Math3402 Problem set 9

Question 1: Show that any linear operator T on a finite dimensional normed space X is bounded. Hint: use the linear combinations lemma from lectures.

Question 2: Let $x, y \in \mathbb{R}^n$, and define an inner product by

$$\langle x, y \rangle = g_{ij} x^i y^j,$$

where we are summing over $1 \leq i, j \leq n$. What properties must the matrix g_{ij} satisfy for this to qualify as an inner product? Can g_{ij} be diagonalised? Show that any inner product on $\mathbb{R}^n$ may be written in this form.

Question 3: Let V be a real inner product space. Show the polarization identity

$$\langle x, y \rangle = \frac{1}{4}(||x + y||^2 - ||x - y||^2).$$

Challenge question: It was shown in lectures that a norm induced by an inner product satisfies the parallelogram equality

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).$$

Show that if a norm on a vector space V satisfies this equality, then the polarization identity may be used to define an inner product on V satisfying $||x|| = \sqrt{\langle x, x \rangle}$. A similar result holds for complex vector spaces.

Hint: To show $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, consider the parallelogram equality with $x = u + v$, then with $x = u - v$, and subtract the latter equation from the former. Rewrite this equation in terms of inner products, and consider what one gets by setting $z = u$, and what one can get by a substitution. To show that $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$, use the fact that $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ to show this result for α a positive integer, then extend to rational numbers, and finally to real numbers (don't forget negative numbers).