Convergence theorems and L^1

Q19 Prove the Monotone Convergence Theorem using Fatou's lemma.

Q20 (Fatou almost everywhere)
Let (X, Σ, μ) be a measure space. Suppose (f_n) is a sequence of non-negative, measurable functions from X to $\mathbb{R}$ which converges almost everywhere to the function f. Show that

$$\int f \leq \liminf \int f_n.$$

Q21 Let (X, Σ, μ) be a measure space and $f\colon X \to \mathbb{R}^*$ be a measurable function. Suppose $f \geq 0$ and $\int f < \infty$. Show that

(a) $\{x \in X \mid f(x) = \infty\}$ is a null-set, and

(b) $\{x \in X \mid f(x) > 0\}$ is a σ–finite set.

Q22 Let (X, Σ, μ) be a measure space. Show that the measure μ is complete if and only if the following two conditions hold:

(a) If f is measurable and $g = f$ almost everywhere, then g is measurable.

(b) If f_n is measurable for all $n \in \mathbb{N}$ and $f_n \to f$ almost everywhere, then f is measurable.

Q23 Let m denote Lebesgue measure on $\mathbb{R}$. Show that there exists $f \in \mathcal{L}^1(m)$ and a sequence $(f_n) \subset \mathcal{L}^1(m)$ such that

$$||f - f_n|| \to 0 \qquad \text{as} \quad n \to \infty,$$

but $f_n(x) \to f(x)$ for no $x \in \mathbb{R}$.
Hint: Choose characteristic functions of intervals I_n such that $m(I_n) \to 0$.

Q24 Let (X, Σ, μ) be a measure space. Show that the equivalence classes of simple functions are dense in $L^1(\mu)$.

Q25 Let μ be counting measure on $\mathbb{N}$. Interpret the three convergence theorems as statements about infinite sequences.

Q26 Suppose $(f_n) \subset L^1(\mu)$ and $f_n \to f$ uniformly.

(a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.

(b) Show that if $\mu(X) = \infty$, then the conlusions of (a) can fail.