Problem Set 1

Ask for hints if you'd like some!

Q1 Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms on a vector space V.
Is $||x|| = \min\{||x||_1, ||x||_2\}$ necessarily a norm on V?

Q2 If $X = (V, || \cdot ||)$ is a normed space, show that

$$d(x, y) := ||x - y||$$

defines a metric on V.

Q3 If V is a vector space and d is a translation-invariant metric on V, show that

$$||x|| := d(0, x)$$

defines a norm on V.

Q4 A subset S of a vector space is *convex* if for all $x, y \in S$, we have

$$\{\alpha x + (1 - \alpha)y \mid 0 \leq \alpha \leq 1\} \subset S.$$

(a) Show that $B(X) = \{x \in X \mid ||x|| \leq 1\}$ is convex, where $X = (V, || \cdot ||)$ is a normed space.

(b) Use (a) to show that the function $|| \cdot ||\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$||(a, b)|| = (\sqrt{|a|} + \sqrt{|b|})^2$$

does not define a norm on $\mathbb{R}^2$.

Q5 Show that l_∞^n and l_∞ are Banach spaces.

Q6 Let $M \subset l_\infty$ be the subspace consisting of all sequences $x = (x_i)$ with at most finitely many non-zero terms. Find a Cauchy sequence in M which does not converge in M and hence conclude that M is not complete.

Q7 For $1 \leq p \leq \infty$, let $|| \cdot ||_p$ be the l_p–norm on $\mathbb{R}^n$ or $\mathbb{C}^n$. Show that if $1 \leq p < q \leq \infty$, then $||x||_p \geq ||x||_q$. For which points x do we have equality?
Prove that for every $\varepsilon > 0$, there is an N such that if $N < p < \infty$, then

$$||x||_\infty \leq ||x||_p \leq (1 + \varepsilon)||x||_\infty.$$