Generators of affine $\mathcal{W}$-algebras

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The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g})$ associated with any simple Lie algebra $\mathfrak{g}$ was constructed by B. Feigin and E. Frenkel, 1990, via the quantum Drinfeld–Sokolov reduction.

More recently, the $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ were introduced by

V. Kac, S.-S. Roan and M. Wakimoto, 2004.

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It plays the role of the Weyl group invariants in the affine

Harish-Chandra isomorphism $\mathfrak{z}(\widehat{\mathfrak{g}}) \cong \mathcal{W}({}^L\mathfrak{g})$

[Feigin–Frenkel, 1992].

Moreover, the Feigin–Frenkel duality provides an isomorphism

$\mathcal{W}^k(\mathfrak{g}) \cong \mathcal{W}^{k'}({}^L\mathfrak{g})$ if $(k+h^\vee)(k'+{}^Lh^\vee)r^\vee = 1$.

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Recent work: representation theory of $\mathcal{W}$-algebras [T. Arakawa]; classical $\mathcal{W}$-algebras and integrable Hamiltonian hierarchies [A. De Sole, V. Kac, D. Valeri].

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[T. Arakawa, 2007, A. De Sole, V. Kac, 2006]

Connection with finite $\mathcal{W}$-algebras:

affine $\mathcal{W}$-algebra — universal affine vertex algebra

$$
\begin{array}{ccc}
\mathcal{W}^k(\mathfrak{g}, f) & \longrightarrow & V^k(\mathfrak{g}) \\
\Big\downarrow \text{ Zhu algebra} & f = 0 & \Big\downarrow \\
\mathcal{W}(\mathfrak{g}, f) & \longrightarrow & \mathrm{U}(\mathfrak{g})
\end{array}
$$

finite $\mathcal{W}$-algebra — universal enveloping algebra

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$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\,\mathbf{1},$$

where $X[r] = X\,t^r$ for any $X \in \mathfrak{b}$ and $r \in \mathbb{Z}$.

The vacuum module $V(\mathfrak{b})$ over $\widehat{\mathfrak{b}}$ is defined by

$$V(\mathfrak{b}) = \mathrm{U}(\widehat{\mathfrak{b}}) \otimes_{\mathrm{U}(\mathfrak{b}[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C},$$

where $\mathbb{C}$ is regarded as the one-dimensional representation of $\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{b}[t]$ acts trivially and $\mathbf{1}$ acts as 1.

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$V(\mathfrak{b})$ is a vertex algebra with the vacuum vector 1, the translation operator $\tau : V(\mathfrak{b}) \to V(\mathfrak{b})$ which is the derivation $\tau = -\partial_t$ of the enveloping algebra $X[-r] \mapsto rX[-r-1]$, and

the following state-field correspondence map

$Y : a \mapsto a(z)$, where $a \in V(\mathfrak{b})$ and

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \qquad a_{(n)} : V(\mathfrak{b}) \to V(\mathfrak{b}).$$

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$$X[-1] \mapsto X(z) = \sum_{n\in\mathbb{Z}} X[n] z^{-n-1},$$

$$X[-r-1] \mapsto \frac{\partial_z^r}{r!} X(z), \qquad r \geqslant 0.$$

Furthermore, if $a = X[-r-1]$ and $b \in V(\mathfrak{b})$ then

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with

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}, \qquad a(z)_- = \sum_{n\geqslant 0} a_{(n)} z^{-n-1}.$$

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- $\tau\,1 = 0$ and $\big[\tau, a(z)\big] = \partial_z a(z)$ for $a \in V(\mathfrak{b})$,
- for any $a, b \in V(\mathfrak{b})$ there exists $N \in \mathbb{Z}_+$ such that $(z-w)^N\,[a(z), b(w)] = 0$.

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This product is quasi-associative:

$$(a_{(-1)}b)_{(-1)}c$$
$$= a_{(-1)}(b_{(-1)}c) + \sum_{j\geqslant 0} a_{(-j-2)}\big(b_{(j)}c\big) + \sum_{j\geqslant 0} b_{(-j-2)}\big(a_{(j)}c\big).$$

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\end{aligned}
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Note that if $a = X[-r-1]$ and $b \in V(\mathfrak{b})$ then

$$a_{(-1)}b = ab$$

so we will omit the (-1)-subscript in such cases.

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Set $\quad \mathfrak{b} = \text{span of } \{e_{ij} \mid i \geqslant j\}, \qquad \mathfrak{m} = \text{span of } \{e_{ij} \mid i > j\}.$

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Set $\quad \mathfrak{b} =$ span of $\{e_{ij} \mid i \geqslant j\}, \qquad \mathfrak{m} =$ span of $\{e_{ij} \mid i > j\}$.

Given $k \in \mathbb{C}$ consider the affinization $\widehat{\mathfrak{b}}$ of $\mathfrak{b}$ with respect to the

form: for $i \geqslant i'$ and $j \geqslant j'$

$$\langle e_{i\,i'}, e_{j\,j'} \rangle = \delta_{i\,i'} \delta_{j\,j'} (k + N) \Big(\delta_{ij} - \frac{1}{N} \Big).$$

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Let $V^k(\mathfrak{b})$ be the corresponding vacuum module over $\widehat{\mathfrak{b}}$.

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To define it, introduce the Lie superalgebra

$$\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_0 \oplus \widehat{\mathfrak{a}}_1 \qquad \text{with} \quad \widehat{\mathfrak{a}}_0 = \widehat{\mathfrak{b}}, \qquad \widehat{\mathfrak{a}}_1 = \mathfrak{m}[t, t^{-1}],$$

with the adjoint action of $\widehat{\mathfrak{a}}_0$ on $\widehat{\mathfrak{a}}_1$, whereas $\widehat{\mathfrak{a}}_1$ is regarded as a supercommutative Lie superalgebra.

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with the adjoint action of $\widehat{\mathfrak{a}}_0$ on $\widehat{\mathfrak{a}}_1$, whereas $\widehat{\mathfrak{a}}_1$ is regarded as a supercommutative Lie superalgebra.

We will write $\psi_{ji}[r] = e_{ji}\, t^{r-1}$ for $e_{ji}\, t^{r-1} \in \mathfrak{m}[t, t^{-1}]$ when it is considered as an element of $\widehat{\mathfrak{a}}_1$.

Let $V^k(\mathfrak{a})$ be the vacuum module for $\widehat{\mathfrak{a}}$ induced from the representation $\mathbb{C}$ of $(\mathfrak{b}[t] \oplus \mathbb{C}\mathbf{1}) \oplus \mathfrak{m}[t]$ where $\mathfrak{b}[t] \subset \widehat{\mathfrak{a}}_0$ and $\mathfrak{m}[t] \subset \widehat{\mathfrak{a}}_1$ act trivially and $\mathbf{1}$ acts as 1.

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$$Q : V^k(\mathfrak{a}) \to V^k(\mathfrak{a})$$

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determined by the following properties.

First, Q commutes with $\tau = -\partial_t$.

Furthermore, $[Q, \mathcal{E}] = (\Psi\,\mathcal{E})^{\rm op} - (\mathcal{E}\,\Psi)^{\rm op}$ and $[Q, \Psi] = \Psi^2$ with

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$$\mathcal{E} = \begin{bmatrix} \alpha\tau + e_{11}[-1] & -1 & 0 & \dots & 0 \\ e_{21}[-1] & \alpha\tau + e_{22}[-1] & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & -1 \\ e_{N1}[-1] & e_{N2}[-1] & \dots & \dots & \alpha\tau + e_{NN}[-1] \end{bmatrix}$$

for $\alpha = k + N - 1$ and

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Explicitly, $[Q, \mathcal{E}] = (\Psi\,\mathcal{E})^{\mathrm{op}} - (\mathcal{E}\,\Psi)^{\mathrm{op}}$ reads

$$\big[Q, e_{ji}[-1]\big] = \sum_{a=i}^{j-1} e_{a\,i}[-1]\,\psi_{j\,a}[0] - \sum_{a=i+1}^{j} \psi_{a\,i}[0]\,e_{j\,a}[-1] + \alpha\,\psi_{ji}[-1] + \psi_{j+1\,i}[0] - \psi_{j\,i-1}[0].$$

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The definition of the $\mathcal{W}$-algebra can be stated in the form

$$\mathcal{W}^k(\mathfrak{g}) = \{v \in V^k(\mathfrak{b}) \mid Q\,v = 0\}.$$

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and write

$$\operatorname{cdet} \mathcal{E} = (\alpha\tau)^N + W^{(1)} (\alpha\tau)^{N-1} + \cdots + W^{(N)}, \qquad W^{(i)} \in V^k(\mathfrak{b}).$$

Explicitly,

$$\operatorname{cdet}\mathcal{E} =$$

$$\sum \big(\alpha\tau + e[-1]\big)_{k_1\, k_0+1} \big(\alpha\tau + e[-1]\big)_{k_2\, k_1+1} \dots \big(\alpha\tau + e[-1]\big)_{k_m\, k_{m-1}+1},$$

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summed over $m = 1, \dots, N$ and $0 = k_0 < k_1 < \dots < k_m = N.$

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summed over $m = 1,\ldots,N$ and $0 = k_0 < k_1 < \cdots < k_m = N.$

Theorem [T. Arakawa–A. M., 2014]

All coefficients $W^{(1)},\ldots,W^{(N)}$ of $\mathrm{cdet}\,\mathcal{E}$ belong to $\mathcal{W}^k(\mathfrak{g})$.

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Theorem [T. Arakawa–A. M., 2014]

All coefficients $W^{(1)}, \dots, W^{(N)}$ of $\mathrm{cdet}\,\mathcal{E}$ belong to $\mathcal{W}^k(\mathfrak{g})$.

Moreover, they generate the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}) \subset V^k(\mathfrak{b})$.

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summed over $m = 1, \dots, N$ and $N = l_0 > l_1 > \dots > l_m = 0.$

Take the reverse determinant of the matrix $\widetilde{\mathcal{E}}$ obtained from $\mathcal{E}$ by replacing α with $\beta = -\alpha = -(k+N-1)$. We have

$$\text{rev-det}\,\widetilde{\mathcal{E}} =$$

$$\sum \big(\beta\tau + e[-1]\big)_{l_0\, l_1+1} \big(\beta\tau + e[-1]\big)_{l_1\, l_2+1} \dots \big(\beta\tau + e[-1]\big)_{l_{m-1}\, l_m+1},$$

summed over $m = 1, \dots, N$ and $N = l_0 > l_1 > \dots > l_m = 0$.

The coefficients $U^{(1)}, \dots, U^{(N)}$ defined by

$$\text{rev-det}\,\widetilde{\mathcal{E}} = (\beta\tau)^N + U^{(1)}\,(\beta\tau)^{N-1} + \dots + U^{(N)},$$

are generators of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g})$.

Example. $\mathcal{W}^k(\mathfrak{sl}_2) = \mathcal{W}^k(\mathfrak{gl}_2)/(W^{(1)} = 0)$.

$$W^{(1)} = e_{11}[-1] + e_{22}[-1],$$

$$W^{(2)} = e_{11}[-1]\, e_{22}[-1] + (k+1)\, e_{22}[-2] + e_{21}[-1].$$

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The coefficients L_n of the series $L(z) = Y(\omega)$ given by

$$L(z) = \sum_{n \in \mathbb{Z}} L_n\, z^{-n-2}, \qquad \omega = -\frac{W^{(2)}}{k+2}, \quad k \neq -2,$$

generate the Virasoro algebra.

Miura map

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Introduce the abelian subalgebra $\mathfrak{l} \subset \mathfrak{gl}_N$ by

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The projection $\mathfrak{b} \to \mathfrak{l}$ induces the vertex algebra homomorphism

$$V^k(\mathfrak{b}) \to V^k(\mathfrak{l}).$$

By restricting to the subalgebra $\mathcal{W}^k(\mathfrak{g}) \subset V^k(\mathfrak{b})$ we get the map

$$\Upsilon : \mathcal{W}^k(\mathfrak{g}) \to V^k(\mathfrak{l}),$$

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For generic k we have [B. Feigin and E. Frenkel]:

$$\mathrm{im}\,\Upsilon = \bigcap_{i=1}^{N-1} \ker V_i,$$

where V_i are the screening operators acting on $V^k(\mathfrak{l})$.

To define the V_i, for $i = 1, \ldots, N-1$ set

$$V_i(z) = \exp\left(\sum_{r<0} \frac{b_i[r]}{r}\, z^{-r}\right) \exp\left(\sum_{r>0} \frac{b_i[r]}{r}\, z^{-r}\right),$$

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For the screening operator we have $V_i = V_i^{(1)}$, where

$$V_i(z) = \sum_{n \in \mathbb{Z}} V_i^{(n)} z^{-n}.$$

Under the Miura map we have

$$\Upsilon : \operatorname{cdet} \mathcal{E} \mapsto \big(\alpha\tau + e_{11}[-1]\big) \dots \big(\alpha\tau + e_{NN}[-1]\big).$$

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Corollary [Fateev and Lukyanov, 1988].

The coefficients $w^{(1)}, \dots, w^{(N)}$ generate the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}) \subset V^k(\mathfrak{l})$.

Suppose that $(k+N)(k'+N) = 1$.

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Corollary [Feigin–Frenkel duality].

$$\mathcal{W}^k(\mathfrak{g})\cong\mathcal{W}^{k'}(\mathfrak{g}).$$

The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{gl}_N, f)$

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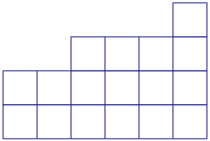
Fix a partition of N and depict it as the right justified pyramid



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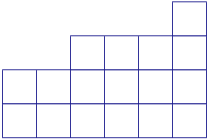

Let $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$ be the lengths of the rows

and $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_l$ be the lengths of the columns.

Number the bricks of a pyramid π by the rule

			8
	3	5	7
1	2	4	6

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Define the corresponding nilpotent element $f \in \mathfrak{gl}_N$ by

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Introduce a grading on $\mathfrak{gl}_N$ by $\deg e_{ij} = \mathrm{col}(j) - \mathrm{col}(i)$.

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$$\mathfrak{gl}_N = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \qquad f \in \mathfrak{g}_{-1}.$$

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$$\mathfrak{b} = \bigoplus_{p \leqslant 0} \mathfrak{g}_p \qquad \text{and} \qquad \mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p.$$

Equip $\mathfrak{b}$ with the symmetric invariant bilinear form

$$\langle X, Y \rangle = \frac{k+N}{2N} \operatorname{tr}_{\mathfrak{b}}(\operatorname{ad} X \operatorname{ad} Y) - \frac{1}{2} \operatorname{tr}_{\mathfrak{g}_0}(\operatorname{ad} X \operatorname{ad} Y).$$

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Introduce the Lie superalgebra

$$\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_0 \oplus \widehat{\mathfrak{a}}_1 \qquad \text{with} \quad \widehat{\mathfrak{a}}_0 = \widehat{\mathfrak{b}}, \qquad \widehat{\mathfrak{a}}_1 = \mathfrak{m}[t, t^{-1}],$$

with the adjoint action of $\widehat{\mathfrak{a}}_0$ on $\widehat{\mathfrak{a}}_1$, whereas $\widehat{\mathfrak{a}}_1$ is regarded as a supercommutative Lie superalgebra.

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We will write $\psi_{ji}[r] = e_{ji}\, t^{r-1}$ for $e_{ji}\, t^{r-1} \in \mathfrak{m}[t, t^{-1}]$.

Equip the vacuum module $V^k(\mathfrak{a})$ with the (-1)-product and introduce its derivation

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$$\big[Q, e_{j\,i}[-1]\big] = \sum_{\mathrm{col}(a)=i}^{j-1} \psi_{j\,a}[0]\, e_{a\,i}[-1] - \sum_{\mathrm{col}(a)=i+1}^{j} e_{j\,a}[-1]\, \psi_{a\,i}[0]$$

$$+ \big(k + N - q_{\mathrm{col}(i)}\big)\, \psi_{j\,i}[-1] + \psi_{j^+\,i}[0] - \psi_{j\,i^-}[0]$$

for dominoes $i^- i$ and $j\, j^+$ occurring in π.

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The $\mathcal{W}$-algebra is defined by

$$\mathcal{W}^k(\mathfrak{g}, f) = \{ v \in V^k(\mathfrak{b}) \mid Q\, v = 0 \}.$$

By a theorem of [V. Kac and M. Wakimoto, 2004]

(also [T. Arakawa 2005]),

there exists a filtration $F_p \mathcal{W}^k(\mathfrak{g}, f)$ such that

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In the principal nilpotent case, the generator $W^{(i)}$ is associated with the element $e_{i\,1} + e_{i+1\,2} + \cdots + e_{N\,N-i+1} \in \mathfrak{g}^f$.

Rectangular pyramids

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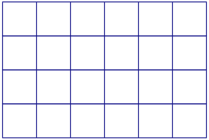
Take a pyramid with n rows and l columns,



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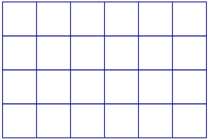

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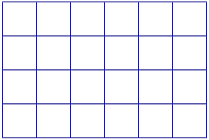

so that $p_1 = \cdots = p_n = l$ are the lengths of the rows

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We have $\dim \mathfrak{g}^f = l\,n^2$.

We will use the isomorphism $\mathfrak{gl}_l \otimes \mathfrak{gl}_n \cong \mathfrak{gl}_N$ such that

$$[e_{ij}]_{i,j=1}^{N} = \begin{bmatrix} e_{11} \otimes E & \dots & e_{1l} \otimes E \\ \dots & \dots & \dots \\ e_{l1} \otimes E & \dots & e_{ll} \otimes E \end{bmatrix},$$

where $E = [e_{ab}]_{a,b=1}^{n}$.

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where $E = [e_{ab}]_{a,b=1}^{n}$.

Explicitly,

$$e_{(i-1)n+a,\,(j-1)n+b} = e_{ij} \otimes e_{ab}.$$

Define the homomorphism from the tensor algebra

$$\mathcal{T} : \mathrm{T}\big(\mathfrak{gl}_l[t^{-1}]t^{-1}\big) \to \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}\big(\mathfrak{gl}_N[t^{-1}]t^{-1}\big),$$

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by setting for $x \in \mathfrak{gl}_l[t^{-1}]t^{-1}$:

$$\mathcal{T}_{ab}(x) = x \otimes e_{ba} \in \mathfrak{gl}_l[t^{-1}]t^{-1} \otimes \mathfrak{gl}_n = \mathfrak{gl}_N[t^{-1}]t^{-1}.$$

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Hence, for any elements x, y of the tensor algebra,

$$\mathcal{T}_{ab}(xy) = \sum_{c=1}^{n} \mathcal{T}_{ac}(x)\,\mathcal{T}_{cb}(y).$$

Consider the matrix

$$\mathcal{E} = \begin{bmatrix} \alpha\tau + e_{11}[-1] & -1 & 0 & \dots & 0 \\ e_{21}[-1] & \alpha\tau + e_{22}[-1] & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & -1 \\ e_{l1}[-1] & e_{l2}[-1] & \dots & \dots & \alpha\tau + e_{ll}[-1] \end{bmatrix}$$

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$$W_{ab} = \mathcal{T}_{ab}\big(\text{cdet}\, \mathcal{E}\big), \qquad a, b = 1, \dots, n.$$

Write

$$W_{ab} = \delta_{ab}(\alpha\tau)^l + W_{ab}^{(1)}(\alpha\tau)^{l-1} + \cdots + W_{ab}^{(l)},$$

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Theorem [T. Arakawa–A. M., 2014]

The coefficients $W_{ab}^{(r)}$ with $a, b \in \{1, \ldots, n\}$ and $r = 1, \ldots, l$ generate the $\mathcal{W}$-algebra $\mathcal{W}^{k}(\mathfrak{g}, f)$.

The Miura map $V^k(\mathfrak{b}) \to V^k(\mathfrak{l})$ with $\mathfrak{l} = \mathfrak{gl}_n \oplus \cdots \oplus \mathfrak{gl}_n$

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Corollary. Under the Miura map we have

$$\Upsilon : \mathcal{T}_{ab}\big(\mathrm{cdet}\,\mathcal{E}\big) \mapsto \mathcal{T}_{ab}\Big(\big(\alpha\tau + e_{1\,1}[-1]\big)\dots\big(\alpha\tau + e_{l\,l}[-1]\big)\Big).$$

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By a general result of [N. Genra, 2016] the image of the Miura map coincides with the intersection of the kernels of screening operators (k is generic).

Classical $\mathcal{W}$-algebras

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Divide by k each row of the matrix

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and set $\bar{e}_{ij}[r] = e_{ij}[r]/k$.

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$$\begin{bmatrix} \alpha\tau + e_{11}[-1] & -1 & 0 & \dots & 0 \\ e_{21}[-1] & \alpha\tau + e_{22}[-1] & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & -1 \\ e_{N1}[-1] & e_{N2}[-1] & \dots & \dots & \alpha\tau + e_{NN}[-1] \end{bmatrix}$$

and set $\bar{e}_{ij}[r] = e_{ij}[r]/k$. Since $\alpha = k + N - 1$, in the limit $k \to \infty$

the column-determinant equals

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$$\big(\tau + \bar{e}_{11}[-1]\big) \dots \big(\tau + \bar{e}_{NN}[-1]\big).$$

We thus recover the (classical) Miura transformation

$$\big(\tau + \bar{e}_{11}[-1]\big)\ldots\big(\tau + \bar{e}_{NN}[-1]\big) = \tau^{N} + u^{(1)}\tau^{N-1} + \cdots + u^{(N)}$$

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The elements $u^{(i)}$ and all their derivatives are algebraically

independent generators of $\mathcal{W}(\mathfrak{gl}_N)$.

At the critical level $k = -N$ the expansion of the column-determinant yields a different presentation of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_N)$.

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Its elements are understood as differential polynomials in the variables $E_{ij} := e_{ij}[-1]$ with $N \geqslant i \geqslant j \geqslant 1$.

Expand the column-determinant

$$\mathrm{cdet}\begin{bmatrix} -\tau+E_{11} & -1 & 0 & 0 & \dots & 0 \\ E_{21} & -\tau+E_{22} & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{N1} & E_{N2} & E_{N3} & \dots & \dots & -\tau+E_{NN} \end{bmatrix}$$

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The elements $w^{(1)},\dots,w^{(N)}$ together with all their derivatives are algebraically independent generators of $\mathcal{W}(\mathfrak{gl}_N)$.

By taking the Zhu algebra of $\mathcal{W}^k(\mathfrak{gl}_N, f)$ for a rectangular pyramid, we recover the generators of the finite $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_N, f)$.

[E. Ragoucy and P. Sorba, 1999];

[J. Brundan and A. Kleshchev, 2006].

Affine Poisson vertex algebra $\mathcal{V}(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and

let $X_1, \ldots, X_d$ be a basis of $\mathfrak{g}$.

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$$\mathcal{V} = \mathbb{C}[X_1^{(r)}, \ldots, X_d^{(r)} \mid r = 0, 1, 2, \ldots] \qquad \text{with} \quad X_i^{(0)} = X_i,$$

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equipped with the derivation ∂,

$$\partial(X_i^{(r)}) = X_i^{(r+1)}$$

for all $i = 1, \ldots, d$ and $r \geqslant 0$.

Introduce the λ-bracket on $\mathcal{V}$ as a linear map

$$\mathcal{V} \otimes \mathcal{V} \to \mathbb{C}[\lambda] \otimes \mathcal{V}, \qquad a \otimes b \mapsto \{a_\lambda b\}.$$

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$$\{a\,{}_\lambda\, bc\} = \{a\,{}_\lambda\, b\}c + \{a\,{}_\lambda\, c\}b.$$

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The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

$$\mathcal{W}(\mathfrak{g}) = \{P \in \mathcal{V}(\mathfrak{b}) \mid \rho\{X_\lambda P\} = 0 \quad \text{for all} \quad X \in \mathfrak{n}_+\}.$$

The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is a Poisson vertex algebra equipped with the λ-bracket

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Motivation: integrable Hamiltonian hierarchies

Drinfeld and Sokolov, 1985;

De Sole, Kac and Valeri, 2013-16.

Generators of $\mathcal{W}(\mathfrak{gl}_N)$

Consider $\mathfrak{gl}_N =$ span of $\{E_{ij} \mid i,j = 1,\ldots,N\}$. Here $\mathfrak{b}$ is the subalgebra of lower triangular matrices.

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The invariant symmetric bilinear form on $\mathfrak{gl}_N$ is defined by

$$(X|Y) = \operatorname{tr} XY, \qquad X, Y \in \mathfrak{gl}_N.$$

Expand the column-determinant with entries in $\mathcal{V}(\mathfrak{b}) \otimes \mathbb{C}[\partial]$,

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Theorem [M.–Ragoucy, 2015], [De Sole–Kac–Valeri, 2015].

All elements $w_1, \dots, w_N$ belong to $\mathcal{W}(\mathfrak{gl}_N)$. Moreover,

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as a differential operator

$$\partial^{2n+1} + w_2\,\partial^{2n-1} + w_3\,\partial^{2n-2} + \cdots + w_{2n+1}, \qquad w_i \in \mathcal{V}(\mathfrak{b}).$$

Theorem [MR]. All elements $w_2, \dots, w_{2n+1}$ belong to $\mathcal{W}(\mathfrak{o}_{2n+1})$.

Moreover,

$$\mathcal{W}(\mathfrak{o}_{2n+1}) = \mathbb{C}\left[w_2^{(r)}, w_4^{(r)}, \dots, w_{2n}^{(r)} \mid r \geqslant 0\right].$$

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Generators of $\mathcal{W}(\mathfrak{sp}_{2n})$

The Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements

$$F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{j'i'}, \qquad i,j = 1,\dots,2n,$$

is the symplectic Lie algebra $\mathfrak{sp}_{2n}$, where

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Introduce the algebra of pseudo-differential operators

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Introduce the algebra of pseudo-differential operators

$\mathcal{V}(\mathfrak{b}) \otimes \mathbb{C}((\partial^{-1}))$,

$$\partial^{-1} F_{ij}^{(r)} = \sum_{s=0}^{\infty} (-1)^s F_{ij}^{(r+s)} \partial^{-s-1}.$$

Take the principal nilpotent element $f \in \mathfrak{o}_{2n}$ in the form

$$f = F_{21} + F_{32} + \dots + F_{n\,n-1} + F_{n'\,n-1}.$$

Remark. Under the embedding $\mathfrak{o}_{2n} \subset \mathfrak{gl}_{2n}$, $f \mapsto \widetilde{f}$,

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$\widetilde{f}$ is not a principal nilpotent in $\mathfrak{gl}_{2n}$:

$$\widetilde{f} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}$$

Expand the column-determinant of the $(2n+1)\times(2n+1)$ matrix

$$\begin{bmatrix} \partial+F_{1\,1} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ F_{2\,1} & \partial+F_{2\,2} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n\,1}-F_{n'\,1} & F_{n\,2}-F_{n'\,2} & \dots & \partial+F_{n\,n} & 0 & -2\partial & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \partial^{-1} & 0 & \dots & 0 & 0 \\ F_{n'\,1} & F_{n'\,2} & \dots & 0 & 0 & \partial+F_{n'\,n'} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ F_{2'\,1} & 0 & \dots & \dots & 0 & F_{2'\,n'}-F_{2'\,n} & \dots & \partial+F_{2'\,2'} & -1 \\ 0 & F_{1'\,2} & \dots & \dots & 0 & F_{1'\,n'}-F_{1'\,n} & \dots & F_{1'\,2'} & \partial+F_{1'\,1'} \end{bmatrix}$$

Expand the column-determinant of the $(2n+1)\times(2n+1)$ matrix

$$\begin{bmatrix} \partial+F_{11} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ F_{21} & \partial+F_{22} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n1}-F_{n'1} & F_{n2}-F_{n'2} & \dots & \partial+F_{nn} & 0 & -2\partial & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \partial^{-1} & 0 & \dots & 0 & 0 \\ F_{n'1} & F_{n'2} & \dots & 0 & 0 & \partial+F_{n'n'} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ F_{2'1} & 0 & \dots & \dots & 0 & F_{2'n'}-F_{2'n} & \dots & \partial+F_{2'2'} & -1 \\ 0 & F_{1'2} & \dots & \dots & 0 & F_{1'n'}-F_{1'n} & \dots & F_{1'2'} & \partial+F_{1'1'} \end{bmatrix}$$

as a pseudo-differential operator

$$\partial^{2n-1}+w_2\,\partial^{2n-3}+w_3\,\partial^{2n-4}+\cdots+w_{2n-1}+(-1)^n\,y_n\,\partial^{-1}\,y_n.$$

Theorem [MR]. All elements $w_2, w_3, \dots, w_{2n-1}$ and y_n belong to $\mathcal{W}(\mathfrak{o}_{2n})$. Moreover,

$$\mathcal{W}(\mathfrak{o}_{2n}) = \mathbb{C}\left[w_2^{(r)}, w_4^{(r)}, \dots, w_{2n-2}^{(r)}, y_n^{(r)} \mid r \geqslant 0\right].$$

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We have

$$y_n = \operatorname{cdet} \begin{bmatrix} \partial + F_{11} & 1 & 0 & 0 & \ldots & 0 \\ F_{21} & \partial + F_{22} & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ F_{n-1\,1} & F_{n-1\,2} & F_{n-1\,3} & \ldots & \ldots & 1 \\ F_{n\,1} - F_{n'\,1} & F_{n\,2} - F_{n'\,2} & F_{n\,3} - F_{n'\,3} & \ldots & \ldots & \partial + F_{n\,n} \end{bmatrix} 1.$$

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$$\phi : \mathcal{V}(\mathfrak{b}) \to \mathcal{V}(\mathfrak{h})$$

denote the homomorphism of differential algebras defined on the generators as the projection $\mathfrak{b} \to \mathfrak{h}$ with the kernel $\mathfrak{n}_-$.

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The restriction of ϕ to $\mathcal{W}(\mathfrak{g})$ is injective. The embedding

$$\phi : \mathcal{W}(\mathfrak{g}) \hookrightarrow \mathcal{V}(\mathfrak{h})$$

is known as the Miura transformation.

For $\mathfrak{g} = \mathfrak{gl}_N$, the image of the column-determinant

$$\mathrm{cdet}\begin{bmatrix} \partial + E_{11} & 1 & 0 & 0 & \dots & 0 \\ E_{21} & \partial + E_{22} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{N-1\,1} & E_{N-1\,2} & E_{N-1\,3} & \dots & \dots & 1 \\ E_{N\,1} & E_{N\,2} & E_{N\,3} & \dots & \dots & \partial + E_{N\,N} \end{bmatrix}$$

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equals

$$(\partial + E_{11}) \dots (\partial + E_{N\,N}) = \partial^{\,N} + w_1\,\partial^{\,N-1} + \dots + w_N.$$

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equals

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Therefore, we recover the Adler–Gelfand–Dickey generators:

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}\left[w_1^{(r)}, \dots, w_N^{(r)} \mid r \geqslant 0\right].$$

Drinfeld–Sokolov generators for $\mathfrak{o}_{2n+1}$:

$$(\partial + F_{11}) \dots (\partial + F_{nn})\, \partial\, (\partial - F_{nn}) \dots (\partial - F_{11})$$

$$= \partial^{2n+1} + w_2\, \partial^{2n-1} + w_3\, \partial^{2n-2} + \dots + w_{2n+1},$$

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Drinfeld–Sokolov generators for $\mathfrak{o}_{2n}$:

$$(\partial + F_{11}) \dots (\partial + F_{nn})\, \partial^{-1} (\partial - F_{nn}) \dots (\partial - F_{11})$$

$$= \partial^{2n-1} + w_2\, \partial^{2n-3} + w_3\, \partial^{2n-4} + \dots + w_{2n-1} + (-1)^n\, y_n\, \partial^{-1}\, y_n.$$

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In particular,

$$y_n = (\partial + F_{11}) \dots (\partial + F_{nn})\, 1.$$

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In particular,

$$y_n = (\partial + F_{11}) \dots (\partial + F_{nn})\, 1.$$

Then

$$\mathcal{W}(\mathfrak{o}_{2n}) = \mathbb{C}\big[w_2^{(r)}, w_4^{(r)}, \dots, w_{2n-2}^{(r)}, y_n^{(r)} \,|\, r \geqslant 0\big].$$