int_0^t \mathbb{E}\left[\|\boldsymbol{H}(s)\|_{\mathbb{H}^1}^2\right]\mathrm{d}s + \int_0^t \mathbb{E}\big[\,\|\nabla\Delta\boldsymbol{u}(s)\|_{\mathbb{L}^2}\,\big]\mathrm{d}s \leq C(1+t),$$

where C depends on $\|\boldsymbol{u_0}\|_{\mathbb{H}^1}$, but is independent of t.

Proof. The proof follows the same line of argument as in Proposition 3.3 and Proposition 3.4, with $\boldsymbol{u}_n$ and $\boldsymbol{H}_n$ replaced by $\boldsymbol{u}$ and $\boldsymbol{H}$, respectively. As such, we will just outline the key steps here, while ensuring that the constant C is independent of t.

Firstly, following the argument leading to (3.11), and assuming $\lambda_r - \lambda_e > 0$ for simplicity of presentation, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + (\lambda_r - \lambda_e)\int_0^t \|\nabla\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_e\int_0^t \|\Delta\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_r\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s \\
&\quad + \lambda_e\int_0^t \||\boldsymbol{u}(s)|\,|\nabla\boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2\,\mathrm{d}s + 2\lambda_e\int_0^t \|\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\le \frac{1}{2}\|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 + \lambda_r\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C\int_0^t \|\boldsymbol{\nu}\|_{\mathbb{L}^2(\mathscr{D};\mathbb{R}^d)}\||\boldsymbol{u}(s)|\,|\nabla\boldsymbol{u}(s)|\|_{\mathbb{L}^2}\,\mathrm{d}s + C\int_0^t \big(1+\|\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\big)\mathrm{d}s \\
&\quad + \frac{1}{2}\sum_{k=1}^n\int_0^t \big(\|\boldsymbol{g}_k\|_{\mathbb{L}^2}^2 + \|\boldsymbol{h}_k\|_{\mathbb{L}^\infty}^2\|\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\big)\mathrm{d}s + \sum_{k=1}^\infty\int_0^t \langle\boldsymbol{g}_k,\boldsymbol{u}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s) \\
&\le \frac{1}{2}\|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 + Ct + C\|\boldsymbol{\nu}\|_{L^2(0,t;\mathbb{L}^2)}^2 + \frac{\lambda_e}{2}\int_0^t \||\boldsymbol{u}(s)|\,|\nabla\boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\quad + \sum_{k=1}^\infty\int_0^t \langle\boldsymbol{g}_k,\boldsymbol{u}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s)
\end{aligned}\tag{6.2}$$

where in the last step we used Young's inequality and the assumptions in 2.2. Therefore, noting the inequality $x^2 \le \epsilon x^4 + 1/(4\epsilon)$ for any $\epsilon > 0$, we can absorb the penultimate term on (6.2) to the left. Furthermore, since

$$\mathbb{E}\left[\sum_{k=1}^\infty\int_0^t \left|\langle\boldsymbol{g}_k,\boldsymbol{u}(s)\rangle_{\mathbb{L}^2}\right|^2\mathrm{d}s\right] \le \left(\sum_{k=1}^\infty\|\boldsymbol{g}_k\|_{\mathbb{L}^2}^2\right)\mathbb{E}\left[\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right] < \infty,$$

the process $t \mapsto \sum_{k=1}^\infty \int_0^t \langle\boldsymbol{g}_k,\boldsymbol{u}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s)$ is a martingale on $[0,T]$. Therefore,

$$\mathbb{E}\left[\int_0^t \langle\boldsymbol{g}_k,\boldsymbol{u}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s)\right] = 0.\tag{6.3}$$

Taking expectation on both sides of (6.2), rearranging the terms, and noting (6.3), we then have

$$\mathbb{E}\left[\|\boldsymbol{u}(t)\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{H}^2}^2\,\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \||\boldsymbol{u}(s)|\,|\nabla\boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right] \le C(1+t).\tag{6.4}$$

Next, we follow the argument leading to (3.16) to obtain

$$\begin{aligned}
&\frac{1}{2}\|\nabla\boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \frac{1}{4}\|\boldsymbol{u}(t)\|_{\mathbb{L}^4}^4 + \lambda_r\int_0^t \|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \lambda_e\int_0^t \|\nabla\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \frac{1}{2}\sum_{k=1}^\infty\int_0^t \|G_k(\boldsymbol{u}(s))\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\le C\|\boldsymbol{u}_{\mathbf{0}}\|_{\mathbb{H}^1}^2 + C(1+t) + \frac{\lambda_r}{2}\int_0^t \|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + C\int_0^t \||\boldsymbol{u}(s)|\,|\nabla\boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2\,\mathrm{d}s \\
&\quad + C\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{H}^1}^2\,\mathrm{d}s + C\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s - \sum_{k=1}^\infty\int_0^t \langle\boldsymbol{g}_k + \gamma\boldsymbol{u}(s)\times\boldsymbol{h}_k,\boldsymbol{H}(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k(s).
\end{aligned}\tag{6.5}$$

Now, note that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^\infty\int_0^t \left|\langle\boldsymbol{g}_k + \gamma\boldsymbol{u}(s)\times\boldsymbol{h}_k,\boldsymbol{H}(s)\rangle_{\mathbb{L}^2}\right|^2\mathrm{d}s\right] \\
&\le \left(\sum_{k=1}^\infty\|\boldsymbol{g}_k\|_{\mathbb{L}^2}^2\right)\mathbb{E}\left[\int_0^t \|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right] + \left(\sum_{k=1}^\infty\|\boldsymbol{h}_k\|_{\mathbb{L}^2}^2\right)\mathbb{E}\left[\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^\infty}^2\,\mathrm{d}s\right] + C\mathbb{E}\left[\int_0^t \|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]
\end{aligned}$$

which is finite by our assumptions in 2.2 and Theorem 2.4. In particular, the process

$$t \mapsto \sum_{k=1}^{\infty} \int_0^t \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}(s) \times \boldsymbol{h}_k, \boldsymbol{H}(s) \rangle_{\mathbb{L}^2} \, \mathrm{d}W_k(s)$$

is a martingale on $[0, T]$. This implies

$$\mathbb{E}\left[\sum_{k=1}^{\infty} \int_0^t \langle \boldsymbol{g}_k + \gamma \boldsymbol{u}(s) \times \boldsymbol{h}_k, \boldsymbol{H}(s) \rangle_{\mathbb{L}^2} \, \mathrm{d}W_k(s) \right] = 0.$$

Therefore, taking expectation on both sides of (6.5) and noting (6.4), we obtain

$$\mathbb{E}\left[\|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 \right] + \mathbb{E}\left[\|\boldsymbol{u}(t)\|_{\mathbb{L}^4}^4 \right] + \mathbb{E}\left[\int_0^t \|\boldsymbol{H}(s)\|_{\mathbb{H}^1}^2 \, \mathrm{d}s \right] \leq C(1+t). \tag{6.6}$$

Finally, note that $\nabla \Delta \boldsymbol{u} = \nabla \boldsymbol{H} - \nabla \boldsymbol{u} + \nabla(|\boldsymbol{u}|^2 \boldsymbol{u})$. By Young's inequality and Sobolev embedding,

$$\begin{aligned}
\int_0^t \mathbb{E}\big[\|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2} \big] \mathrm{d}s &\leq \int_0^t \mathbb{E}\big[\|\boldsymbol{H}(s)\|_{\mathbb{H}^1} + \|\boldsymbol{u}(s)\|_{\mathbb{H}^1} + 2 \, \| |\boldsymbol{u}(s)| \, |\nabla \boldsymbol{u}(s)| \|_{\mathbb{L}^2} \, \|\boldsymbol{u}(s)\|_{\mathbb{L}^\infty} \big] \, \mathrm{d}s \\
&\leq C \int_0^t 1 + \mathbb{E}\left[\|\boldsymbol{H}(s)\|_{\mathbb{H}^1}^2 + \|\boldsymbol{u}(s)\|_{\mathbb{H}^1}^2 + \| |\boldsymbol{u}(s)| \, |\nabla \boldsymbol{u}(s)| \|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}(s)\|_{\mathbb{H}^2}^2 \right] \mathrm{d}s \\
&\leq C(1+t),
\end{aligned}$$

where in the last step we used (6.4) and (6.6). This completes the proof of the lemma. □

Lemma 6.2. Let $\boldsymbol{u}$ be the solution of (1.3) given by Theorem 2.4. Then there exists a positive constant C such that for all $t \geq 0$,

$$\int_0^t \mathbb{E}\left[\log\left(1 + \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \right) \right] \mathrm{d}s \leq C \|\boldsymbol{u_0}\|_{\mathbb{H}^2}^2 + C(1+t),$$

where C depends on the coefficients of the equation, but is independent of t.

Proof. Note that Proposition 3.3, Proposition 3.4, and Proposition 3.6 imply that for any $t \geq 0$,

$$\mathbb{E}\left[\|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2 \right] + \int_0^t \mathbb{E}\left[\left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \right] \mathrm{d}s \leq C \|\boldsymbol{u_0}\|_{\mathbb{H}^2}^2 + C(1+e^{Ct}),$$

where C is a constant depending only on σ_g, σ_h, and the coefficients of the equation. By Jensen's inequality, noting that the function $x \mapsto \log x$ is concave, we obtain

$$\int_0^t \mathbb{E}\left[\log\left(1 + \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \right) \right] \mathrm{d}s \leq \log\left(\int_0^t \mathbb{E}\left[1 + \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \right] \mathrm{d}s \right) \leq C \|\boldsymbol{u_0}\|_{\mathbb{H}^2}^2 + C(1+t),$$

as required. □

Lemma 6.3. Let $\boldsymbol{u}$ be the solution of (1.3) given by Theorem 2.4. Then there exists a positive constant C independent of t such that for all $t \geq 0$,

$$\mathbb{E}\left[\sup_{\tau \in [0,t]} \tau \, \|\nabla \Delta \boldsymbol{u}(\tau)\|_{\mathbb{L}^2}^2 \right] \leq C(1+e^{Ct}),$$

where C is a positive constant depending on $\|\boldsymbol{u_0}\|_{\mathbb{H}^2}$.

Proof. We will use (4.12) and estimate $\mathbb{E}\left[t \left\| A^{\frac{3}{4}} I_j(t) \right\|_{\mathbb{L}^2}^2 \right]$ for $j = 0, 1, \ldots, 6$. Firstly, we have

$$\mathbb{E}\left[t \left\| A^{\frac{3}{4}} I_0(t) \right\|_{\mathbb{L}^2}^2 \right] = \mathbb{E}\left[t \left\| A^{\frac{1}{4}} e^{-tA} A^{\frac{1}{2}} \boldsymbol{u_0} \right\|_{\mathbb{L}^2}^2 \right] \leq \sqrt{t} \left\| A^{\frac{1}{2}} \boldsymbol{u_0} \right\|_{\mathbb{L}^2}^2 \tag{6.7}$$

The rest of the estimates follows the same argument as in the proof of Lemma 4.10, noting Proposition 3.3 and Proposition 3.6. Therefore, we have

$$\mathbb{E}\left[t \left\| A^{\frac{3}{4}} I_j(t) \right\|_{\mathbb{L}^2}^2 \right] \leq C(1+e^{Ct}), \tag{6.8}$$

for $j = 1, 2, \dots, 6$, where C depends on $\|\boldsymbol{u_0}\|_{\mathbb{H}^2}$. The required estimate follows from (6.7) and (6.8). $\square$

To prove the existence of an invariant measure, we adopt the techniques in [16]. First, we prove a continuous dependence estimate and some strong moment bounds to establish the Feller property of the transition semigroup P_t defined in (6.1). The existence of an invariant measure then follows from the well-known theorem of Krylov–Bogoliubov.

In the following, we will use the stopping time defined for any $S > 0$ by

$$\sigma_S(\boldsymbol{v_0}) := \inf\{t \geq 0 : \|\boldsymbol{u}(t; \boldsymbol{v_0})\|_{\mathbb{H}^2}^2 > S\}. \tag{6.9}$$

The next lemma shows a continuous dependence result up to a stopping time.

Lemma 6.4. Let $\boldsymbol{u}^m$ and $\boldsymbol{u}$ be the solution of (1.3) with initial data $\boldsymbol{u_0^m}$ and $\boldsymbol{u_0}$ respectively. Let $\sigma_S := \sigma_S(\boldsymbol{u_0^m}) \wedge \sigma_S(\boldsymbol{u_0})$. Then for any $t \in [0, T]$ and $R > 0$,

$$\mathbb{E}\left[\sup_{\tau\in[0,t\wedge\sigma_S]} \|\boldsymbol{u}^m(\tau) - \boldsymbol{u}(\tau)\|_{\mathbb{H}^2}^2\right] + \mathbb{E}\left[\int_0^{t\wedge\sigma_S} \|\boldsymbol{u}^m(s) - \boldsymbol{u}(s)\|_{\mathbb{H}^4}^2 \,\mathrm{d}s\right] \leq C_{S,T}\left\|\boldsymbol{u_0^m} - \boldsymbol{u_0}\right\|_{\mathbb{H}^2}^2,$$

where $C_{S,T}$ is a constant depending on S and T.

Proof. Let $\boldsymbol{v}(t) := \boldsymbol{u}^m(t) - \boldsymbol{u}(t)$. By Itô's lemma, following the same argument as in the proof of Theorem 2.5, we obtain an inequality similar to (5.7), namely

$$\begin{aligned}\mathbb{E}&\left[\sup_{\tau\in[0,t\wedge\sigma_S]} \|\boldsymbol{v}(\tau)\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\int_0^{t\wedge\sigma_S} \|\Delta\boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right]\\ &\leq C\|\boldsymbol{v}(0)\|_{\mathbb{L}^2}^2 + C(1+S^2)\int_0^t \mathbb{E}\left[\sup_{\tau\in[0,s\wedge\sigma_S]} \|\boldsymbol{v}(\tau)\|_{\mathbb{L}^2}^2\right]\mathrm{d}s.\end{aligned} \tag{6.10}$$

Subsequently, for clarity we will suppress the dependence of the functions on t. Applying Itô's lemma to the function $\psi(\boldsymbol{v}) = \frac{1}{2}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2$ then integrating with respect to t (cf. proof of Proposition 3.6), we obtain

$$\begin{aligned}&\frac{1}{2}\|\Delta\boldsymbol{v}(t)\|_{\mathbb{L}^2}^2 + \lambda_e\int_0^t \left\|\Delta^2\boldsymbol{v}\right\|_{\mathbb{L}^2}^2 \mathrm{d}s + (\lambda_r - \lambda_e)\int_0^t \|\nabla\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\\ &= \frac{1}{2}\|\Delta\boldsymbol{v}(0)\|_{\mathbb{L}^2}^2 + \lambda_r\int_0^t \|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\\ &\quad - \lambda_r\int_0^t \left\langle\Delta\big(|\boldsymbol{u}^m|^2\boldsymbol{v}\big), \Delta\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}s - \lambda_r\int_0^t \left\langle\Delta\big(((\boldsymbol{u}^m + \boldsymbol{u})\cdot\boldsymbol{v})\boldsymbol{u}\big), \Delta\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}s\\ &\quad + \lambda_e\int_0^t \left\langle\Delta\big(|\boldsymbol{u}^m|^2\boldsymbol{v}\big), \Delta^2\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}s + \lambda_r\int_0^t \left\langle\Delta\big(((\boldsymbol{u}^m + \boldsymbol{u})\cdot\boldsymbol{v})\boldsymbol{u}\big), \Delta^2\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}s\\ &\quad + \int_0^t \left\langle R(\boldsymbol{u}^m) - R(\boldsymbol{u}), \Delta^2\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}s + \int_0^t \left\langle L(\boldsymbol{u}^m) - L(\boldsymbol{u}), \Delta^2\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}s\\ &\quad + \sum_{k=1}^\infty \left\langle G_k(\boldsymbol{u}_1) - G_k(\boldsymbol{u}_2), \Delta^2\boldsymbol{v}\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(s)\\ &=: \frac{1}{2}\|\Delta\boldsymbol{v}(0)\|_{\mathbb{L}^2}^2 + I_2(t) + I_3(t) + \ldots + I_9.\end{aligned}$$

We will estimate $I_j(t\wedge\sigma_S)$ for $j = 2, \dots, 9$ as follows. Firstly, for the terms $I_2(t\wedge\sigma_S)$ and $I_3(t\wedge\sigma_S)$, by (2.5),

$$\begin{aligned}|I_2(t\wedge\sigma_S)| &\leq C\int_0^t \|\boldsymbol{u}^m\|_{\mathbb{H}^2}^2 \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \,\mathrm{d}s \leq CS\int_0^t \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \,\mathrm{d}s,\\ |I_3(t\wedge\sigma_S)| &\leq C\int_0^t \|\boldsymbol{u}^m + \boldsymbol{u}\|_{\mathbb{H}^2} \|\boldsymbol{u}\|_{\mathbb{H}^2} \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \,\mathrm{d}s \leq CS\int_0^t \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \,\mathrm{d}s.\end{aligned}$$

Similarly, for the next term, by (2.5) and Young's inequality,

$$|I_4(t\wedge\sigma_S)| \le C\int_0^t \|\boldsymbol{u}^m\|_{\mathbb{H}^2}^2 \|\boldsymbol{v}\|_{\mathbb{H}^2} \left\|\Delta^2\boldsymbol{v}\right\|_{\mathbb{L}^2} \mathrm{d}s \le CS\int_0^t \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \mathrm{d}s + \frac{\lambda_e}{8}\int_0^t \left\|\Delta^2\boldsymbol{v}\right\|_{\mathbb{L}^2}^2 \mathrm{d}s.$$

The rest of the terms can be estimated in a similar manner, which we omit for brevity. Taking expected value, we obtain the inequality

$$\mathbb{E}\left[\sup_{\tau\in[0,t\wedge\sigma_S]} \|\Delta\boldsymbol{v}(\tau)\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\int_0^{t\wedge\sigma_S} \left\|\Delta^2\boldsymbol{v}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right] \le C\|\Delta\boldsymbol{v}(0)\|_{\mathbb{L}^2}^2 + C(1+S)\int_0^t \mathbb{E}\left[\sup_{\tau\in[0,s\wedge\sigma_S]} \|\boldsymbol{v}(\tau)\|_{\mathbb{H}^2}^2\right] \mathrm{d}s. \tag{6.11}$$

Adding (6.10) and (6.11), then applying Gronwall's inequality yields

$$\mathbb{E}\left[\sup_{\tau\in[0,t\wedge\sigma_S]} \|\boldsymbol{v}(\tau)\|_{\mathbb{H}^2}^2\right] + \mathbb{E}\left[\int_0^{t\wedge\sigma_S} \left\|\Delta^2\boldsymbol{v}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right] \le C\|\boldsymbol{v}(0)\|_{\mathbb{H}^2}^2 e^{C(1+S^2)T},$$

as required. □

We introduce another stopping time related to the instant parabolic regularisation provided by the equation (cf. Lemma 4.10). For any $R>0$ and $\boldsymbol{v_0}\in\mathbb{H}^2$, define

$$\rho_R(\boldsymbol{v_0}) := \inf\{t\ge 0 : t\|\boldsymbol{v}(t;\boldsymbol{v_0})\|_{\mathbb{H}^3}^2 > R\}. \tag{6.12}$$

Lemma 6.5. Let $\boldsymbol{v_0}\in\mathbb{H}^2$ and let $\sigma_S(\boldsymbol{v_0})$ and $\rho_R(\boldsymbol{v_0})$ be as defined in (6.9) and (6.12), respectively. Then

$$\mathbb{P}(\sigma_S(\boldsymbol{v_0})<t) \le \frac{C}{S}(1+e^{Ct}), \tag{6.13}$$

$$\mathbb{P}(\rho_R(\boldsymbol{v_0})<t) \le \frac{C}{R}(1+e^{Ct}), \tag{6.14}$$

where C depends on $\|\boldsymbol{v_0}\|_{\mathbb{H}^2}$.

Proof. By Proposition 3.3, Proposition 3.6, and Markov's inequality, we have

$$\mathbb{P}(\sigma_S(\boldsymbol{v_0})<t) = \mathbb{P}\left(\sup_{\tau\in[0,t]} \|\boldsymbol{u}(\tau;\boldsymbol{v_0})\|_{\mathbb{H}^2}^2 > S\right) \le \frac{1}{S}\mathbb{E}\left[\sup_{\tau\in[0,t]} \|\boldsymbol{u}(\tau)\|_{\mathbb{H}^2}^2\right] \le \frac{C}{S}(1+e^{Ct}).$$

Similarly, inequality (6.14) follows from Lemma 6.3 and Markov's inequality. □

Next, define

$$\sigma_S := \sigma_S(\boldsymbol{u_0^m})\wedge\sigma_S(\boldsymbol{u_0}), \tag{6.15}$$

$$\rho_R := \rho_R(\boldsymbol{u_0^m})\wedge\rho_R(\boldsymbol{u_0}). \tag{6.16}$$

We can now show the Feller property of the transition semigroup P_t.

Proposition 6.6. For each $\phi\in C_b(\mathbb{H}^2)$ and $t>0$, the map $\boldsymbol{v}\mapsto P_t\phi(\boldsymbol{v})$ is continuous.

Proof. Let $\epsilon>0$ be given and fix $t>0$ and $\boldsymbol{u_0}\in\mathbb{H}^2$. Let $\boldsymbol{u_0^m}\in\mathbb{H}^2$ be such that $\left\|\boldsymbol{u_0^m}-\boldsymbol{u_0}\right\|_{\mathbb{H}^2}<1$. With σ_S and ρ_R as defined in (6.15) and (6.16) respectively, we write

$$\begin{aligned}
\left|P_t\phi(\boldsymbol{u_0^m}) - P_t\phi(\boldsymbol{u_0})\right| &= \left|\mathbb{E}\left[\phi(\boldsymbol{u}(t;\boldsymbol{u_0^m})) - \phi(\boldsymbol{u}(t;\boldsymbol{u_0}))\right]\right| \\
&\le \left|\mathbb{E}\left[\phi(\boldsymbol{u}(t;\boldsymbol{u_0^m})) - \phi(\boldsymbol{u}(t;\boldsymbol{u_0}))\right]\mathbb{I}_{\{\sigma_S<t\}}\right| \\
&\quad + \left|\mathbb{E}\left[\phi(\boldsymbol{u}(t;\boldsymbol{u_0^m})) - \phi(\boldsymbol{u}(t;\boldsymbol{u_0}))\right]\mathbb{I}_{\{\rho_R<t\}}\right| \\
&\quad + \left|\mathbb{E}\left[\phi(\boldsymbol{u}(t;\boldsymbol{u_0^m})) - \phi(\boldsymbol{u}(t;\boldsymbol{u_0}))\right]\mathbb{I}_{\{\sigma_S\ge t\}}\mathbb{I}_{\{\rho_R\ge t\}}\right| =: E_1+E_2+E_3.
\end{aligned}$$

We will estimate each term in the following. To this end, let $\|\phi\|_\infty := \sup_{\boldsymbol{v}\in\mathbb{H}^2}|\phi(\boldsymbol{v})|$. By Lemma 6.5,

$$E_1 \le 2\|\boldsymbol{\phi}\|_\infty\left(\mathbb{P}(\sigma_S(\boldsymbol{u_0^m})<t) + \mathbb{P}(\sigma_S(\boldsymbol{u_0})<t)\right) \le \frac{C}{S}\|\phi\|_\infty(1+e^{Ct}).$$

Similarly,

$$E_2 \leq \frac{C}{R} \|\phi\|_\infty (1+e^{Ct}),$$

where C depends on $\|\boldsymbol{u_0}\|_{\mathbb{H}^2}$. By taking S and R sufficiently large, we can ensure that $E_1 + E_2 < \epsilon/2$.

Next, we will estimate E_3. Note that on the set $\{\rho_R \geq t\}$, we have $\boldsymbol{u_0^m}, \boldsymbol{u_0} \in \boldsymbol{B} := \boldsymbol{B}_3\left((R/t)^{\frac{1}{2}}\right)$, where $\boldsymbol{B}_\alpha(r)$ is the closed ball in $\mathbb{H}^\alpha$ with radius r centred at $\mathbf{0}$. On this ball $\boldsymbol{B}$ (which is compact in $\mathbb{H}^2$ since the embedding $\mathbb{H}^3 \hookrightarrow \mathbb{H}^2$ is compact), we can approximate the given $\phi \in C_b(\mathbb{H}^2)$ by a Lipschitz function φ with Lipschitz constant L_φ such that

$$\sup_{\boldsymbol{v}\in\boldsymbol{B}} |\phi(\boldsymbol{v}) - \varphi(\boldsymbol{v})| < \frac{\epsilon}{8}.$$

Therefore, by the triangle inequality, Jensen's inequality, and Lemma 6.4, we have

$$\begin{aligned} E_3 &\leq 2\sup_{\boldsymbol{v}\in\boldsymbol{B}} |\phi(\boldsymbol{v}) - \varphi(\boldsymbol{v})| + L_\varphi\, \mathbb{E}\left[\left\|(\boldsymbol{u}(t;\boldsymbol{u_0^m})) - \boldsymbol{u}(t;\boldsymbol{u_0}))\mathbb{I}_{\{\sigma_S\geq t\}}\right\|_{\mathbb{H}^2}\right] \\ &\leq \frac{\epsilon}{4} + L_\varphi \left(\mathbb{E}\left[\sup_{s\in[0,t\wedge\sigma_S]} \left\|\boldsymbol{u}(s;\boldsymbol{u_0^m}) - \boldsymbol{u}(s;\boldsymbol{u_0})\right\|_{\mathbb{H}^2}^2\right]\right)^{\frac{1}{2}} \\ &\leq \frac{\epsilon}{4} + C_S L_\varphi \left\|\boldsymbol{u_0^m} - \boldsymbol{u_0}\right\|_{\mathbb{H}^2}, \end{aligned}$$

which can be made less than $\epsilon/2$ whenever $\left\|\boldsymbol{u_0^m} - \boldsymbol{u_0}\right\|_{\mathbb{H}^2} < \epsilon/(4C_S L_\varphi) =: \delta$. Hence, we have shown that if $\left\|\boldsymbol{u_0^m} - \boldsymbol{u_0}\right\|_{\mathbb{H}^2} < \delta$, we have $\left|P_t\phi(\boldsymbol{u_0^m}) - P_t\phi(\boldsymbol{u_0})\right| < \epsilon$. This completes the proof of the statement. ☐

Theorem 2.6 is an immediate consequence of the above lemmas, which we show below.

Proof of Theorem 2.6. Proposition 6.6 implies that the transition semigroup $\{P_t\}_{t\geq 0}$ is Feller in $\mathbb{H}^2$. Next, we show tightness of the family of probability measures $\{\mu_T\}_{T\geq 1}$ given by

$$\mu_T(\cdot) := \frac{1}{T}\int_0^T P_t(\boldsymbol{u_0}, \cdot)\,\mathrm{d}t.$$

Let $\boldsymbol{B}_3(R)$ denote the closed ball in $\mathbb{H}^3$ with radius R centred at $\mathbf{0}$. Then, by Markov's inequality and Lemma 6.1, we have

$$\begin{aligned} \sup_{T\geq 1} \mu_T\left(\mathbb{H}^2 \setminus \boldsymbol{B}_3(R)\right) &= \sup_{T\geq 1} \frac{1}{T}\int_0^T \mathbb{P}\big(\{\|\boldsymbol{u}(s;\boldsymbol{u_0})\|_{\mathbb{H}^3} \geq R\}\big)\,\mathrm{d}s \\ &\leq \sup_{T\geq 1} \frac{1}{TR}\int_0^T \mathbb{E}\left[\|\boldsymbol{u}(s)\|_{\mathbb{H}^3}\right]\mathrm{d}s \\ &\leq \sup_{T\geq 1} \frac{C(1+T)}{TR} \leq \frac{2C}{R} \to 0 \ \text{ as } R\to\infty. \end{aligned}$$

Since the embedding $\mathbb{H}^3 \hookrightarrow \mathbb{H}^2$ is compact, the family of probability measures $\{\mu_T\}_{T\geq 1}$ is tight on $\mathbb{H}^2$. This yields the existence of at least one invariant measure by the Krylov–Bogoliubov theorem.

We now show that such an invariant measure μ is supported on $\mathbb{H}^4$. By the invariance property of μ, for any $T>0$ and $\phi \in C_b(\mathbb{H}^2)$ we have

$$\int_{\mathbb{H}^2} \phi(\boldsymbol{u_0})\,\mathrm{d}\mu(\boldsymbol{u_0}) = \int_{\mathbb{H}^2}\int_{\mathbb{H}^2} \frac{1}{T}\int_0^T P_t(\boldsymbol{u_0}, \mathrm{d}\boldsymbol{u})\phi(\boldsymbol{u})\,\mathrm{d}t\,\mathrm{d}\mu(\boldsymbol{u}_0). \tag{6.17}$$

For any $n\in\mathbb{N}$, $R>0$, and $\boldsymbol{v}\in\mathbb{H}^2$, let

$$\Psi_{n,R}(\boldsymbol{v}) := \left(\log(1 + \left\|\Delta^2 \Pi_n \boldsymbol{v}\right\|_{\mathbb{L}^2}^2)\right) \wedge R,$$

where Π_n is the projection operator defined in (3.1). Note that for any $\boldsymbol{u_0} \in \boldsymbol{B}_2(\alpha)$, the ball of radius α about the origin in $\mathbb{H}^2$, by Lemma 6.2 we have

$$\left|\frac{1}{T}\int_0^T \int_{\mathbb{H}^2} P_t(\boldsymbol{u}_0, \mathrm{d}\boldsymbol{u})\Psi_{n,R}(\boldsymbol{u})\,\mathrm{d}t\right| = \left|\frac{1}{T}\int_0^T \mathbb{E}\Psi_{n,R}(\boldsymbol{u}(t;\boldsymbol{u_0}))\,\mathrm{d}t\right| \le C\left(1+\frac{1+\|\boldsymbol{u_0}\|_{\mathbb{H}^2}^2}{T}\right) \le C\left(1+\frac{\alpha^2}{T}\right).$$

Then by (6.17) and the above inequality, we have

$$\begin{aligned}\int_{\mathbb{H}^2} \Psi_{n,R}(\boldsymbol{u_0})\,\mathrm{d}\mu(\boldsymbol{u_0}) &\le \int_{\mathbb{H}^2}\left|\frac{1}{T}\int_0^T \int_{\mathbb{H}^2} P_t(\boldsymbol{u}_0, \mathrm{d}\boldsymbol{u})\Psi_{n,R}(\boldsymbol{u})\,\mathrm{d}t\right| \mathrm{d}\mu(\boldsymbol{u}_0) \\ &= \int_{\boldsymbol{B}_2(\alpha)}\left|\frac{1}{T}\int_0^T \mathbb{E}\Psi_{n,R}(\boldsymbol{u}(t;\boldsymbol{u_0}))\,\mathrm{d}t\right| \mathrm{d}\mu(\boldsymbol{u_0}) \\ &\quad + \int_{\mathbb{H}^2\setminus \boldsymbol{B}_2(\alpha)}\left|\frac{1}{T}\int_0^T \mathbb{E}\Psi_{n,R}(\boldsymbol{u}(t;\boldsymbol{u_0}))\,\mathrm{d}t\right| \mathrm{d}\mu(\boldsymbol{u_0}) \\ &\le C(1+\alpha^2 T^{-1})\,\mu(\boldsymbol{B}_2(\alpha)) + R\,\mu(\mathbb{H}^2\setminus \boldsymbol{B}_2(\alpha)),\end{aligned}$$

where C is a constant which does not depend on T, α, and R. First, we take α to be sufficiently large (depending on R) so that $R\,\mu(\mathbb{H}^2\setminus \boldsymbol{B}_2(\alpha)) \le 1$. Then we choose T sufficiently large (depending on α), so that $\alpha T^{-1} \le 1$. This implies

$$\int_{\mathbb{H}^2} \Psi_{n,R}(\boldsymbol{u_0})\,\mathrm{d}\mu(\boldsymbol{u_0}) \le C,$$

where C is independent of n and R. Letting $n \to \infty$, by Fatou's lemma we have

$$\int_{\mathbb{H}^2} \Big(\big(\log(1+\left\|\Delta^2\boldsymbol{u_0}\right\|_{\mathbb{L}^2}^2)\big)\wedge R\Big)\,\mathrm{d}\mu(\boldsymbol{u_0}) \le C,$$

where C is independent of R. By the monotone convergence theorem, taking $R\uparrow\infty$, we obtain

$$\int_{\mathbb{H}^2} \log\left(1+\left\|\Delta^2\boldsymbol{u_0}\right\|_{\mathbb{L}^2}^2\right)\mathrm{d}\mu(\boldsymbol{u_0}) < \infty.$$

In particular, this implies that μ is supported on $\mathbb{H}^4$. This completes the proof of the theorem. ☐

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