Classical Lie algebras and Yangians

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Lecture 2. Yangians: algebraic structure

Recall $E = [E_{ij}]$ with $i, j \in \{1, \dots, N\}$. We have

$$[E_{ij}, (E^s)_{kl}] = \delta_{kj}(E^s)_{il} - \delta_{il}(E^s)_{kj}.$$

This implies that $\mathrm{tr}\, E^s$ are Casimir elements for $\mathfrak{gl}_N$ (the Gelfand invariants).

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More generally, we have

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kj} (E^s)_{il} - (E^s)_{kj} (E^r)_{il},$$

where $r, s \geqslant 0$ and $E^0 = 1$ is the identity matrix.

Yangian for $\mathfrak{gl}_N$

Definition

The Yangian for $\mathfrak{gl}_N$ is the associative algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)}, \dots$ where $i, j = 1, \dots, N$, and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)},$$

where $r, s = 0, 1, \dots$ and $t_{ij}^{(0)} = \delta_{ij}$.

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$$[t_{ij}^{(r+1)},t_{kl}^{(s)}]-[t_{ij}^{(r)},t_{kl}^{(s+1)}]=t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)},$$

where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$.

This algebra is denoted by $\mathrm{Y}(\mathfrak{gl}_N)$.

Introduce the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \dots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]].$$

The defining relations take the form

$$(u - v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u).$$

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The defining relations take the form

$$(u - v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u).$$

The defining relations are equivalent to

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min\{r,s\}} \Big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \Big).$$

Proposition

The assignment

$$\pi_N : t_{ij}(u) \mapsto \delta_{ij} + E_{ij}\, u^{-1}$$

defines a surjective homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$. *Moreover, the assignment*

$$E_{ij} \mapsto t_{ij}^{(1)}$$

defines an embedding $\mathrm{U}(\mathfrak{gl}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$.

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We may regard $\mathrm{U}(\mathfrak{gl}_N)$ as a subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$.

Matrix form of the defining relations

Introduce the $N \times N$ matrix $T(u)$ whose ij-th entry is the series $t_{ij}(u)$. We regard $T(u)$ as an element of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$. Then

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u),$$

where $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$ are the standard matrix units.

For any positive integer m consider the algebra

$$(\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathrm{Y}(\mathfrak{gl}_N).$$

For any $a \in \{1, \dots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra.

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For any $a \in \{1, \dots, m\}$ denote by $T_a(u)$ the matrix $T(u)$ which corresponds to the a-th copy of the algebra $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra. That is, $T_a(u)$ is a formal power series in u^{-1} given by

$$T_a(u) = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes t_{ij}(u),$$

where 1 is the identity matrix.

If

$$C = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, e_{ij} \otimes e_{kl} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N,$$

then for any two indices $a, b \in \{1, \dots, m\}$ such that $a < b$, define the element C_{ab} of the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes m}$ by

$$C_{ab} = \sum_{i,j,k,l=1}^{N} c_{ijkl}\, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{kl} \otimes 1^{\otimes(m-b)}.$$

The tensor factors e_{ij} and e_{kl} belong to the a-th and b-th copies of $\mathrm{End}\,\mathbb{C}^N$, respectively.

Consider now the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N.$$

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The rational function

$$R(u) = 1 - P u^{-1}$$

with values in $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ is called the Yang R-matrix.

Proposition

In the algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes 3}(u,v)$ *we have the identity*

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v) = R_{23}(v)\,R_{13}(u+v)\,R_{12}(u).$$

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This relation is known as the Yang–Baxter equation. The Yang R-matrix is its simplest nontrivial solution.

Proposition

The defining relations of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ *can be written in the equivalent form*

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v).$$

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Here $T_1(u)$ and $T_2(v)$ as formal power series with the coefficients in the algebra

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The matrix relation is called the *RTT* relation (or ternary relation).

Symmetries of $\mathrm{Y}(\mathfrak{gl}_N)$

Let $f(u)$ be a formal power series in u^{-1} of the form

$$f(u) = 1 + f_1\, u^{-1} + f_2\, u^{-2} + \dots \in \mathbb{C}[[u^{-1}]].$$

Let $c \in \mathbb{C}$ and let B be any nonsingular complex $N \times N$ matrix.

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Let $c \in \mathbb{C}$ and let B be any nonsingular complex $N \times N$ matrix.

Proposition. Each of the mappings

$$T(u) \mapsto f(u)\, T(u), \tag{1}$$

$$T(u) \mapsto T(u - c), \tag{2}$$

$$T(u) \mapsto B\, T(u)\, B^{-1} \tag{3}$$

defines an automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

Proposition. Each of the mappings

$$\sigma_N : T(u) \mapsto T(-u),$$
$$t : T(u) \mapsto T^{\,t}(u),$$
$$\mathrm{S} : T(u) \mapsto T^{-1}(u)$$

defines an anti-automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

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Corollary. The mapping

$$\omega_N : T(u) \mapsto T^{-1}(-u)$$

defines an involutive automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$.

Poincaré–Birkhoff–Witt theorem

Theorem

Given an arbitrary linear order on the set of generators $t_{ij}^{(r)}$*, any element of the algebra* $\mathrm{Y}(\mathfrak{gl}_N)$ *can be uniquely written as a linear combination of ordered monomials in these generators.*

Poincaré–Birkhoff–Witt theorem

Theorem

Given an arbitrary linear order on the set of generators $t_{ij}^{(r)}$, any element of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of ordered monomials in these generators.

Corollary. Consider the ascending filtration on $\mathrm{Y}(\mathfrak{gl}_N)$ defined by

$$\deg t_{ij}^{(r)} = r.$$

The graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$ is an algebra of polynomials.

Hopf algebra structure

A coalgebra (over the field $\mathbb{C}$) is a vector space A equipped with linear maps $\Delta : A \to A \otimes A$, the comultiplication, and $\varepsilon : A \to \mathbb{C}$, the counit, satisfying some axioms; e.g.,

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes \Delta} \uparrow & & \uparrow {\scriptstyle \Delta} \\ A \otimes A & \xleftarrow[\Delta]{} & A \end{array}$$

the coassociativity of Δ.

A bialgebra is an associative unital algebra A equipped with a coalgebra structure, such that Δ and ε are algebra homomorphisms. In particular, then we have $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$.

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A bialgebra A is called a Hopf algebra, if it is also equipped with an anti-automorphism $\mathrm{S} : A \to A$, the antipode, such that the following two diagrams commute:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{S}\otimes\mathrm{id}} & A \otimes A \\ \Delta \uparrow & & \downarrow \mu \\ A & \xrightarrow[\delta\circ\varepsilon]{} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id}\otimes\mathrm{S}} & A \otimes A \\ \Delta \uparrow & & \downarrow \mu \\ A & \xrightarrow[\delta\circ\varepsilon]{} & A \end{array}$$

Theorem

The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ *is a Hopf algebra with comultiplication*

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{N} t_{ik}(u) \otimes t_{kj}(u),$$

the antipode

$$\mathrm{S} : T(u) \mapsto T^{-1}(u),$$

and the counit $\varepsilon : T(u) \mapsto 1$.

Quantum determinant

For any $m \geqslant 2$ introduce the rational function $R(u_1, \dots, u_m)$ with values in the tensor product algebra $(\mathrm{End}\,\mathbb{C}^N)^{\otimes m}$ by

$$R(u_1, \dots, u_m) = (R_{m-1,m})(R_{m-2,m}R_{m-2,m-1}) \dots (R_{1m} \dots R_{12}),$$

where $u_1, \dots, u_m$ are independent complex variables and we abbreviate $R_{ij} = R_{ij}(u_i - u_j) = 1 - P_{ij}(u_i - u_j)^{-1}$.

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where $u_1, \dots, u_m$ are independent complex variables and we abbreviate $R_{ij} = R_{ij}(u_i - u_j) = 1 - P_{ij}(u_i - u_j)^{-1}$.

Using the Yang–Baxter equation, we get

$$R(u_1, \dots, u_m) = (R_{12} \dots R_{1m}) \dots (R_{m-2,m-1}R_{m-2,m})(R_{m-1,m}).$$

Applying the RTT relation repeatedly,

we come to the fundamental relation

$$R(u_1,\dots,u_m)\,T_1(u_1)\dots T_m(u_m)=T_m(u_m)\dots T_1(u_1)\,R(u_1,\dots,u_m).$$

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Lemma

If $u_i - u_{i+1} = 1$ *for all* $i = 1, \dots, m-1$ *then*

$$R(u_1, \dots, u_m) = A_m,$$

the image of the anti-symmetrizer $\sum_{p \in \mathfrak{S}_m} \operatorname{sgn} p \cdot p \in \mathbb{C}[\mathfrak{S}_m]$ *in the algebra* $\mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$*.*

Hence, we have

$$A_m\, T_1(u) \dots T_m(u - m + 1) = T_m(u - m + 1) \dots T_1(u)\, A_m.$$

Hence, we have

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If $m = N$ then the operator A_N on $(\mathbb{C}^N)^{\otimes N}$ is one-dimensional.

Definition

The quantum determinant of the matrix $T(u)$ with the coefficients in $\mathrm{Y}(\mathfrak{gl}_N)$ is the formal series

$$\mathrm{qdet}\, T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \dots$$

such that both sides of the above relation with $m = N$, are equal to $A_N \, \mathrm{qdet}\, T(u)$.

Proposition

For any permutation $q \in \mathfrak{S}_N$ we have

$$\begin{aligned}
\operatorname{qdet} T(u) &= \operatorname{sgn} q \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1),q(1)}(u) \dots t_{p(N),q(N)}(u - N + 1) \\
&= \operatorname{sgn} q \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{q(1),p(1)}(u - N + 1) \dots t_{q(N),p(N)}(u).
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In particular,

$$\operatorname{qdet} T(u) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{p(1),1}(u) \dots t_{p(N),N}(u - N + 1)$$

$$= \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot t_{1,p(1)}(u - N + 1) \dots t_{N,p(N)}(u).$$

Example

For $N = 2$ we have

$$\begin{aligned}
\mathrm{qdet}\, T(u) &= t_{11}(u)\, t_{22}(u-1) - t_{21}(u)\, t_{12}(u-1) \\
&= t_{22}(u)\, t_{11}(u-1) - t_{12}(u)\, t_{21}(u-1) \\
&= t_{11}(u-1)\, t_{22}(u) - t_{12}(u-1)\, t_{21}(u) \\
&= t_{22}(u-1)\, t_{11}(u) - t_{21}(u-1)\, t_{12}(u).
\end{aligned}$$

Assuming that $m \leqslant N$ is arbitrary, define

the $m \times m$ quantum minors $t\,^{a_1 \dots\, a_m}_{b_1 \dots\, b_m}(u)$ so that each side of

$$A_m\, T_1(u) \dots T_m(u - m + 1) = T_m(u - m + 1) \dots T_1(u)\, A_m$$

equals

$$\sum e_{a_1 b_1} \otimes \dots \otimes e_{a_m b_m} \otimes t\,^{a_1 \dots\, a_m}_{b_1 \dots\, b_m}(u),$$

summed over the indices $a_i, b_i \in \{1, \dots, N\}$.

Proposition

The images of quantum minors under the comultiplication are given by

$$\Delta\big(t^{\,a_1\dots\, a_m}_{\,b_1\dots\, b_m}(u)\big) = \sum_{c_1<\dots<c_m} t^{\,a_1\dots\, a_m}_{\,c_1\dots\, c_m}(u) \otimes t^{\,c_1\dots\, c_m}_{\,b_1\dots\, b_m}(u),$$

summed over all subsets of indices $\{c_1, \dots, c_m\}$ *from* $\{1, \dots, N\}$.

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summed over all subsets of indices $\{c_1, \dots, c_m\}$ *from* $\{1, \dots, N\}$.

In particular, as $\mathrm{qdet}\, T(u) = t^{1\dots N}_{1\dots N}(u)$,

$$\Delta : \mathrm{qdet}\, T(u) \mapsto \mathrm{qdet}\, T(u) \otimes \mathrm{qdet}\, T(u).$$

Center of $\mathrm{Y}(\mathfrak{gl}_N)$

Proposition

We have the relations

$$(u-v)\,[\,t_{kl}(u), t^{\,a_1\dots\,a_m}_{\,b_1\dots\,b_m}(v)] = \sum_{i=1}^{m} t_{a_i l}(u)\, t^{\,a_1\dots\,k\,\dots\,a_m}_{\,b_1\;\,\dots\;\;\,b_m}(v) - \sum_{i=1}^{m} t^{\,a_1\;\,\dots\;\;\,a_m}_{\,b_1\dots\,l\,\dots\,b_m}(v)\, t_{kb_i}(u)$$

where the indices k and l in the quantum minors replace a_i and b_i, respectively.

Theorem

The coefficients $d_1, d_2, \dots$ of the series $\mathrm{qdet}\, T(u)$ belong to the center $\mathrm{ZY}(\mathfrak{gl}_N)$ of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Moreover, these elements are algebraically independent and generate $\mathrm{ZY}(\mathfrak{gl}_N)$.

Proof.

The first part follows from the Proposition. For the second part introduce another filtration on $\mathrm{Y}(\mathfrak{gl}_N)$ by setting

$$\deg' t_{ij}^{(r)} = r - 1$$

for every $r \geqslant 1$. Then the corresponding graded algebra $\mathrm{gr}' \,\mathrm{Y}(\mathfrak{gl}_N)$ is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N[z])$. □

Yangian for $\mathfrak{sl}_N$

For any series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ consider

the automorphism $\mu_f : T(u) \mapsto f(u)\, T(u)$ of $\mathrm{Y}(\mathfrak{gl}_N)$.

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The Yangian for $\mathfrak{sl}_N$ is the subalgebra $\mathrm{Y}(\mathfrak{sl}_N)$ of $\mathrm{Y}(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms μ_f.

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The Yangian for $\mathfrak{sl}_N$ is the subalgebra $\mathrm{Y}(\mathfrak{sl}_N)$ of $\mathrm{Y}(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms μ_f.

Theorem

We have the isomorphism

$$\mathrm{Y}(\mathfrak{gl}_N) = \mathrm{ZY}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{sl}_N).$$

In particular, the center of $\mathrm{Y}(\mathfrak{sl}_N)$ *is trivial.*

Corollary

The algebra $\mathrm{Y}(\mathfrak{sl}_N)$ *is isomorphic to the quotient of* $\mathrm{Y}(\mathfrak{gl}_N)$ *by the ideal generated by the elements* $d_1, d_2, \dots,$ *i.e.,*

$$\mathrm{Y}(\mathfrak{sl}_N) \cong \mathrm{Y}(\mathfrak{gl}_N)/(\mathrm{qdet}\ T(u) = 1).$$

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Proposition

The subalgebra $\mathrm{Y}(\mathfrak{sl}_N)$ *of* $\mathrm{Y}(\mathfrak{gl}_N)$ *is a Hopf algebra whose comultiplication, antipode and counit are obtained by restricting those from* $\mathrm{Y}(\mathfrak{gl}_N)$.

Quantum Liouville formula

The quantum comatrix $\widehat{T}(u)$ is defined by

$$\widehat{T}(u)\, T(u - N + 1) = \mathrm{qdet}\, T(u).$$

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Proposition

The entries $\widehat{t}_{ij}(u)$ of the matrix $\widehat{T}(u)$ are given by

$$\widehat{t}_{ij}(u) = (-1)^{i+j}\, t^{\,1\dots\widehat{j}\dots N}_{\,1\dots\widehat{i}\dots N}(u),$$

where the hats on the right hand side indicate the indices to be omitted. Moreover, we have the relation

$$\widehat{T}^{\,t}(u-1)\, T^{\,t}(u) = \mathrm{qdet}\, T(u).$$

Consider the series $z(u)$ with coefficients from $\mathrm{Y}(\mathfrak{gl}_N)$ given by the formula

$$z(u)^{-1} = \frac{1}{N}\,\mathrm{tr}\,\Big(T(u)\,T^{-1}(u-N)\Big),$$

so that

$$z(u) = 1 + z_2 u^{-2} + z_3 u^{-3} + \dots \quad \text{where} \quad z_i \in \mathrm{Y}(\mathfrak{gl}_N).$$

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Theorem

We have the relation

$$z(u) = \frac{\operatorname{qdet} T(u-1)}{\operatorname{qdet} T(u)}.$$

Proof.

We have

$$z(u)^{-1} = \frac{1}{N}\,\mathrm{tr}\,\big(T(u)\,\widehat{T}(u-1)\,(\mathrm{qdet}\,T(u-1))^{-1}\big).$$

Using the centrality of $\mathrm{qdet}\,T(u)$ we get

$$T^{\,t}(u)\,\widehat{T}^{\,t}(u-1) = \mathrm{qdet}\,T(u)$$

and so

$$\mathrm{tr}\,\big(T(u)\,\widehat{T}(u-1)\big) = N\,\mathrm{qdet}\,T(u),$$

implying the formula. □

Theorem

The square of the antipode S *is the automorphism of* $\mathrm{Y}(\mathfrak{gl}_N)$ *given by*

$$\mathrm{S}^2 : T(u) \mapsto z(u+N)\, T(u+N).$$

In particular, $\mathrm{qdet}\, T(u)$ *is stable under* S^2*.*

Application to $\mathfrak{gl}_N$

Recall the evaluation homomorphism $\pi_N : T(u) \mapsto 1 + E\, u^{-1}$:

$$\pi_N : z(-u+N)^{-1} \mapsto \frac{1}{N}\,\mathrm{tr}\Big((1 - E\,(u-N)^{-1})(1 - E\,u^{-1})^{-1}\Big)$$
$$= 1 - \frac{1}{u-N}\sum_{k=1}^{\infty} \mathrm{tr}\, E^k\, u^{-k}.$$

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$$= 1 - \frac{1}{u-N} \sum_{k=1}^{\infty} \operatorname{tr} E^k\, u^{-k}.$$

The quantum Liouville formula gives

$$z(u+1)^{-1} = \frac{\operatorname{qdet} T(u+1)}{\operatorname{qdet} T(u)}.$$

Applying the evaluation homomorphism to both sides of this relation, we get Newton's formulas (see Lecture 1).

Factorization of the quantum determinant

Let $A = [a_{ij}]$ be an $N \times N$ matrix over a ring with 1.

The ij-th quasideterminant of A is defined by

$$|A|_{ij} = \big((A^{-1})_{ji}\big)^{-1}.$$

Example

For a 2×2 matrix A the four quasideterminants are

$$|A|_{11} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21}, \qquad |A|_{12} = a_{12} - a_{11}\, a_{21}^{-1}\, a_{22},$$

$$|A|_{21} = a_{21} - a_{22}\, a_{12}^{-1}\, a_{11}, \qquad |A|_{22} = a_{22} - a_{21}\, a_{11}^{-1}\, a_{12}.$$

For $m = 1, \dots, N$ denote by $T^{(m)}(u)$ the submatrix of $T(u)$ corresponding to the first m rows and columns.

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Theorem

The quantum determinant $\mathrm{qdet}\, T(u)$ *admits the factorization in the algebra* $\mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]]$

$$\mathrm{qdet}\, T(u) = t_{11}(u) \, \big| T^{(2)}(u-1) \big|_{22} \ldots \big| T^{(N)}(u-N+1) \big|_{NN}.$$

Moreover, the N *factors on the right hand side of this equality pairwise commute.*

Set

$$\widetilde{\mathcal{C}}(q) = \sum_{p \in \mathfrak{S}_N} \operatorname{sgn} p \cdot \big(1 + q\, E\big)_{p(1),1} \dots \big(1 + q\,(E - N + 1)\big)_{p(N),N}.$$

Then $\quad \widetilde{\mathcal{C}}(q) = q^N\, \mathcal{C}(q^{-1}), \quad$ where $\mathcal{C}(u)$ is the Capelli determinant.

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Then $\quad \widetilde{\mathcal{C}}(q) = q^N\, \mathcal{C}(q^{-1}), \quad$ where $\mathcal{C}(u)$ is the Capelli determinant.

Apply the evaluation homomorphism to the decomposition of the Theorem to get

$$\widetilde{\mathcal{C}}(q) = \big|1 + q\, E^{(1)}\big|_{11} \dots \big|1 + q\,(E^{(N)} - N + 1)\big|_{NN},$$

where $E^{(m)}$ is the submatrix of E corresponding to the first m rows and columns.

For the Harish-Chandra image of $\widetilde{\mathcal{C}}(q)$ we have

$$\chi\big(\widetilde{\mathcal{C}}(q)\big) = (1 + q\, l_1)\dots(1 + q\, l_N), \qquad l_i = \lambda_i - i + 1.$$

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Hence, if we define the Casimir elements Φ_k by

$$\sum_{k=1}^{\infty} \Phi_k\, q^{k-1} = -\frac{d}{dq}\, \log \widetilde{\mathcal{C}}(-q),$$

then

$$\chi(\Phi_k) = l_1^{\,k} + \dots + l_N^{\,k}.$$

On the other hand, by the quasideterminant decomposition,

$$\sum_{k=1}^{\infty} \Phi_k \, q^{k-1} = - \sum_{m=1}^{N} \frac{d}{dq} \, \log \, \big| 1 - q \, (E^{(m)} - m + 1) \big|_{mm}.$$

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$$\sum_{k=1}^{\infty} \Phi_k \, q^{k-1} = -\sum_{m=1}^{N} \frac{d}{dq} \log \left|1 - q\,(E^{(m)} - m + 1)\right|_{mm}.$$

Therefore,

$$\Phi_k = \Phi_k^{(1)} + \cdots + \Phi_k^{(N)},$$

where

$$\sum_{k=1}^{\infty} \Phi_k^{(m)} \, q^{k-1} = -\frac{d}{dq} \log \left|1 - q\,(E^{(m)} - m + 1)\right|_{mm}.$$

Quantum Sylvester theorem

Suppose that $A = [a_{ij}]$ is a numerical $(M+N) \times (M+N)$ matrix. For any indices $i, j = 1, \ldots, N$ introduce the minors c_{ij} of A corresponding to the rows $1, \ldots, M, M+i$ and columns $1, \ldots, M, M+j$ so that

$$c_{ij} = a^{\,1\ldots\, M, M+i}_{\,1\ldots\, M, M+j}.$$

Let $A^{(M)}$ be the submatrix of A determined by the first M rows and columns. The classical Sylvester theorem provides a formula for the determinant of the matrix $C = [c_{ij}]$:

$$\det C = \det A \cdot \big(\det A^{(M)}\big)^{N-1}.$$

Introduce the series with coefficients in $\mathrm{Y}(\mathfrak{gl}_{M+N})$ by

$$t^{\sharp}_{ij}(u) = t^{\,1\dots\, M, M+i}_{\,1\dots\, M, M+j}(u)$$

and set $T^{\sharp}(u) = [\, t^{\sharp}_{ij}(u)]$.

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and set $T^{\sharp}(u) = [\, t^{\sharp}_{ij}(u)]$.

Theorem

The mapping

$$t_{ij}(u) \mapsto t^{\sharp}_{ij}(u), \qquad 1 \leqslant i, j \leqslant N,$$

defines a homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{Y}(\mathfrak{gl}_{M+N})$. *Moreover,*

$$\mathrm{qdet}\, T^{\sharp}(u) = \mathrm{qdet}\, T(u) \cdot \mathrm{qdet}\, T^{(M)}(u-1) \dots \mathrm{qdet}\, T^{(M)}(u-N+1).$$

Twisted Yangians

Consider the orthogonal Lie algebra $\mathfrak{o}_N$ as the subalgebra of $\mathfrak{gl}_N$ spanned by the skew-symmetric matrices. The elements $F_{ij} = E_{ij} - E_{ji}$ with $i < j$ form a basis of $\mathfrak{o}_N$. Introduce the $N \times N$ matrix F whose ij-th entry is F_{ij}.

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The matrix elements of the powers of the matrix F are known to satisfy the relations

$$[F_{ij}, (F^s)_{kl}] = \delta_{kj}(F^s)_{il} - \delta_{il}(F^s)_{kj} - \delta_{ik}(F^s)_{jl} + \delta_{lj}(F^s)_{ki}.$$

Introduce the generating series

$$f_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} (F^r)_{ij} \Big(u + \frac{N-1}{2} \Big)^{-r}.$$

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Then we have the relations

$$\begin{aligned}(u^2 - v^2)\,[f_{ij}(u), f_{kl}(v)] &= (u+v)\,\big(f_{kj}(u)\,f_{il}(v) - f_{kj}(v)\,f_{il}(u)\big) \\ &\quad - (u-v)\,\big(f_{ik}(u)\,f_{jl}(v) - f_{ki}(v)\,f_{lj}(u)\big) \\ &\quad + f_{ki}(u)\,f_{jl}(v) - f_{ki}(v)\,f_{jl}(u).\end{aligned}$$

More generally, equip $\mathbb{C}^N$ with a nonsingular bilinear form which may be either symmetric or alternating. The alternating case can only occur if N is even. Let $G = [g_{ij}]$ be the corresponding matrix so that G is nonsingular with $G^t = \pm G$.

More generally, equip $\mathbb{C}^N$ with a nonsingular bilinear form which may be either symmetric or alternating. The alternating case can only occur if N is even. Let $G = [g_{ij}]$ be the corresponding matrix so that G is nonsingular with $G^t = \pm\, G$.

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the symmetric case and the lower sign to the alternating case. Introduce the elements F_{ij} of the Lie algebra $\mathfrak{gl}_N$ by the formulas

$$F_{ij} = \sum_{k=1}^{N} (E_{ik}\, g_{kj} \mp E_{jk}\, g_{ki}).$$

Obviously,

$$F_{ji} = \mp F_{ij}$$

and the elements F_{ij} satisfy the commutation relations

$$[F_{ij}, F_{kl}] = g_{kj} F_{il} - g_{il} F_{kj} - g_{ik} F_{jl} + g_{lj} F_{ki}.$$

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The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements F_{ij} is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$ in the symmetric case and to the symplectic Lie algebra $\mathfrak{sp}_N$ in the alternating case. This Lie algebra will be denoted by $\mathfrak{g}_N$.

The twisted Yangian $\mathrm{Y}_G(\mathfrak{g}_N)$ is an associative algebra with generators $s_{ij}^{(1)},\ s_{ij}^{(2)},\dots$ where $1\leqslant i,j\leqslant N$, and the defining relations written in terms of the generating series

$$s_{ij}(u)=g_{ij}+s_{ij}^{(1)}u^{-1}+s_{ij}^{(2)}u^{-2}+\dots$$

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$$s_{ij}(u) = g_{ij} + s_{ij}^{(1)} u^{-1} + s_{ij}^{(2)} u^{-2} + \dots$$

as follows

$$\begin{aligned}(u^2 - v^2)\,[s_{ij}(u), s_{kl}(v)] &= (u+v)\,\big(s_{kj}(u)s_{il}(v) - s_{kj}(v)s_{il}(u)\big)\\ &\quad - (u-v)\,\big(s_{ik}(u)s_{jl}(v) - s_{ki}(v)s_{lj}(u)\big)\\ &\quad + s_{ki}(u)s_{jl}(v) - s_{ki}(v)s_{jl}(u)\end{aligned}$$

and

$$s_{ji}(-u) = \pm\, s_{ij}(u) + \frac{s_{ij}(u) - s_{ij}(-u)}{2u}.$$

If G and G' are two nonsingular symmetric (respectively, skew-symmetric) $N \times N$-matrices then the algebras $\mathrm{Y}_G(\mathfrak{g}_N)$ and $\mathrm{Y}_{G'}(\mathfrak{g}_N)$ are isomorphic to each other.

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Proposition

The assignment

$$s_{ij}(u) \mapsto g_{ij} + F_{ij}\Big(u \pm \frac{1}{2}\Big)^{-1}$$

defines an algebra epimorphism $\varrho_N : \mathrm{Y}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$*. Moreover, the assignment*

$$F_{ij} \mapsto s_{ij}^{(1)}$$

defines an embedding $\mathrm{U}(\mathfrak{g}_N) \hookrightarrow \mathrm{Y}(\mathfrak{g}_N)$*.*

Matrix form of the defining relations

Introduce the $N \times N$ matrix $S(u)$ by

$$S(u) = \sum_{i,j=1}^{N} e_{ij} \otimes s_{ij}(u) \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{g}_N)[[u^{-1}]]$$

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Proposition

The defining relations of $\mathrm{Y}(\mathfrak{g}_N)$ have the form

$$R(u-v)\, S_1(u)\, R^t(-u-v)\, S_2(v) = S_2(v)\, R^t(-u-v)\, S_1(u)\, R(u-v)$$

and

$$S^t(-u) = \pm\, S(u) + \frac{S(u) - S(-u)}{2u}.$$

Here

$$R(u) = 1 - P u^{-1}$$

is the Yang R-matrix, while

$$R^t(u) = 1 - Q\, u^{-1}, \qquad Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}.$$

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Theorem

The mapping

$$S(u) \mapsto T(u)\, G\, T^{\,t}(-u)$$

defines an embedding $\mathrm{Y}(\mathfrak{g}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$*.*

Sklyanin determinant

The Sklyanin determinant is a series in u^{-1} defined by

$$\operatorname{sdet} S(u) = \gamma_{n,G}(u)\,\operatorname{qdet} T(u)\,\operatorname{qdet} T(-u+N-1),$$

where

$$\gamma_{n,G}(u) = \begin{cases} \det G & \text{if} \quad \mathfrak{g}_N = \mathfrak{o}_N, \\ \dfrac{2u+1}{2u-2n+1}\,\det G & \text{if} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}. \end{cases}$$

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All coefficients of $\mathrm{sdet}\, S(u)$ are contained in $\mathrm{Y}(\mathfrak{g}_N)$ and belong to the center of $\mathrm{Y}(\mathfrak{g}_N)$.

Introduce the scalar $\gamma_n(u)$ by

$$\gamma_n(u) = \begin{cases} 1 & \text{if} \quad \mathfrak{g}_N = \mathfrak{o}_N, \\ (-1)^n \dfrac{2u+1}{2u-2n+1} & \text{if} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}. \end{cases}$$

Theorem

We have

$$\begin{aligned} &\mathrm{sdet}\, S(u) \\ &\qquad = \gamma_n(u) \sum_{p \in \mathfrak{S}_N} \mathrm{sgn}\, p\, p' \cdot s^t_{p(1),p'(1)}(-u) \dots s^t_{p(n),p'(n)}(-u+n-1) \\ &\qquad\qquad \times s_{p(n+1),p'(n+1)}(u-n) \dots s_{p(N),p'(N)}(u-N+1). \end{aligned}$$

Here we denote the matrix elements of the transposed matrix $S^t(u)$ by $s^t_{ij}(u)$, and for any permutation $p \in \mathfrak{S}_N$ we denote by p' its image under the map $\varphi_N : \mathfrak{S}_N \to \mathfrak{S}_N$ (Lecture 1).

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Example

For $N = 2$ we have

$$\operatorname{sdet} S(u) = \frac{1 \mp 2u}{1 - 2u} \big(s^t_{11}(-u)\, s_{22}(u-1) - s^t_{21}(-u)\, s_{12}(u-1) \big).$$

Here we denote the matrix elements of the transposed matrix $S^{\,t}(u)$ by $s^t_{ij}(u)$, and for any permutation $p \in \mathfrak{S}_N$ we denote by p' its image under the map $\varphi_N : \mathfrak{S}_N \to \mathfrak{S}_N$ (Lecture 1).

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If $\quad N = 3 \quad$ then $\quad \mathrm{sdet}\, S(u) =$

$$s^t_{22}(-u)\, s_{11}(u-1)\, s_{33}(u-2) + s^t_{12}(-u)\, s_{31}(u-1)\, s_{23}(u-2)$$
$$+ s^t_{21}(-u)\, s_{32}(u-1)\, s_{13}(u-2) - s^t_{12}(-u)\, s_{21}(u-1)\, s_{33}(u-2)$$
$$- s^t_{32}(-u)\, s_{11}(u-1)\, s_{23}(u-2) - s^t_{31}(-u)\, s_{22}(u-1)\, s_{13}(u-2).$$

The center of the twisted Yangian

Theorem

All coefficients of the series

$$\mathrm{sdet}\, S(u) = c_0 + c_1 u^{-1} + c_2 u^{-2} + \dots$$

belong to the center of the algebra $\mathrm{Y}(\mathfrak{g}_N)$*. Moreover, the even coefficients* $c_2, c_4, \dots$ *are algebraically independent and generate the center of* $\mathrm{Y}(\mathfrak{g}_N)$*.*

Coideal property

Theorem

The subalgebra $\mathrm{Y}(\mathfrak{g}_N)$ *is a left coideal of the Hopf algebra* $\mathrm{Y}(\mathfrak{gl}_N)$, *i.e.,*

$$\Delta\big(\mathrm{Y}(\mathfrak{g}_N)\big) \subset \mathrm{Y}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{g}_N).$$

Moreover,

$$\Delta : s_{ij}(u) \mapsto \sum_{a,b=1}^{N} t_{ia}(u)\, t_{jb}(-u) \otimes s_{ab}(u).$$

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- Characteristic identities