Cauchy identity for double Schur functions

and its applications

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Symmetric functions

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Denote by Λ_n the ring of symmetric polynomials in $x_1, \dots, x_n$

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Consider the evaluation maps

$$\varphi_n : \Lambda_n^{\leqslant k} \to \Lambda_{n-1}^{\leqslant k}, \qquad P(x_1, \dots, x_n) \mapsto P(x_1, \dots, x_{n-1}, a_n)$$

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and the corresponding inverse limit

$$\Lambda^{\leqslant k} = \varprojlim \Lambda_n^{\leqslant k}, \qquad n \to \infty.$$

The elements of $\Lambda^{\leqslant k}$ are sequences

$$P = (P_0, P_1, P_2, \ldots), \qquad P_n \in \Lambda_n^{\leqslant k}$$

such that

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Then the union $\quad \Lambda^a = \bigcup_{k \geqslant 0} \Lambda^{\leqslant k} \quad$ is a ring with the product

$$PQ = (P_0 Q_0, P_1 Q_1, P_2 Q_2, \dots), \qquad Q = (Q_0, Q_1, Q_2, \dots)$$

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Examples. We have

$$\varphi_n : \sum_{i=1}^{n}(x_i^k - a_i^k) \mapsto \sum_{i=1}^{n-1}(x_i^k - a_i^k)$$

hence

$$p_k(x \,\|\, a) = \sum_{i=1}^{\infty}(x_i^k - a_i^k) \in \Lambda^a,$$

the power sums symmetric function.

Proposition. The following are elements of Λ^a:

$$e_k(x \,\|\, a) = \sum_{i_1 > \cdots > i_k} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k + k - 1}),$$

$$h_k(x \,\|\, a) = \sum_{i_1 \geqslant \cdots \geqslant i_k} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k - k + 1}).$$

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Proof.

$$1 + \sum_{k=1}^{\infty} \frac{e_k(x\,\|\,a)\,t^k}{(1+a_1\,t)\dots(1+a_k\,t)} = \prod_{i=1}^{\infty} \frac{1+x_i\,t}{1+a_i\,t},$$

$$1 + \sum_{k=1}^{\infty} \frac{h_k(x\,\|\,a)\,t^k}{(1-a_0\,t)\dots(1-a_{-k+1}\,t)} = \prod_{i=1}^{\infty} \frac{1-a_i\,t}{1-x_i\,t}.$$

A diagram (or partition) is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers λ_i such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, depicted as an array of unit boxes.

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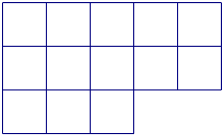
Example. The diagram $\lambda = (5, 5, 3)$ is



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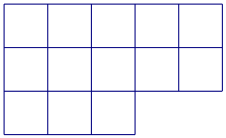

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The number of nonzero rows is its length, denoted $\ell(\lambda)$.

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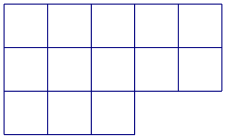

$|\lambda| = 13$ $\qquad$ $\ell(\lambda) = 3$

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A reverse λ-tableau T is obtained by filling in the boxes of λ with the numbers $1, 2, \ldots$ in such a way that the entries weakly decrease along the rows and strictly decrease down the columns. If $\alpha = (i, j)$ is a box of λ we let $T(\alpha) = T(i, j)$ denote the entry of T in the box α.

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Example. A reverse λ-tableau for $\lambda = (5, 5, 3)$:

8	8	4	2	2
4	3	2	1	1
2	1	1		

Double Schur functions

For any diagram λ define the (double) Schur function by

$$s_\lambda(x\,\|\,a) = \sum_T \prod_{\alpha\in\lambda} (x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}),$$

summed over the reverse λ-tableaux T,

$c(\alpha) = j - i$ is the content of the box $\alpha = (i, j)$.

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Proposition.

$s_\lambda(x\,\|\,a) \in \Lambda^a \quad$ for all diagrams λ.

Example. For $\lambda = (2,1)$ the reverse tableaux are

$$\begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array} \quad \text{with} \quad i \geqslant j \quad \text{and} \quad i > k$$

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Hence

$$s_{(2,1)}(x \,\|\, a) = \sum_{i \geqslant j,\ i > k} (x_i - a_i)(x_j - a_{j-1})(x_k - a_{k+1}).$$

We have $s_{(k)}(x \,\|\, a) = h_k(x \,\|\, a), \qquad s_{(1^k)}(x \,\|\, a) = e_k(x \,\|\, a).$

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Hence,

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However, for $n = 1, 2, \dots$ the polynomials

$$\sum_{i=1}^{n}(x_i - a_{i-1})$$

are not consistent with respect to the evaluation maps $x_n := a_n$.

For any partition $\lambda = (\lambda_1, \dots, \lambda_n)$ set

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Theorem. Each of the sets $\{p_\lambda(x \,\|\, a)\}$, $\{e_\lambda(x \,\|\, a)\}$, $\{h_\lambda(x \,\|\, a)\}$ and $\{s_\lambda(x \,\|\, a)\}$, parameterized by the set $\mathcal{P}$ of all partitions λ, forms a basis of Λ^a over $\mathbb{Q}[a]$.

Under the specialization $a = (0)$ the ring Λ^a becomes the ring Λ of symmetric functions. It admits six distinguished bases, parameterized by partitions:

$\{p_\lambda(x)\}$, $\{e_\lambda(x)\}$, $\{h_\lambda(x)\}$, $\{s_\lambda(x)\}$, $\{m_\lambda(x)\}$ and $\{f_\lambda(x)\}$.

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$$m_\lambda(x) = \sum x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(n)}^{\lambda_n},$$

summed over permutations σ of the x_i which give distinct monomials.

There is an involution $\omega : \Lambda \to \Lambda$ such that

$$\omega : e_\lambda(x) \mapsto h_\lambda(x), \qquad h_\lambda(x) \mapsto e_\lambda(x),$$

$$m_\lambda(x) \mapsto f_\lambda(x), \qquad f_\lambda(x) \mapsto m_\lambda(x),$$

$$p_\lambda(x) \mapsto \varepsilon_\lambda\, p_\lambda(x), \qquad \varepsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)},$$

$$s_\lambda(x) \mapsto s_{\lambda'}(x),$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ and λ'_i is the number of boxes in column i of λ.

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Answers.

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Answers.

- Yes.
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- Work in progress for $\{s_\lambda(x \,\|\, a)\}$ and $\{h_\lambda(x \,\|\, a)\}$ (Alex Fun).

Main instrument: an analogue of the classical Cauchy identity

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}} s_\lambda(x)\, s_\lambda(y),$$

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and

$$\prod_{i,j} \frac{1}{1 - x_i\, y_j} = \sum_{\lambda \in \mathcal{P}} h_\lambda(x)\, m_\lambda(y) = \sum_{\lambda \in \mathcal{P}} e_\lambda(x)\, f_\lambda(y).$$

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Theorem. We have the expansion

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where $s_\lambda(y, a)$ is the dual Schur function,

$$s_\lambda(y, a) = \sum_{T} \prod_{\alpha \in \lambda} Y_{T(\alpha)}(a_{-c(\alpha)+1}, a_{-c(\alpha)}),$$

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$$Y_i(a, b) = \frac{y_i\,(1 - a\, y_{i-1}) \dots (1 - a\, y_1)}{(1 - b\, y_i) \dots (1 - b\, y_1)}.$$

Examples.

$$s_{(1)}(y,a) = \sum_i Y_i(a_1,a_0) = \sum_i \frac{y_i\,(1-a_1\,y_{i-1})\ldots(1-a_1\,y_1)}{(1-a_0\,y_i)\ldots(1-a_0\,y_1)}.$$

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The dual symmetric functions $s_\lambda(y,a)$, $e_\lambda(y,a)$, $h_\lambda(y,a)$, $m_\lambda(y,a)$ are elements of the ring $\widehat{\Lambda}^a$ of formal series of the (usual) symmetric functions in y with coefficients in $\mathbb{Q}[a]$,

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$$\widehat{\Lambda}^a = \Big\{ \sum_{\lambda \in \mathcal{P}} c_\lambda(a)\, m_\lambda(y) \mid c_\lambda(a) \in \mathbb{Q}[a] \Big\}.$$

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Example.

$$s_{(1)}(y,a) = \sum_{k=1}^{\infty} (a_0 - a_1)^{k-1}\, m_{(1^k)}(y,a)$$

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$$s_{(1)}(y,a) = \sum_{k=1}^{\infty} (a_0 - a_1)^{k-1}\, m_{(1^k)}(y,a)$$

$$= \sum_{\alpha,\beta\geqslant 0} (-1)^\beta a_0^\alpha\, a_1^\beta\, s_{(\alpha|\beta)}(y),$$

where $(\alpha|\beta)$ is the hook diagram with arm α and leg β.

Introduce the bases $m_\lambda(x \,\|\, a)$ and $f_\lambda(x \,\|\, a)$ of Λ^a by

$$\prod_{i,j} \frac{1 - a_i\, y_j}{1 - x_i\, y_j} = \sum_{\lambda \in \mathcal{P}} m_\lambda(x \,\|\, a)\, h_\lambda(y, a) = \sum_{\lambda \in \mathcal{P}} f_\lambda(x \,\|\, a)\, e_\lambda(y, a).$$

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Pairing between Λ^a and $\widehat{\Lambda}^a$

Define the bilinear pairing

$$\langle\ ,\ \rangle : \Lambda^a \otimes \widehat{\Lambda}^a \to \mathbb{Q}[a], \qquad \big\langle h_\lambda(x \,\|\, a), m_\mu(y, a)\big\rangle = \delta_{\lambda\mu}.$$

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Then the Cauchy identity implies

$$\big\langle s_\lambda(x \,\|\, a), s_\mu(y, a)\big\rangle = \delta_{\lambda\mu},$$

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The skew Schur functions in Λ^a and $\widehat{\Lambda}^a$ can now be defined

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$$\big\langle s_{\nu/\mu}(x\,\|\,a),\ s_\lambda(y,a)\big\rangle = \big\langle s_\nu(x\,\|\,a),\ s_\lambda(y,a)\,s_\mu(y,a)\big\rangle$$

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Example. We have

$$s_{(2)/(1)}(x\,\|\,a) = s_{(1)}(x\,\|\,a) = \sum_{i=1}^{\infty}(x_i - a_i),$$

$$s_{(2)/(1)}(y,a) = \sum_{i=1}^{\infty} Y_i(a_0, a_{-1}).$$

Define the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ and the dual Littlewood–Richardson polynomials $c_{\lambda\mu}'^{\,\nu}(a)$ by the expansions

$$s_\lambda(x\,\|\,a)\,s_\mu(x\,\|\,a)=\sum_\nu c_{\lambda\mu}^{\nu}(a)\,s_\nu(x\,\|\,a)\qquad\text{(finite sum)},$$

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Interpolation approach

For any partition μ introduce the sequence

$$a_\mu = (a_{1-\mu_1}, a_{2-\mu_2}, \dots).$$

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For instance,

$$s_{(1)}(a_\mu \,\|\, a) = \sum_{i=1}^{\infty} (a_{i-\mu_i} - a_i) = |a_\mu| - |a_\emptyset|.$$

Universal interpolation formula.

For $P(x) \in \Lambda^a$ consider the expansion

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Moreover,

$$c^{\nu}_{P,\mu} = \sum_{R} \sum_{k=0}^{l} \frac{P(a_{\rho(k)})}{(|a_{\rho(k)}| - |a_{\rho(0)}|) \ldots \wedge \ldots (|a_{\rho(k)}| - |a_{\rho(l)}|)},$$

Moreover,

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summed over all sequences of partitions R of the form

$$\mu = \rho^{(0)} \rightarrow \rho^{(1)} \rightarrow \cdots \rightarrow \rho^{(l-1)} \rightarrow \rho^{(l)} = \nu,$$

$\rho \rightarrow \sigma$ means σ is obtained from ρ by adding one box.

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and use it to prove the generalized Cauchy identity

$$\prod_{i,j} \frac{1 - a_{i-\mu_i}\, y_j}{1 - x_i\, y_j}\, s_{\mu}(x \,\|\, a) = \sum_{\nu \in \mathcal{P},\ \mu \subseteq \nu} s_{\nu}(x \,\|\, a)\, s_{\nu/\mu}(y, a).$$

The application of the recurrence relation relies on the bijection

$$\bigcup_{\mu^+} \{\text{reverse } \nu/\mu^+\text{–tableaux}\} \longleftrightarrow \bigcup_{\nu^-} \{\text{reverse } \nu^-/\mu\text{–tableaux}\}$$

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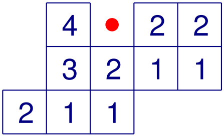

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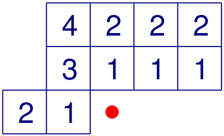

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Isomorphism ω_a

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Introduce the isomorphism $\omega_a : \Lambda^a \to \Lambda^{a'}$ by

$$\omega_a : e_k(x \,\|\, a) \mapsto h_k(x \,\|\, a').$$

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$$\omega_a : s_\lambda(x \,\|\, a) \mapsto s_{\lambda'}(x \,\|\, a'),$$

$$p_\lambda(x \,\|\, a) \mapsto \varepsilon_\lambda \, p_\lambda(x \,\|\, a'), \qquad \varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)},$$

$$m_\lambda(x \,\|\, a) \mapsto f_\lambda(x \,\|\, a'),$$

$$h_\lambda(x \,\|\, a) \mapsto e_\lambda(x \,\|\, a').$$

Corollary. We have the identities

$$\prod_{i,j} \frac{1 + x_i\, y_j}{1 + a_i\, y_j} = \sum_{\lambda \in \mathcal{P}} s_\lambda(x \,\|\, a)\, s_{\lambda'}(y, a'),$$

$$\prod_{i,j} \frac{1 + x_i\, y_j}{1 + a_i\, y_j} = \sum_{\lambda \in \mathcal{P}} e_\lambda(x \,\|\, a)\, m_\lambda(y, a'),$$

$$\prod_{i,j} \frac{1 + x_i\, y_j}{1 + a_i\, y_j} = \sum_{\lambda \in \mathcal{P}} \frac{\varepsilon_\lambda}{z_\lambda}\, p_\lambda(x \,\|\, a)\, p_\lambda(y),$$

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Irreducible characters

Define the polynomials $\chi^\lambda_\mu(a) \in \mathbb{Q}[a]$ by the expansion

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Hence, if $|\mu| = |\lambda| = n$, then, by the interpolation formula, the values χ^λ_μ of the irreducible character χ^λ of the symmetric group $\mathfrak{S}_n$ can be found by

$$\chi^\lambda_\mu = \sum_R \sum_{k=1}^{n} \frac{p_\mu(a_{\rho(k)} \,\|\, a)}{(|a_{\rho(k)}| - |a_{\rho(0)}|) \ldots \wedge \ldots (|a_{\rho(k)}| - |a_{\rho(n)}|)},$$

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Example.

$$\chi^{\lambda}_{(1^n)} = \sum_{R} \sum_{k=1}^{n} \frac{(|a_{\rho(k)}| - |a_{\rho(0)}|)^{n-1}}{(|a_{\rho(k)}| - |a_{\rho(1)}|)\ldots \wedge \ldots (|a_{\rho(k)}| - |a_{\rho(n)}|)}$$

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$=$ number of standard λ-tableaux.

Consider the specialization $a_i = i$ for all $i \in \mathbb{Z}$.

For partitions $\mu = (1^{m_1} 2^{m_2} \dots r^{m_r})$ and $\rho = (\rho_1, \dots, \rho_l)$ set

$$\pi_\mu(\rho) = \prod_{k=1}^{r} \Big((1 - \rho_1)^k + \dots + (l - \rho_l)^k - 1^k - \dots - l^k \Big)^{m_k}.$$

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For any skew diagram θ denote by $\dim\theta$ the number of standard θ-tableaux with entries in $\{1, 2, \dots, |\theta|\}$ and set

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If θ is normal (non-skew), then h_θ is the product of hooks of θ.

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Hence,

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