# A Modified Kähler-Ricci Flow

Zhou Zhang
Department of Mathematics
University of Michigan, at Ann Arbor

**Abstract**

In this note, we study a Kähler-Ricci flow modified from the classic version. In the non-degenerate case, strong convergence at infinite time is achieved. The main focus should be on degenerate case, where some partial results are presented.

## 1 Set-up and Motivation

Kähler-Ricci flow, which is nothing but Ricci flow with initial metric being Kähler, enjoys the same debut as Ricci flow in R. Hamilton's original paper [5]. H. D. Cao's paper, [1], can be taken as the first work devoted to the study of Kähler-Ricci flow and the alternative proof of Calabi Conjecture presented there has been bringing great interests to this object.

Though it is essentially Ricci flow, the cohomology meaning coming with Kähler condition makes it possible to transform the metric flow to an equivalent scalar (potential) flow [1], which is much simpler-looking and more flexible to work with. One motivation of this note is to give a flavor of this flexibility.

Let $\omega_0$ be any Kähler metric over a closed manifold $X$ with $dim_{\mathbb{C}} X = n \geqslant 2$, and $\omega_\infty$ is any smooth real closed $(1,1)$-form.

Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ and consider the following flow at the level of metric potential for space-time

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega}, \quad u(0, \cdot) = 0, \tag{1.1}$$

where $\Omega$ is a smooth volume form over $X$. This flow looks very much like the flow studied in [1], which can be considered as another motivation of this work.

Let $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ and the corresponding flow at the level of metric is as follows

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\text{Ric}(\widetilde{\omega}_t) + \text{Ric}(\Omega) - e^{-t}(\omega_0 - \omega_\infty), \quad \widetilde{\omega}_0 = \omega_0, \tag{1.2}$$

[1]This statement makes use of the uniqueness and short time existence results of Ricci flow.

where the meaning of the form, $\mathrm{Ric}(\Omega)$, as in [11], is a natural generalization from the Ricci form for a Kähler metric, i.e.

$$\mathrm{Ric}(\Omega) = -\sqrt{-1}\partial\bar{\partial}\log\frac{\Omega}{V_{Eul}},$$

where $V_{Eul}$ is the Euclidean volume form with respect to some choice of local coordinates. [2]

**Remark 1.1.** *The equation (1.2) doesn't look so natural at the first sight when $\omega_0 \neq \omega_\infty$, but it's essentially still a Kähler-Ricci flow, and the extra term in comparison to the flow studied in [1], which is exponentially decaying, should in principle not bring too much difference to the behavior.*

A major motivation to study this flow is to solve the following complex Monge-Ampère equation (with $[\omega_\infty]^n = \int_X \Omega$)

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = \Omega, \tag{1.3}$$

using flow techniques. This has been done in the case of $[\omega_\infty]$ being Kähler in [1] by choosing the initial metric to be in the same Kähler class as $[\omega_\infty]$, which gives rise to an alternative proof of Calabi Conjecture.

One can also solve (1.3) for some degenerate $[\omega_\infty]$ (for example, semi-ample and big) by method of continuity using other (more direct) perturbations, which seems to be less delicate than Kähler-Ricci flow as described in [17] and [12].

The main idea is to allow the change of cohomology class along the flow, which is important for the consideration of $[\omega_\infty]$ not being Kähler. The modification of original Kähler-Ricci flow by such a term as in (1.2) is inevitable from simple cohomology consideration.

The main results of this note can be summarized in the following theorem.

**Theorem 1.2.** *The modified Kähler-Ricci flow (1.1) (or (1.2) equivalently) exists smoothly as long as the cohomology class, $[\omega_t]$ remains Kähler.*

*1) If $[\omega_\infty]$ is Kähler, the flow converges smoothly to the unique solution of the corresponding complex Monge-Ampère equation (1.3).*

*2) If $[\omega_\infty]$ is semi-ample and big, we have degenerate metric estimates, i.e. uniform estimates in any compact set away from the stable base locus set of $[\omega_\infty]$ along the flow, and the volume form, $\widetilde{\omega}_t^n$, is bounded both from above and away from 0 for all time.*

*3) If $[\omega_\infty]$ is big but out of the closure of Kähler cone (in other words, the flow exists up to a finite time $T$), and further assume $[\omega_T]$ is semi-ample, we have local convergence in $C^\infty$ topology of the flow away from the stable base locus set of $[\omega_T]$ as $t \to T$.*

The definition of "stable base locus set" will appear later. The rest part of this note is devoted to the proof of this theorem.

[2] This is nothing but using the volume form $\Omega$ instead of the volume form for some Kähler metric in the expression of Ricci form in sight of the classic computation in Kähler geometry.

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## 2 General Facts and Basic Computations

The equation (1.1) is clearly still parabolic, and so short time existence and uniqueness of the solution is classic. It's also easy to see that the smooth solution exists as long as $[\omega_t]$ remains Kähler which has already being described in [17]. Simply speaking, when arguing locally in time, $\omega_t$ can be made uniform as metric which makes life very easy to follow Cao and Yau's argument as in [1] and [15] to get the estimates uniform on $X$ (but local in time). So the existence part of Theorem 1.2 is justified.

Convergence, or estimate uniform for maximal existence time, is the main concern from now on. Let's begin with some basic computation for $t$-derivatives of (1.1).

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \langle\widetilde{\omega}_t, \frac{\partial\widetilde{\omega}_t}{\partial t}\rangle = \Delta_{\widetilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle.$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) = \langle\widetilde{\omega}_t, \frac{\partial^2\widetilde{\omega}_t}{\partial t^2}\rangle - \left(\frac{\partial\widetilde{\omega}_t}{\partial t}, \frac{\partial\widetilde{\omega}_t}{\partial t}\right)_{\widetilde{\omega}_t} \leqslant \Delta_{\widetilde{\omega}_t}\left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t}\langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle,$$

where the notation $\langle\cdot,\cdot\rangle$ means taking trace for the right term using the left term which is always a metric.

**As a convention, the same $C$ might stand for different positive constant even in the same equation.**

Take summation of the above two and apply standard maximum principle argument to get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \leqslant C,$$

and from this, it's easy to see

$$\frac{\partial u}{\partial t} \leqslant C,$$

which gives the $L^\infty$ measure bound for $\widetilde{\omega}_t^n = e^{\frac{\partial u}{\partial t}}\Omega$. This allows us to apply the results of Kolodziej's (as in [8] and [9]) and our generalization (as in [16]) under respective assumptions on $[\omega_t]$ as in Theorem 1.2, which provides the uniform bound for the metric potential along the flow after routine normalization for the corresponding cases under consideration, i.e. for $v = u - \frac{\int_X u\Omega}{\int_X \Omega}$,

$$|v| \leqslant C.$$

For the readers' convenience, the related results are summarized below.

**Theorem 2.1.** *Let $X$ be a closed Kähler manifold with $dim_{\mathbb{C}} = n \geqslant 2$. Suppose we have a holomorphic map $F : X \to \mathbb{CP}^N$ with the image $F(X)$ of the same dimension as $X$. Let $\omega_M$ be any Kähler form over some neighborhood of $F(X)$ in $\mathbb{CP}^N$. For the following equation of Monge-Ampère type:*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega,$$

*where $\omega = F^*\omega_M$, $\Omega$ is a fixed smooth (non-degenerate) volume form over $X$ and $f$ is a nonnegative function in $L^p(X)$ for some $p > 1$ with the correct total integral over $X$, i.e. $\int_X f\Omega = \int_X (F^*\omega_M)^n$, then we have the following:*

*(1) (A priori estimate) If $u$ is a weak solution in $PSH_\omega(X) \cap L^\infty(X)$ of the equation with the normalization $sup_X u = 0$, then there is a constant $C$ such that $\|u\|_{L^\infty} \leqslant C\|f\|^n_{L^p}$ where $C$ only depends on $F$, $\omega$ and $p$.*

*(2) (Existence of bounded solution) There exists a bounded (weak) solution for this equation.*

*(3) (Continuity and uniqueness of bounded solution) If $F$ is locally birational, any bounded solution is actually the unique continuous solution.*

**Remark 2.2.** *Even in the case of $[\omega_\infty]$ being Kähler, the result from pluripotential theory as in [8] is used here for the normalized metric potential bound, so the logic line of the argument below is not quite the same as that in [1]. It would also be interesting to see whether the original argument there can be carried through more directly.*

We also need to derive some kind of lower bound for $\frac{\partial u}{\partial t}$, i.e. the volume form, in search for the metric bound. The following equation would be very useful for that

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta_{\widetilde{\omega}_t}\left(\frac{\partial u}{\partial t} + u\right) - n + \frac{\partial u}{\partial t} + \langle \widetilde{\omega}_t, \omega_\infty \rangle. \tag{2.1}$$

# 3 The Baby Version

Let's start with the situation when there is no degeneration on the cohomology classes, i.e. $[\omega_\infty]$ is Kähler. This is Statement 1) in Theorem 1.2, which is a natural generalization of the main result in [1] by allowing the change of cohomology class along the flow and any choice of initial metric $\omega_0$.

## 3.1 Uniform Estimates and Global Existence

Global existence of the flow only needs estimates local in time as illustrated before. Now we have to go for estimates uniform for all time.

The most essential part is the $C^0$ estimates. In sight of all the estimates listed in Section 2, there is only the lower bound for $\frac{\partial u}{\partial t}$ left.

To begin with, let's assume $\omega_\infty > 0$, which will not change the problem in any essential way. We'll remove this simplification later.

Let's assume $\int_X \Omega = 1$ for the simplicity of notations.

As discussed in Section 2, by the measure bound from the previous section, we already know that

$$|v| \leqslant C,$$

where $v = u - \int_X u\Omega$. It's clear that

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \int_X \frac{\partial u}{\partial t}\Omega \geqslant \frac{\partial u}{\partial t} - C$$

from the upper bound of $\frac{\partial u}{\partial t}$, and so the lower bound of $\frac{\partial u}{\partial t}$ would give that for $\frac{\partial v}{\partial t}$. In fact the converse is also true as seen below. (2.1) can be transformed to be

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + v\right) = \Delta_{\widetilde{\omega}_t}\left(\frac{\partial u}{\partial t} + v\right) - n + \frac{\partial v}{\partial t} + \langle \widetilde{\omega}_t, \omega_\infty \rangle. \tag{3.1}$$

Assuming $\frac{\partial v}{\partial t} \geqslant -C$, we can get a lower bound for $\frac{\partial u}{\partial t}$ by applying maximum principle as H. Tsuji did in [13] using the control of volume by the control of trace, which is nothing but the classic algebraic-geometric mean value inequality.

Indeed, we can get the lower bound of $\frac{\partial u}{\partial t}$ by a more careful maximum principle argument following the same idea.

Consider the minimum value point, $p$, for $\frac{\partial u}{\partial t} + v$ for $X \times [0, T]$ with any fixed $0 < T < \infty$. Clearly, one only needs to study the case when $p$ is not at the initial time since the situation for the initial time is well under control. At $p$, from (3.1), one has

$$n - \frac{\partial v}{\partial t} \geqslant \langle \widetilde{\omega}_t, \omega_\infty \rangle \geqslant n \cdot \left(\frac{\omega_\infty^n}{\widetilde{\omega}_t^n}\right)^{\frac{1}{n}} = n \cdot \left(\frac{\omega_\infty^n}{e^{\frac{\partial u}{\partial t}}\Omega}\right)^{\frac{1}{n}} > 0,$$

and so $(1 - \frac{1}{n}\frac{\partial v}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geqslant C > 0$. Using $\frac{\partial v}{\partial t} \geqslant \frac{\partial u}{\partial t} - C$, one arrives at

$$(C - \frac{\partial u}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geqslant C > 0$$

with $C - \frac{\partial u}{\partial t} > 0$, which gives $\frac{\partial u}{\partial t} \geqslant -C$ at $p$.

Combining with the uniform bound for $v$, this gives the lower bound of $\frac{\partial u}{\partial t}$ over $X$ (and also for $\frac{\partial v}{\partial t}$) through the definition of point $p$.

There is another way of doing the maximum principle argument which might seems to be more direct in this case, which is also a very classic point of view using $ODE$. One examines the evolution of space-direction extremal value along the flow. This function, now only depending on time, would be (locally) Lipschitz simply by definition, and so it is legitimate to consider the first order ordinary differential inequality.

This kind of argument, in principle, would be more delicate than what is used previously, but for the differential inequality of interest here, the study would be as rough as before. Let's illustrate the idea below.

Define $A(t) := \min_{X\times\{t\}}(\frac{\partial u}{\partial t} + v)$. Let's also take some $x(t)$ where the value $A(t)$ is achieved, but we do not assume (or need) any regularity of $x(t)$ with respect to $t$. Using the sign of Laplacian, we can derive the following differential inequality for the function $A(t)$

$$\begin{aligned}\frac{\partial A}{\partial t} &\geqslant -n + \frac{\partial v}{\partial t}\,(x(t)) + \langle\widetilde{\omega}_t, \omega_\infty\rangle(x(t)) \\ &\geqslant -C + (\frac{\partial u}{\partial t} + v)\,(x(t)) + Ce^{-(\frac{\partial u}{\partial t}+v)(x(t))} \\ &= -C + A + Ce^{-A}.\end{aligned}$$

From this inequality, we can see that when $A$ is sufficiently small (i.e., very negative), $\frac{\partial A}{\partial t}$ would be big (very positive). It won't be hard to get a lower bound for $A$ from this mechanism. All the pieces from the previous argument are also applied here, but this looks a bit more straightforward).

**Remark 3.1.** *Clearly, in the degenerate version of maximum principle argument as what will appear later, this point of view still works as long as the point $x(t)$ is always in the regular part for the discussion.*

Then using classic second order estimate, we can have uniform control for the trace of $\widetilde{\omega}_t$ (i.e., Lapacian). And so, together with the volume lower bound from the bound of $\frac{\partial u}{\partial t}$, we have flow metric controlled uniformly as metric, i.e.

$$C^{-1}\omega_0 \leqslant \widetilde{\omega}_t \leqslant C\omega_0.$$

Finally, high order derivatives are also uniformly controlled using classic estimates for parabolic PDEs (including Yau's computation and parabolic Schauder estimates). These are very standard arguments for the current situation.

Now we remove the assumption that $\omega_\infty > 0$ imposed at the beginning. Instead, we always have $\omega_\infty + \sqrt{-1}\partial\bar{\partial}f > 0$ for some smooth function $f$ over $X$ as $[\omega_\infty]$ is Kähler. Also recall that $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$. Now set

$$\bar{\omega}_t = (\omega_\infty + \sqrt{-1}\partial\bar{\partial}f) + e^{-t}(\omega_0 - (\omega_\infty + \sqrt{-1}\partial\bar{\partial}f)) = \omega_t + (1-e^{-t})\sqrt{-1}\partial\bar{\partial}f,$$

and clearly $\widetilde{\omega}_t = \bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}(u - (1-e^{-t})f)$.

Define $w := u - (1-e^{-t})f$ and we have $\widetilde{\omega}_t = \bar{\omega}_t + \sqrt{-1}\partial\bar{\partial}w$. Clearly $\frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - e^{-t}f$ and taking $t$-derivative gives

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) = \Delta_{\widetilde{\omega}_t}(\frac{\partial w}{\partial t}) - e^{-t}\langle \widetilde{\omega}_t, \omega_0 - \omega_\infty - \sqrt{-1}\partial\bar{\partial}f\rangle + e^{-t}f.$$

We still need the following equation

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t} + \bar{w}\right) = \Delta_{\widetilde{\omega}_t}\left(\frac{\partial w}{\partial t} + \bar{w}\right) - n + \langle \widetilde{\omega}_t, \omega_\infty + \sqrt{-1}\partial\bar{\partial}f\rangle + \frac{\partial \bar{w}}{\partial t} + e^{-t}f,$$

where $\bar{w}$ is the normalization of $w$ in the same manner as $v$ for $u$.

We can now apply maximum principle for the above equation at the (local in time) minimum value point of $\frac{\partial w}{\partial t} + \bar{w}$. At that point (if not at time 0), we have

$$n - \frac{\partial \bar{w}}{\partial t} \geqslant e^{-t}f + \langle \widetilde{\omega}_t, \omega_\infty + \sqrt{-1}\partial\bar{\partial}f\rangle.$$

Without loss of generality, we can make sure $f > 0$, and so one arrives at

$$\begin{aligned} n - \frac{\partial \bar{w}}{\partial t} &\geqslant \langle \widetilde{\omega}_t, \omega_\infty + \sqrt{-1}\partial\bar{\partial}f\rangle \\ &\geqslant n \cdot (\frac{(\omega_\infty + \sqrt{-1}\partial\bar{\partial}f)^n}{\widetilde{\omega}_t^n})^{\frac{1}{n}} \\ &= n \cdot \left(\frac{(\omega_\infty + \sqrt{-1}\partial\bar{\partial}f)^n}{e^{\frac{\partial u}{\partial t}}\Omega}\right)^{\frac{1}{n}} > 0, \end{aligned}$$

which gives $(1 - \frac{1}{n}\frac{\partial \bar{w}}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geqslant C > 0$. As we also have $\frac{\partial \bar{w}}{\partial t} \geqslant \frac{\partial w}{\partial t} - C \geqslant \frac{\partial u}{\partial t} - C$, we can conclude

$$(C - \frac{\partial u}{\partial t})^n \cdot e^{\frac{\partial u}{\partial t}} \geqslant C > 0$$

with $C - \frac{\partial u}{\partial t} > 0$, and so $\frac{\partial u}{\partial t} \geqslant -C$ at that point. Thus $\frac{\partial w}{\partial t} \geqslant -C$ at that point.

It's clear that $|\bar{w}| \leqslant C$ from the estimates for $v$ before since we do not assume $\omega_\infty > 0$ there. Hence we see $\frac{\partial w}{\partial t} + \bar{w} \geqslant -C$ globally, which gives the uniform lower bound for $\frac{\partial w}{\partial t}$ and so for $\frac{\partial u}{\partial t}$ as they differ only by a bounded term $e^{-t}f$.

The standard argument for uniform higher derivatives goes through in the same way as before.

**Remark 3.2.** *The main philosophy of the above argument is that a choice of representative in a cohomology class boils down to terms like $f$ or $e^{-t}f$ for a smooth function $f$ over $X$ which is clearly controlled along the flow and so should not bring any trouble. This observation is also useful when trying to apply Yau's Laplacian estimate (as in [1]) to get second order derivative control for our concern.*

Up to now, we have got the global existence of the flow and the uniformity of the estimates allows us to apply the classic Ascoli-Azela's Theorem to get convergence for sequences of metrics along the flow. Just as in Cao's work, we should expect convergence in a much stronger sense, which is the topic for the next subsection.

### 3.2 Flow Convergence

The argument in [1] for convergence which makes use of Li-Yau's Harnack Inequality should be easy to get carried through here since for the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_{\widetilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle,$$

$\langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle$ has been uniformly controlled, and so the extra term in comparison to the equation in [1] is exponentially decreasing. Let's illustrate some main points when adjusting his argument to the current situation below.

For the exponential decreasing of the oscillation of $\frac{\partial u}{\partial t}$, we'll use Cao's argument for the following family of auxiliary functions

$$(\frac{\partial}{\partial t} - \Delta_{\widetilde{\omega}_t})\phi_{T_0} = 0, \quad \phi_{T_0}(T_0, \cdot) = \frac{\partial u}{\partial t}(T_0, \cdot)$$

over $[T_0, \infty) \times X$ where $T_0 \in [0, \infty)$. As we have already obtained the uniform estimates for $\frac{\partial u}{\partial t}$ and $\widetilde{\omega}_t$ for all time, using Li-Yau's Harnack Inequality as in [1], we have

$$\text{osc}_X \phi_{T_0}(t) \leqslant Ce^{-a(t-T_0)}, \quad t \in [T_0, \infty)$$

where the positive constants are uniform for all $T_0$. Also, from the uniform estimates along the flow, we have

$$(\frac{\partial}{\partial t} - \Delta_{\widetilde{\omega}_t})\left(\frac{\partial u}{\partial t} + Ce^{-t}\right) \leqslant 0,$$

$$(\frac{\partial}{\partial t} - \Delta_{\widetilde{\omega}_t})\left(\frac{\partial u}{\partial t} - Ce^{-t}\right) \geqslant 0.$$

Meanwhile, one has the following equations from above

$$(\frac{\partial}{\partial t} - \Delta_{\widetilde{\omega}_t})(\phi_{T_0} + Ce^{-T_0}) = 0,$$

$$(\frac{\partial}{\partial t} - \Delta_{\widetilde{\omega}_t})(\phi_{T_0} - Ce^{-T_0}) = 0.$$

Comparing them and applying maximum principle, we get the decreasing of

$$\max_X(\frac{\partial u}{\partial t} + Ce^{-t} - \phi_{T_0} - Ce^{-T_0})$$

and the increasing of

$$\min_X(\frac{\partial u}{\partial t} - Ce^{-t} - \phi_{T_0} + Ce^{-T_0})$$

as time increases (starting from $t = T_0$).

The values at $t = T_0$ for both quantities are 0, and so we have for $t \in [T_0, \infty)$,

$$\frac{\partial u}{\partial t} \leqslant \phi_{T_0} + Ce^{-T_0} - Ce^{-t},$$

$$\frac{\partial u}{\partial t} \geqslant \phi_{T_0} - Ce^{-T_0} + Ce^{-t}.$$

Hence $\mathrm{osc}_X \frac{\partial u}{\partial t} \leqslant \mathrm{osc}_X \phi_{T_0} + Ce^{-T_0}$ for $t \in [T_0, \infty)$. Using the result for $\phi_{T_0}$ stated above, we have $\mathrm{osc}_X \frac{\partial u}{\partial t} \leqslant Ce^{-a(t-T_0)} + Ce^{-T_0}$ for $t \geqslant T_0$. Taking $t = 2T_0$ and noticing this is uniform for all $T_0$, we finally arrive at

$$\mathrm{osc}_X \frac{\partial u}{\partial t} \leqslant Ce^{-at}$$

for all time. Here the $a$ should differ from the previous one, but it's still a positive constant.

This is exactly one of the essential results needed to draw the convergence for $t \to \infty$ as in [1].

Set $\psi = \frac{\partial u}{\partial t} - \frac{\int_X \frac{\partial u}{\partial t} \widetilde{\omega}_t^n}{\int_X \widetilde{\omega}_t^n}$. Clearly its difference from $\frac{\partial v}{\partial t}$ is controlled by $Ce^{-at}$, but it is more convenient for the following consideration.

We can have similar computation as in [1], for the energy,

$$G(t) = \int_X \psi^2 \widetilde{\omega}_t^n,$$

to derive a differential inequality. There are more terms coming out, but they will all be terms controlled by $Ce^{-t}$ from the uniform estimates along the flow. Notice that though the volume is also changing along the flow, the variation is also well under control. In all, we get

$$\frac{dG(t)}{dt} \leqslant -CG(t) + Ce^{-t}$$

for large $t$. The reason to get only for large $t$ is that we need the smallness of $\psi$ from the control of oscillator of $\frac{\partial u}{\partial t}$. From this differential inequality, we can still conclude the exponential decaying of $G(t)$. [3]

The final computation and argument in [1] to derive the $L^1$ convergence of the normalized metric potential can be carried through line by line in sight of the above controls. One can justify the exponential convergence of the flow with little extra effort (just as what is carried out carefully in [17]).

**Remark 3.3.** *In this situation, we now have a somewhat natural flow from one Ricci-flat metric to another Ricci-flat metric (of course in different Kähler classes) when* $c_1(X) = 0$. *One just needs to choose* $\Omega$ *such that* $\mathrm{Ric}(\Omega) = 0$ *for the flow (1.1).*

[3] Obviously, the exponential decaying of $G$ can be deduced from the decaying of the oscillation of $\frac{\partial u}{\partial t}$ in a more direct manner, but the argument above applying the differential inequality is more delicate and can easily be adjusted for higher order Sobolev estimates.

# 4 Main Interest: Degenerate Case

Of course, our main interest is when $[\omega_\infty]$ is degenerate as Kähler class. In [17], we have discussed the corresponding Monge-Ampère equation using other perturbations to set up method of continuity. Now we want to see whether the modified flow (1.1) can be applied to construct a solution for (1.3) as the limiting equation.

Now one can assume the manifold $X$ to be projective to get into algebraic geometry context for the notions of semi-ample and big, or one can use the setting in [16].

So far, we have the existence of the smooth flow as long as $[\omega_t]$ remains Kähler. There are two cases, i.e. the flow exists up to infinite time and up to finite time. We discuss them separately and finish the proof of Theorem 1.2.

## 4.1 Infinite Time Case

We prove Statement 2) in Theorem 1.2 in this part. Let's assume that $[\omega_\infty]$ is semi-ample and big. We still have the $L^\infty$ bound of the normalized metric potential $v$ as before using the result non-degenerate Monge-Ampère equation (from [2] and [16]). Now (3.1) can be modified to be

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+v-\epsilon\log|\sigma|^2\right) = \Delta_{\widetilde{\omega}_t}\left(\frac{\partial u}{\partial t}+v-\epsilon\log|\sigma|^2\right) - n + \frac{\partial v}{\partial t} + \\ + \langle\widetilde{\omega}_t, \omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle \end{aligned} \tag{4.1}$$

with $\omega_\infty + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0$, where constant $\epsilon > 0$ can be as close to 0 as possible, $\sigma$ is the defining section of a holomorphic line bundle $E$ mentioned Lemma 4.1 below and $|\cdot|$ is a properly chosen Hermitian metric for this line bundle.

Such introduction of a singular term, as far as I know, was initiated by H. Tsuji in [13], which gives a natural and simple description of the algebraic geometry fact listed below in the analysis of related PDEs.

The following lemmas are classic results in algebraic geometry (for example, see in [6] and [7] for related discussion). Lemma 4.2 is called Kodaira's Lemma (as in [14]) and will be applied for the finite time case later. The way to translate these results to the analytic setting as above is very standard fact in complex geometry (as in [4]).

**Lemma 4.1.** *Let $L$ be a divisor in a projective manifold $X$. If $L$ is nef. and big, then there are an effective divisor $E$ and a constant $a > 0$ such that $L - \epsilon E$ is Kähler for any $\epsilon \in (0, a)$.*

**Lemma 4.2.** *Let $L$ be a divisor in a projective manifold $X$. If $L$ is big, then there are an effective divisor $E$ and constants $a, b > 0$ such that $L - \epsilon E$ is Kähler for $\epsilon \in (a, b)$.*

Nef. means numerically effective, which means non-negative intersection with curves in $X$, or on the boundary of the Kähler cone for $X$.

Similar argument as for (3.1) would give a degenerate lower bound as

$$\frac{\partial u}{\partial t} \geqslant -C + \epsilon \log|\sigma|^2,$$

where the positive constant $C$ below might depend on the other positive constant $\epsilon$, but this won't bring up any confusion later. Let's briefly go through the argument below.

One still has $\frac{\partial v}{\partial t} \geqslant \frac{\partial u}{\partial t} - C$. Then by considering the minimum value point of the term whose evolution is described by (4.1), we know $\frac{\partial u}{\partial t}$ could not be too small at that point using the same way to deduce a contradiction as in Subsection 3.1. That would give the degenerate lower bound above.

Now the degenerate second and higher order estimates would be obtained as before. More specifically, for the second order estimate, one considers the following equation

$$(\omega_t + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 + \sqrt{-1}\partial\bar{\partial}(v - \epsilon\log|\sigma|^2))^n = e^{\frac{\partial u}{\partial t}}\Omega.$$

Applying Yau's computation in [15] and using degenerate maximum principle argument in the same way as in [11], we can get the degenerate Laplacian bound [4]. Combining with the degenerate control for volume, we have achieved degenerate bound for metrics along the flow, i.e. for some constant $\alpha > 0$,

$$C|\sigma|^{\alpha}\omega_0 \leqslant \widetilde{\omega}_t \leqslant C|\sigma|^{-\alpha}\omega_0.$$

The treatment for higher derivatives is then standard. Some detail would be provided in the last section.

The above argument can be done for any $\sigma$ for such a divisor $E$ and so the estimates are uniform away from the intersection of all those $\{\sigma = 0\}$ which is called the **stable base locus set** of $[\omega_\infty]$.

**Remark 4.3.** *There is a big difference from the situation in [11] which needs to be pointed out. The metric potential along the flow can be bounded (though in a degenerate way) simply from the flow argument there, but we can not do that here at this moment. The bound for (normalized) metric potential is achieved using results from pluripotential theory. That's why we need semi-ample (not just nef.) for the current case.*

Though our estimates are uniform for all time now, which gives sequence convergence for the flow, there is still this big issue about convergence along the flow which is crucial to describe the limit itself. As discussed in the baby

[4] In fact, one only needs the uniform upper bound of $\frac{\partial u}{\partial t}$ to carry through the Laplacian estimate by noticing the dominance of $e^{-\frac{1}{n}\frac{\partial u}{\partial t}}$ over $-\frac{\partial u}{\partial t}$ when $\frac{\partial u}{\partial t}$ is small.

version, the counterpart in [1] makes use of Li-Yau's Harnack Inequality, which can be applied for the non-degenerate case as shown in Subsection 3.2. In the current situation, it is very different. It seems that new method needs to be introduced for this purpose. Let's make the following conjecture about the flow convergence which is naturally be expected to be true.

**Conjecture 4.4.** *For $[\omega_\infty]$ semi-ample and big, as $t \to \infty$, The solution for (1.1) converges weakly over $X$ and locally smoothly out of the stable base locus set of $[\omega_\infty]$ to the unique (bounded) solution of the limiting degenerate complex Monge-Ampère equation (1.3).*

In the following, we prove the rest part of Statement 2) of Theorem 1.2, i.e. the volume form has uniform lower bound for all time. This might be helpful for proving the above conjecture. Moreover, it is a nice application of a similar result for the following more canonical Kähler-Ricci flow discussed in [11].

Set $\widehat{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}\phi$. At the level of potential, consider the flow

$$\frac{\partial \phi}{\partial t} = \log \frac{\widehat{\omega}_t^n}{\Omega} - \phi, \quad \phi(0, \cdot) = 0, \tag{4.2}$$

whose corresponding flow at the level of metric is the following,

$$\frac{\partial \widehat{\omega}_t}{\partial t} = -\mathrm{Ric}(\widehat{\omega}_t) + \mathrm{Ric}(\Omega) - \widehat{\omega}_t + \omega_\infty, \quad \widehat{\omega}_0 = \omega_0.$$

At this moment, we have $[\omega_\infty]$ is big and semi-ample, so as discussed in [11] and [17], the following estimates are available [5]

$$|\phi| \leqslant C, \quad |\frac{\partial \phi}{\partial t}| \leqslant C,$$

which give a lower bound for the volume form, $\widehat{\omega}_t^n$ for all time and we are looking for a similar thing for $\widetilde{\omega}_t^n$.

**Remark 4.5.** *The uniform volume lower bound is interesting because the class $[\omega_\infty]$ is not Kähler, but somehow we have that the cohomology statement, $[\omega_\infty]^n > 0$, from semi-ample and big assumption also makes sense in a point-wise fashion. Of course, method of continuity using other perturbation also indicates this phenomenon.*

Let's recall the following equations already appeared in this note. $v$ is the normalization of $u$.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - \langle \widetilde{\omega}_t, e^{-t}(\omega_0 - \omega_\infty)\rangle,$$

[5]The lower bound of $\frac{\partial \phi}{\partial t}$ is in [17] and [18]. It makes use of the essential decreasing of the volume form and the fact that for infinite time limit, the derivative of potential has to go to 0 (in the regular part).

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t} + v\right) - n + \langle \widetilde{\omega}_t, \omega_\infty \rangle.$$

For some constant $T_1 > 0$, multiplying the first equation with $e^{-T_1}$ and taking the difference of these two, we have

$$\frac{\partial}{\partial t}\left((1 - e^{-T_1})\frac{\partial u}{\partial t}\right) = \Delta\left((1 - e^{-T_1})\frac{\partial u}{\partial t} + v\right) - n + \langle \widetilde{\omega}_t, \omega_{t+T_1} \rangle.$$

Using the solution for the other flow, $\phi$, this equation can be transformed as follows

$$\frac{\partial}{\partial t}\left((1 - e^{-T_1})\frac{\partial u}{\partial t}\right) = \Delta\left((1 - e^{-T_1})\frac{\partial u}{\partial t} + v - \phi(t + T_1)\right) - n + \langle \widetilde{\omega}_t, \widehat{\omega}_{t+T_1} \rangle$$

with some emphasize on the time parameter. The Laplacian is still with respect to $\widetilde{\omega}_t$. Using the principle as discussed above, we modify it to be

$$\begin{aligned}\frac{\partial}{\partial t}\left((1 - e^{-T_1})\frac{\partial u}{\partial t} + v - \phi(t + T_1)\right) = \Delta\left((1 - e^{-T_1})\frac{\partial u}{\partial t} + v - \phi(t + T_1)\right) - n + \\ + \frac{\partial v}{\partial t} - \frac{\partial \phi(t + T_1)}{\partial t} + \langle \widetilde{\omega}_t, \widehat{\omega}_{t+T_1} \rangle.\end{aligned}$$

Define $B := (1-e^{-T_1})\frac{\partial u}{\partial t} + v - \phi(t+T_1)$. Using the following known estimates

$$\frac{\partial v}{\partial t} \geqslant \frac{\partial u}{\partial t} - C, \quad \frac{\partial \phi(t + T_1)}{\partial t} \geqslant -C, \quad \widehat{\omega}_t^n \geqslant C\Omega,$$

one arrives at

$$\frac{\partial B}{\partial t} \geqslant \Delta B + \frac{\partial u}{\partial t} - C + C \cdot e^{-\frac{1}{n}\frac{\partial u}{\partial t}}.$$

By maximum principle argument, one can conclude the lower bound for $B$, and so for $\frac{\partial u}{\partial t}$, which gives the uniform lower bound for the volume form $\widetilde{\omega}_t^n$.

**Remark 4.6.** *In this above argument, the translation of time by $T_1$ is crucial which makes the infinite time situation special. For example, we do not have uniform volume lower bound for finite time case for either flows (at least at this moment and in fact we do not expect this to happen [6]).*

## 4.2 Finite Time Limit

We now prove Statement 3) of Theorem 1.2. $[\omega_\infty]$ is now big but not in the closure of the Kähler cone. More specifically, the flow exists smoothly up to some finite time $T$. We also require $[\omega_T]$, which is clearly nef. and big, to be semi-ample. [7]

[6] This has been justified in more recent work [19].

[7] This is not such a restrictive assumption as it is the case, when $[\omega_\infty] = K_X$ and $[\omega_0]$ is rational, from classic algebraic geometry result.

The argument used in the previous subsection can be carried through for most part. More precisely, those degenerate estimates would still be available, though the $\epsilon$ can't be too small now in sight of the difference between Lemmas 4.2 and 4.1.

The advantage about finite time is that the metric potential $u$ is (degenerately) bounded by itself (without normalization) from the bound for its time derivative, $\frac{\partial u}{\partial t}$, and it also decreases after controllable normalization (by $-Ct$) in sight of the uniform upper bound for $\frac{\partial u}{\partial t}$. So as in [11], the (local) convergence for $t \to T$ is achieved.

This local convergence would be out of the stable base locus set of $[\omega_\infty]$ as explained in the previous subsection. Clearly, it would be more satisfying to replace the class $[\omega_\infty]$ by $[\omega_T]$. This is indeed the case as presented below. [8] The key is to introduce a virtual time.

We begin with searching for the crucial estimate for $\frac{\partial u}{\partial t}$. Recall the following two equations,

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+v\right)=\Delta_{\widetilde{\omega}_t}\left(\frac{\partial u}{\partial t}+v\right)-n+\frac{\partial v}{\partial t}+\langle\widetilde{\omega}_t,\omega_\infty\rangle,$$

$$\frac{\partial}{\partial t}\left(e^{t-T}\frac{\partial u}{\partial t}\right)=\Delta_{\widetilde{\omega}_t}\left(e^{t-T}\frac{\partial u}{\partial t}\right)+e^{t-T}\frac{\partial u}{\partial t}-e^{-T}\langle\widetilde{\omega}_t,\omega_0-\omega_\infty\rangle.$$

Take the difference to get

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t}+v\right)=\Delta_{\widetilde{\omega}_t}((1-e^{t-T})\frac{\partial u}{\partial t}+v)-n+\frac{\partial v}{\partial t}-e^{t-T}\frac{\partial u}{\partial t}+\langle\widetilde{\omega}_t,\omega_T\rangle.$$

As usual, take $\sigma$ for $[\omega_T]$ such that $\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2>0$ for some $\epsilon>o$ small enough. Using this to perturb the above equation, one arrives at

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t}+v-\epsilon\log|\sigma|^2\right)=\Delta_{\widetilde{\omega}_t}\left((1-e^{t-T})\frac{\partial u}{\partial t}+v-\epsilon\log|\sigma|^2\right)-n+ \\ +\frac{\partial v}{\partial t}-e^{t-T}\frac{\partial u}{\partial t}+\langle\widetilde{\omega}_t,\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle.$$

Define $D:=(1-e^{t-T})\frac{\partial u}{\partial t}+v-\epsilon\log|\sigma|^2$. We have, out of $\{\sigma=0\}$,

$$\frac{\partial D}{\partial t}\geqslant\Delta_{\widetilde{\omega}_t}D-C+(1-e^{t-T})\frac{\partial u}{\partial t}+Ce^{-\frac{1}{n}\frac{\partial u}{\partial t}}.$$

Now apply maximum principle argument. Recall that $t$ is in a finite interval $[0,T)$. At the (local in time) minimum value point of $D$, assuming it is not at the initial time, which is clearly out of $\{\sigma=0\}$, we have $\frac{\partial u}{\partial t}$ can not be too small (negative). Thus $D$ can not be too small there, i.e.

$$(1-e^{t-T})\frac{\partial u}{\partial t}+v-\epsilon\log|\sigma|^2\geqslant -C.$$

[8] One can improve the corresponding result in [11] a little bit in exactly the same way.

There is a problem coming from the fact that $1 - e^{t-T}$ would go to 0 as $t \to T$, which can be solved by using a "virtual" time $T_\epsilon > T$ still satisfying $\omega_{T_\epsilon} + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0$ for some fixed $\epsilon > 0$. Clearly, the same argument gives

$$(1 - e^{t-T_\epsilon})\frac{\partial u}{\partial t} + v - \epsilon\log|\sigma|^2 \geqslant -C,$$

in other words,

$$\frac{\partial u}{\partial t} \geqslant -C_\epsilon + C_\epsilon\log|\sigma|^2.$$

Notice that now the $\sigma$ is for the class $[\omega_T]$ in Lemma 4.2.

Up to now, we have achieved the following controls

$$|u| \leqslant C, \quad -C_\epsilon + C_\epsilon \log|\sigma|^2 \leqslant \frac{\partial u}{\partial t} \leqslant C.$$

Following the exact argument in the previous subsection, one can have degenerate estimates for $\widetilde{\omega}_t$ and its derivatives away from $\{\sigma = 0\}$, and so away from the stable base locus set of $[\omega_T]$.

Finally, we can conclude the local convergence because one can consider the potential $u - Ct$ which decreases for $C$ sufficiently positive. In finite time case, $|u - Ct| \leqslant C$ and the convergence as $t \to T$ locally in $C^\infty$ topology away from the stable base locus set of $[\omega_T]$ using the Interpolation Inequalities as in [3].

One can also apply maximum principle argument in another flavor just as what is done in Section 3. It has to be done in a more careful way as follows.

Recall the differential inequality for $D = (1 - e^{t-T})\frac{\partial u}{\partial t} + v - \epsilon\log|\sigma|^2$,

$$\frac{\partial D}{\partial t} \geqslant \Delta_{\widetilde{\omega}_t} D - C + (1 - e^{t-T})\frac{\partial u}{\partial t} + Ce^{-\frac{1}{n}\frac{\partial u}{\partial t}}.$$

In the following, we control the last two terms by functions on $D$ with the desirable direction.

The last term on the right can be treated with ease because

$$e^{t-T}\frac{\partial u}{\partial t} - v + \epsilon\log|\sigma|^2 \leqslant C,$$

which gives $-\frac{\partial u}{\partial t} \geqslant -D + C$. But the other term makes the situation trickier since $-\epsilon\log|\sigma|^2$ can not be bounded from above over $X$ by any constant. We proceed as follows.

Define $H := (1 - e^{t-T})\frac{\partial u}{\partial t} + v$. As $H \geqslant \frac{\partial u}{\partial t} - C$, one has

$$(1 - e^{t-T})\frac{\partial u}{\partial t} + Ce^{-\frac{1}{n}\frac{\partial u}{\partial t}} \geqslant -C + H + Ce^{-\frac{H}{n}}.$$

The function over $H$, $H + Ce^{-\frac{H}{n}}$ would be decreasing with respect to $H$ for small enough $H$ by simple computation, and so for $H$ small enough, i.e.

$(1-e^{t-T})\frac{\partial u}{\partial t}$ sufficiently small, we can change $H$ to $\widetilde{H} := H - \epsilon\log|\sigma|^2 \geqslant H - C$ and draw the conclusion.

Hence we have proved Statement 3) of Theorem 1.2 and finished the proof of the theorem.

# 5 Appendix: Higher Order Derivatives

We provide detailed discussion on the degenerate third and higher order derivative estimates for Kähler-Ricci flow over a closed manifold, $X$. It works for the modified flow (1.1) in this note and others (as the one used in Subsection 4.1 discussed in [11]) with $[\omega_\infty]$ being big.

The flow equation on the potential level is

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \quad u(0,\cdot) = 0, \tag{5.1}$$

or without the $-u$ term on the right hand side, where $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ with $\omega_0$ being the initial Kähler metric, $\omega_\infty$ being a smooth representative for the (formal) infinite limiting class and $\Omega$ being a smooth volume form over $X$. $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ is the solution of the flow at the level of metric.

The class $[\omega_\infty]$ is big and $T \leqslant \infty$ is the time of singularity from cohomology consideration with $[\omega_T]$ being nef. and big. The following estimates have been obtained:

$$\epsilon\log|\sigma|^2 - C_\epsilon \leqslant u \leqslant C, \quad C\log|\sigma|^2 - C \leqslant \frac{\partial u}{\partial t} \leqslant C, \quad \langle\omega_0, \widetilde{\omega}_t\rangle \leqslant C|\sigma|^{-l},$$

where $E = \{\sigma = 0\}$ is a proper chosen devisor such that $[\omega_T] - \epsilon E$ is Kähler, $\sigma$ is a holomorphic section of the line bundle, and $|\cdot|$ is a proper Hermitian metric for the line bundle. Clearly, $1|\sigma|^2$ is a smooth function valued in $[0, C]$.

The higher estimates are discussed briefly to achieve the full local regularity. We begin with the so-called third order estimate a little more carefully below. The computation in [15] is the key for this business. As in [15], define $S := \widetilde{g}^{i\bar{j}}\widetilde{g}^{k\bar{l}}\widetilde{g}^{m\bar{n}}u_{i\bar{l}m}u_{\bar{j}k\bar{n}}$, where the covariant derivative is taken with respect to the uniform background metric. The metric norm used below is with respect to the flow metric. $|\cdot|$ stands for either this norm or Hermitian metric for line bundle $E$, which is not going to bring up any confusion.

If the flow metric control is uniform, then the parabolic version of Yau's computation gives

$$(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})S \geqslant -C \cdot S - C$$

as described in [1] and [17].

In order to adjust this inequality for our situation, one only needs to observe that the flow metric has been controlled (uniformly in time and degenerate in space) as follows

$$|\sigma|^{\beta}\omega_0 \leqslant \widetilde{\omega}_t \leqslant |\sigma|^{-\beta}\omega_0$$

for some (large) constant $\beta > 0$. Then we know by the computation in [15] that

$$|\sigma|^{2N}(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})S \geqslant -C|\sigma|^{2N-\beta} \cdot S - C$$

with $N$ chosen large enough to dominate all the degenerate terms.

Now let's see how the term $|\sigma|^{2N}S$ is acted on by the heat operator along the flow. The only additional part is from the action of $\Delta_{\widetilde{\omega}_t}$. There are two terms. One is clearly $2\mathrm{Re}(\nabla|\sigma|^{2N}, \nabla S)_{\widetilde{\omega}_t}$, the other one is $\Delta_{\widetilde{\omega}_t}|\sigma|^{2N} \cdot S$.

For the first one, $\nabla S = \nabla(|\sigma|^{2N}S \cdot |\sigma|^{-2N}) = |\sigma|^{-2N}\nabla(|\sigma|^{2N}S) - N|\sigma|^{-2}S\nabla|\sigma|^2$.

For the second one,

$$\begin{aligned}
\Delta_{\widetilde{\omega}_t}|\sigma|^{2N} &= \Delta_{\widetilde{\omega}_t}(e^{N\log|\sigma|^2}) = (|\sigma|^{2N}N(\log|\sigma|^2)_{\bar{i}})_i \\
&= N^2|\sigma|^{2N}|\nabla\log|\sigma|^2|^2 + N|\sigma|^{2N}\langle\widetilde{\omega}_t, \sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle \\
&\geqslant -N|\sigma|^{2N}\langle\widetilde{\omega}_t, -\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle.
\end{aligned}$$

Out of $E = \{\sigma = 0\}$, $-\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2$ is nothing but the curvature form of the corresponding line bundle, still denoted by $E$. Using the degenerate metric bound, we have

$$\Delta_{\widetilde{\omega}_t}|\sigma|^{2N} \geqslant -N|\sigma|^{2N}\langle\widetilde{\omega}_t, E\rangle \geqslant -C|\sigma|^{2N-\beta}.$$

**Remark 5.1.** *If we are in semi-ample case, with proper choice of the Hermitian metric for the bundle (the $|\cdot|$ above), we can make sure that $\Phi - \epsilon E > 0$ (since the corresponding cohomology class is Kähler) where $\Phi$ is the pullback of a Kähler metric from the image of the holomorphic map which is constructed from the semi-ample class $[\omega_T]$. In this case, we also have better 0-th order bound. Moreover, for the flow in [11], using parabolic Schwarz estimates as in [10] (or [18]), we can have $\langle\widetilde{\omega}_t, \Phi\rangle \leqslant C$, and so*

$$\Delta_{\widetilde{\omega}_t}|\sigma|^{2N} \geqslant -N|\sigma|^{2N}\langle\widetilde{\omega}_t, E\rangle \geqslant -C|\sigma|^{2N}\langle\widetilde{\omega}_t, \Phi\rangle \geqslant -C|\sigma|^{2N},$$

*which is better. But it not going to make too much difference in the discussion below.*

Now we do computation as follows

$$\begin{aligned}
(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})(|\sigma|^{2N}S) &\geqslant -C|\sigma|^{2N-\beta} \cdot S - C \\
&\quad + 2\mathrm{Re}\left(\nabla|\sigma|^{2N}, |\sigma|^{-2N}\nabla(|\sigma|^{2N}S) - N|\sigma|^{-2}S\nabla|\sigma|^2\right)_{\widetilde{\omega}_t} \\
&= -C|\sigma|^{2N-\beta}S - C + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N}), \nabla(|\sigma|^{2N}S)\right)_{\widetilde{\omega}_t} \\
&\quad - N^2|\sigma|^{2N-4}S|\nabla|\sigma|^2|^2. \\
&\geqslant -C|\sigma|^{2N-2-\beta}S - C + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N}), \nabla(|\sigma|^{2N}S)\right)_{\widetilde{\omega}_t},
\end{aligned}$$

where $|\nabla|\sigma|^2|^2 \leqslant C|\sigma|^{2-\beta}$ is applied for the last step.

Also as in [15], we consider the $\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle$ acted by the heat operator where $\omega_{t,\epsilon}$ is the perturbation for the background form $\omega_t$.

Had the metric control been uniform, we would have

$$(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})(\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle) \geqslant C \cdot S - C.$$

For our case, similar to $S$, we have the following inequality instead

$$|\sigma|^{2N}(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})(\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle) \geqslant C|\sigma|^{2N+\beta} S - C.$$

Exactly the same computation as already done for $S$ gives us

$$\begin{aligned}
&(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})(|\sigma|^{2N}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle) \\
&\geqslant C|\sigma|^{2N+\beta} \cdot S - C - C|\sigma|^{2N-\beta}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle \\
&\quad + 2\mathrm{Re}\left(\nabla|\sigma|^{2N}, |\sigma|^{-2N}\nabla(|\sigma|^{2N}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle) - N|\sigma|^{-2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle\nabla|\sigma|^2\right)_{\widetilde{\omega}_t} \\
&\geqslant C|\sigma|^{2N+\beta} S - C + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N}), \nabla(|\sigma|^{2N}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle)\right)_{\widetilde{\omega}_t} \\
&\quad - C|\sigma|^{2N-2-\beta}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle.
\end{aligned}$$

Properly choosing large constants $N_1 > N_2 > 0$ and $C$'s, we have

$$\begin{aligned}
&(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t})(|\sigma|^{2N_1} S + C|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle) \\
&\geqslant C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-\beta}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N_1}), \nabla(|\sigma|^{2N_1} S)\right)_{\widetilde{\omega}_t} \\
&\quad + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N_2}), \nabla(C|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle)\right)_{\widetilde{\omega}_t}.
\end{aligned}$$

For the choice of $N_1$ and $N_2$, we only need $2N_1 - 2 - \beta \geqslant 2N_2 + \beta$ at this moment, but they will be fixed later to be large in principle.

Now apply maximum principle argument. At the (local in time) maximum value point, for $|\sigma|^{2N_1} S + C|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle$, which clearly exists out of $\{\sigma = 0\}$ and is assumed not to be at the initial time, one has

$$\nabla(|\sigma|^{2N_1} S) = -\nabla(C|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle),$$

$$\begin{aligned}
0 &\geqslant C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-\beta}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle \\
&\quad + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N_1}), \nabla(|\sigma|^{2N_1} S)\right)_{\widetilde{\omega}_t} + 2\mathrm{Re}\left(\nabla(\log|\sigma|^{2N_2}), \nabla(C|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle)\right)_{\widetilde{\omega}_t} \\
&= C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-\beta}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle \\
&\quad + 2\mathrm{Re}\left(-\nabla(\log|\sigma|^{2N_1}) + \nabla(\log|\sigma|^{2N_2}), \nabla(C|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle)\right)_{\widetilde{\omega}_t} \\
&\geqslant C|\sigma|^{2N_2+\beta} \cdot S - C - C|\left(\nabla(\log|\sigma|^2), \nabla(|\sigma|^{2N_2}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle)\right)_{\widetilde{\omega}_t}|.
\end{aligned}$$

For the last term above, we have

$$
\begin{aligned}
&\left| \left(\nabla(\log|\sigma|^2), \nabla(|\sigma|^{2N_2}\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle)\right)_{\widetilde{\omega}_t} \right| \\
&= \left| \left(|\sigma|^{-2}\nabla|\sigma|^2, N_2|\sigma|^{2N_2-2}\nabla|\sigma|^2\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle + |\sigma|^{2N_2}\nabla\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle\right)_{\widetilde{\omega}_t} \right| \\
&\leqslant \left| \left(|\sigma|^{-2}\nabla|\sigma|^2, N_2|\sigma|^{2N_2-2}\nabla|\sigma|^2\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle\right)_{\widetilde{\omega}_t} \right| + \left| \left(|\sigma|^{-2}\nabla|\sigma|^2, |\sigma|^{2N_2}\nabla\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle\right)_{\widetilde{\omega}_t} \right| \\
&\leqslant C|\sigma|^{2N_2-2-2+1+1-\beta-\beta} + |\sigma|^{2N_2-2}|\nabla|\sigma|^2| \cdot |\nabla\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle| \\
&\leqslant C|\sigma|^{2N_2-2-2\beta} + C|\sigma|^{2N_2-2+1-\frac{\beta}{2}} \cdot |\nabla\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle|.
\end{aligned}
$$

Now one observes that

$$|\nabla\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle| = |\nabla(F + \Delta_{\omega_{t,\epsilon}} u)| \leqslant |\nabla F| + |\nabla\Delta_{\omega_{t,\epsilon}}(u)| \leqslant |\sigma|^{-\frac{\beta}{2}} + C|\sigma|^{-2\beta}S^{\frac{1}{2}}$$

with $F$ being a well controlled function. Combining all this, we have at that maximum point,

$$0 \geqslant C|\sigma|^{2N_2+\beta} \cdot S - C - C|\sigma|^{2N_2-2-2\beta} - C|\sigma|^{2N_2-1-\beta} - C|\sigma|^{2N_2-1-\frac{5\beta}{2}} \cdot S^{\frac{1}{2}}.$$

For large enough $N_2$, we have

$$0 \geqslant |\sigma|^{2N_2+\beta} \cdot S - C(|\sigma|^{2N_2+\beta} \cdot S)^{\frac{1}{2}} - C,$$

and so $|\sigma|^{2N_2+\beta} \cdot S \leqslant C$.

For $N_1$ even larger, we have uniform upper bound for $|\sigma|^{2N_1}S+C|\sigma|^{2N_2}\langle \omega_{t,\epsilon}, \widetilde{\omega}_t\rangle$ at that point, and so it is true globally which provides the desirable bound

$$S \leqslant C|\sigma|^{-2N_1}.$$

So far, we have achieved $C^{2,\alpha}$ bound for the scalar potential flow solution. Standard parabolic version of Schauder estimates can then be carried though to provide all the local higher order derivative estimates by considering the space derivative of (5.1).

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