Higher order Hamiltonians for the trigonometric Gaudin model

Alexander Molev

University of Sydney

Joint work with Eric Ragoucy

Rational Gaudin model

Rational Gaudin model

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Rational Gaudin model

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Consider the polynomial current Lie algebra $t^{-1}\mathfrak{g}[t^{-1}]$ which spanned by the elements $X[r] = Xt^r$ with $X \in \mathfrak{g}$ and $r < 0$.

Rational Gaudin model

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Consider the polynomial current Lie algebra $t^{-1}\mathfrak{g}[t^{-1}]$ which spanned by the elements $X[r] = Xt^r$ with $X \in \mathfrak{g}$ and $r < 0$.

Given any element $\chi \in \mathfrak{g}^*$ and a nonzero $z \in \mathbb{C}$, the mapping

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g}), \qquad X[r] \mapsto Xz^r + \delta_{r,-1}\,\chi(X),$$

defines a (shifted) evaluation homomorphism.

Using the coassociativity of the standard coproduct on $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined by

$$\Delta : Y \mapsto Y \otimes 1 + 1 \otimes Y, \qquad Y \in t^{-1}\mathfrak{g}[t^{-1}],$$

Using the coassociativity of the standard coproduct on $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined by

$$\Delta : Y \mapsto Y \otimes 1 + 1 \otimes Y, \qquad Y \in t^{-1}\mathfrak{g}[t^{-1}],$$

for any $\ell \geqslant 1$ we get the homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\otimes \ell}$$

as an iterated coproduct map.

Now fix distinct complex numbers $z_1, \ldots, z_\ell$ and let u be a complex parameter.

Now fix distinct complex numbers $z_1, \ldots, z_\ell$ and let u be a complex parameter. Applying the evaluation homomorphisms to the tensor factors, we get another homomorphism

$$\Psi : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g})^{\otimes \ell},$$

Now fix distinct complex numbers $z_1, \dots, z_\ell$ and let u be a complex parameter. Applying the evaluation homomorphisms to the tensor factors, we get another homomorphism

$$\Psi : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g})^{\otimes \ell},$$

given by

$$\Psi : X[r] \mapsto \sum_{a=1}^{\ell} X_a (z_a - u)^r + \delta_{r,-1}\, \chi(X) \in \mathrm{U}(\mathfrak{g})^{\otimes \ell},$$

where $X_a = 1^{\otimes(a-1)} \otimes X \otimes 1^{\otimes(\ell-a)}$.

Feigin–Frenkel center

Feigin–Frenkel center

Consider the affine Kac—Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K.$$

Feigin–Frenkel center

Consider the affine Kac—Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K.$$

Identify $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ with the vacuum module $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)\mathbf{1}$ over $\widehat{\mathfrak{g}}$ at the critical level:

Feigin–Frenkel center

Consider the affine Kac—Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K.$$

Identify $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ with the vacuum module $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)\mathbf{1}$ over $\widehat{\mathfrak{g}}$ at the critical level:

it is generated by the vacuum vector $\mathbf{1}$ such that

$$\mathfrak{g}[t]\,\mathbf{1} = 0 \qquad \text{and} \qquad K\,\mathbf{1} = -h^{\vee}\,\mathbf{1}.$$

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined as the subalgebra of invariants:

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)\mathbf{1} \mid \mathfrak{g}[t]\,v = 0\}.$$

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined as the subalgebra of invariants:

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)\mathbf{1} \mid \mathfrak{g}[t]\,v = 0\}.$$

Elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ are known as Segal–Sugawara vectors.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined as the subalgebra of invariants:

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)\mathbf{1} \mid \mathfrak{g}[t]\,v = 0\}.$$

Elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ are known as Segal–Sugawara vectors.

Higher Gaudin Hamiltonians are obtained by taking the images of Segal–Sugawara vectors under the homomorphism

$$\Psi : \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g})^{\otimes \ell}$$

[FFR 1994, Rybnikov 2006, FFTL 2010].

In particular, the quadratic Gaudin Hamiltonian arises from the canonical Segal–Sugawara vector

$$S = \sum_{a=1}^{d} J_a[-1]J^a[-1],$$

where $J_1, \ldots, J_d$ and $J^1, \ldots, J^d$ are dual bases of $\mathfrak{g}$ with respect to the normalized Killing form.

Explicit generators of the Feigin–Frenkel center were found in

type A by A. Chervov and D. Talalaev (2006),

in types B, C, D by A. M. (2013) and

in type G_2 by A. M., E. Ragoucy and N. Rozhkovskaya (2016).

Explicit generators of the Feigin–Frenkel center were found in

type A by A. Chervov and D. Talalaev (2006),

in types B, C, D by A. M. (2013) and

in type G_2 by A. M., E. Ragoucy and N. Rozhkovskaya (2016).

This yields explicit higher Gaudin Hamiltonians in those cases

and reproduces Talalaev's formulas in type A (2006).

Higher Gaudin Hamiltonians in type A

Higher Gaudin Hamiltonians in type A

A family of higher Gaudin Hamiltonians for $\mathfrak{g} = \mathfrak{gl}_N$ arises from the coefficients of the differential operators

$$\operatorname{tr}\big(\partial_u + E(u)\big)^k, \qquad k = 1, 2, \dots$$

which form a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$.

Higher Gaudin Hamiltonians in type A

A family of higher Gaudin Hamiltonians for $\mathfrak{g} = \mathfrak{gl}_N$ arises from the coefficients of the differential operators

$$\mathrm{tr}\big(\partial_u + E(u)\big)^k, \qquad k = 1, 2, \dots$$

which form a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$.

Here $E(u) = [E_{ij}(u)]$ is the matrix with the entries

$$E_{ij}(u) = \sum_{r<0} E_{ij}[r]\, u^{-r-1}, \qquad E_{ij}[r] = E_{ij}\, t^r.$$

Trigonometric Gaudin model

Trigonometric Gaudin model

Now $t^{-1}\mathfrak{gl}_N[t^{-1}]$ is replaced by the extended Lie algebra

$$\widehat{\mathfrak{g}}^{+} = \mathfrak{b}^{+} \oplus t^{-1}\mathfrak{gl}_N[t^{-1}],$$

where $\mathfrak{b}^{+}$ is the subalgebra of $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqslant j$.

Trigonometric Gaudin model

Now $t^{-1}\mathfrak{gl}_N[t^{-1}]$ is replaced by the extended Lie algebra

$$\widehat{\mathfrak{g}}^+ = \mathfrak{b}^+ \oplus t^{-1}\mathfrak{gl}_N[t^{-1}],$$

where $\mathfrak{b}^+$ is the subalgebra of $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqslant j$. Accordingly, $E(u)$ is replaced by the matrix $\mathcal{L}^+(u) = [\mathcal{L}^+_{ij}(u)]$ such that

Trigonometric Gaudin model

Now $t^{-1}\mathfrak{gl}_N[t^{-1}]$ is replaced by the extended Lie algebra

$$\widehat{\mathfrak{g}}^+ = \mathfrak{b}^+ \oplus t^{-1}\mathfrak{gl}_N[t^{-1}],$$

where $\mathfrak{b}^+$ is the subalgebra of $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqslant j$. Accordingly, $E(u)$ is replaced by the matrix $\mathcal{L}^+(u) = [\mathcal{L}^+_{ij}(u)]$ such that

$$\mathcal{L}^+(u) = -\begin{pmatrix} E_{11} & \dots & 2\,E_{1N} \\ \vdots & \ddots & \vdots \\ 0 & \dots & E_{NN} \end{pmatrix} - 2\sum_{n=1}^{\infty} E[-n]\,u^n.$$

The coefficients of the series $\operatorname{tr} \mathcal{L}^+(u)^2$ are pairwise commuting elements of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$ [Sklyanin (1987), Jurčo (1989)].

The coefficients of the series $\operatorname{tr}\mathcal{L}^+(u)^2$ are pairwise commuting elements of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$ [Sklyanin (1987), Jurčo (1989)].

As with the rational Gaudin model, this series is the generating function for quadratic Hamiltonians: taking the image in the tensor product of the vector representations, we get

$$\mathcal{L}^+(u) \mapsto r_{01}(u/a_1) + \cdots + r_{0l}(u/a_l)$$

for some parameters a_i,

The coefficients of the series $\operatorname{tr}\mathcal{L}^+(u)^2$ are pairwise commuting elements of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$ [Sklyanin (1987), Jurčo (1989)].

As with the rational Gaudin model, this series is the generating function for quadratic Hamiltonians: taking the image in the tensor product of the vector representations, we get

$$\mathcal{L}^+(u) \mapsto r_{01}(u/a_1) + \cdots + r_{0l}(u/a_l)$$

for some parameters a_i, where

$$r(x) = \sum_{i,j=1}^{N} \Big(\frac{1+x}{1-x} + \operatorname{sgn}(j-i) \Big)\, e_{ij} \otimes e_{ji}.$$

The function $r(x)$ is a trigonometric classical r-matrix.

The function $r(x)$ is a trigonometric classical r-matrix. It satisfies the classical Yang–Baxter equation

$$\circlearrowleft_{1,2,3} \left[r_{12}(x_1/x_2), r_{23}(x_2/x_3) \right] = 0$$

The function $r(x)$ is a trigonometric classical r-matrix. It satisfies the classical Yang–Baxter equation

$$\circlearrowleft_{1,2,3} \left[r_{12}(x_1/x_2), r_{23}(x_2/x_3) \right] = 0$$

together with the skew-symmetry condition

$$r_{12}(x) + r_{21}(1/x) = 0.$$

The function $r(x)$ is a trigonometric classical r-matrix. It satisfies the classical Yang–Baxter equation

$$\circlearrowleft_{1,2,3} \left[r_{12}(x_1/x_2), r_{23}(x_2/x_3) \right] = 0$$

together with the skew-symmetry condition

$$r_{12}(x) + r_{21}(1/x) = 0.$$

Taking the residue at a_i, we recover the i-th Gaudin Hamiltonian

$$\operatorname*{res}_{u=a_i} \operatorname{tr} \mathcal{L}^+(u)^2 = 2a_i \sum_{j \neq i} r_{ij}(a_i/a_j),$$

assuming the parameters a_i are all distinct and nonzero.

Our goal is to construct higher order Hamiltonians for the trigonometric Gaudin model. They are obtained from a commuting family of elements of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^{+})$.

Our goal is to construct higher order Hamiltonians for the trigonometric Gaudin model. They are obtained from a commuting family of elements of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^+)$.

This commuting family is analogous to the one produced from the differential operators $\mathrm{tr}\big(\partial_u + E(u)\big)^k$:

Our goal is to construct higher order Hamiltonians for the trigonometric Gaudin model. They are obtained from a commuting family of elements of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^{+})$.

This commuting family is analogous to the one produced from the differential operators $\operatorname{tr}\big(\partial_u + E(u)\big)^k$:

the highest degree term of the corresponding operator coincides with $\operatorname{tr}\mathcal{L}^{+}(u)^k$.

Introduce the function $T(y)$ in a variable y with values in $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ by

$$T(y) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1}{1-y} \sum_{i<j} e_{ij} \otimes e_{ji} + \frac{1}{1+y} \sum_{i>j} e_{ij} \otimes e_{ji}.$$

Introduce the function $T(y)$ in a variable y with values in

$\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ by

$$T(y) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1}{1-y} \sum_{i<j} e_{ij} \otimes e_{ji} + \frac{1}{1+y} \sum_{i>j} e_{ij} \otimes e_{ji}.$$

For any $1 \leqslant a < b \leqslant s$ we let $T_{ab}(y)$ denote the function $T(y)$

regarded as an element of the algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{s} \otimes \mathrm{U}(\widehat{\mathfrak{g}}^{+}),$$

Introduce the function $T(y)$ in a variable y with values in

$\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ by

$$T(y) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \frac{1}{1-y} \sum_{i<j} e_{ij} \otimes e_{ji} + \frac{1}{1+y} \sum_{i>j} e_{ij} \otimes e_{ji}.$$

For any $1 \leqslant a < b \leqslant s$ we let $T_{ab}(y)$ denote the function $T(y)$

regarded as an element of the algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{s} \otimes \mathrm{U}(\widehat{\mathfrak{g}}^+),$$

associated with the a-th and b-th copies of $\mathrm{End}\,\mathbb{C}^N$ and as the

identity element in all the remaining tensor factors.

Define differential operators $\theta_m \in \mathrm{U}(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$ by means of the generating function

$$\sum_{m=1}^{\infty} \theta_m y^m = \sum_{s=1}^{\infty} y^s \, \mathrm{tr}_{1,\dots,s} \, T_{s-1\, s}(y) \dots T_{1\, 2}(y) \, \mathcal{L}_1 \dots \mathcal{L}_s,$$

Define differential operators $\theta_m \in \mathrm{U}(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$ by means of the generating function

$$\sum_{m=1}^{\infty} \theta_m y^m = \sum_{s=1}^{\infty} y^s \, \mathrm{tr}_{1,\dots,s} \, T_{s-1\, s}(y) \dots T_{1\, 2}(y) \, \mathcal{L}_1 \dots \mathcal{L}_s,$$

where $\mathcal{L} = 2u\partial_u - \mathcal{L}^+(u)$ and the trace is taken over all s copies of $\mathrm{End}\, \mathbb{C}^N$.

We have

$$\theta_1 = \operatorname{tr}\mathcal{L} = 2Nu\partial_u - \operatorname{tr}\mathcal{L}^+(u).$$

We have

$$\theta_1 = \operatorname{tr} \mathcal{L} = 2Nu\partial_u - \operatorname{tr} \mathcal{L}^+(u).$$

Furthermore,

$$\theta_2 = \operatorname{tr}_{1,2} P_{12}\mathcal{L}_1\mathcal{L}_2 = \operatorname{tr} \mathcal{L}^2 = \operatorname{tr}\big(2u\partial_u - \mathcal{L}^+(u)\big)^2$$

$$= 4Nu^2\partial_u^2 - 4u\big(\operatorname{tr} \mathcal{L}^+(u) - N\big)\partial_u - 2u \operatorname{tr} \mathcal{L}^+(u)' + \operatorname{tr} \mathcal{L}^+(u)^2$$

We have

$$\theta_1 = \operatorname{tr} \mathcal{L} = 2Nu\partial_u - \operatorname{tr} \mathcal{L}^+(u).$$

Furthermore,

$$\theta_2 = \operatorname{tr}_{1,2} P_{12}\mathcal{L}_1\mathcal{L}_2 = \operatorname{tr} \mathcal{L}^2 = \operatorname{tr}\big(2u\partial_u - \mathcal{L}^+(u)\big)^2$$

$$= 4Nu^2\partial_u^2 - 4u\big(\operatorname{tr} \mathcal{L}^+(u) - N\big)\partial_u - 2u \operatorname{tr} \mathcal{L}^+(u)' + \operatorname{tr} \mathcal{L}^+(u)^2$$

and

$$\theta_3 = \operatorname{tr}\big(2u\partial_u - \mathcal{L}^+(u)\big)^3 + \sum_{i,j=1}^{N} \operatorname{sgn}(i-j)\mathcal{L}^+_{ij}(u)\mathcal{L}^+_{ji}(u).$$

For any $m \geqslant 1$ the differential operator θ_m takes the form

$$\theta_m = \theta_m^{(0)}\,\partial_u^m + \cdots + \theta_m^{(m-1)}\,\partial_u + \theta_m^{(m)},$$

For any $m \geqslant 1$ the differential operator θ_m takes the form

$$\theta_m = \theta_m^{(0)}\,\partial_u^m + \dots + \theta_m^{(m-1)}\,\partial_u + \theta_m^{(m)},$$

where each $\theta_m^{(k)}$ is a power series in u with coefficients in the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^+)$.

For any $m \geqslant 1$ the differential operator θ_m takes the form

$$\theta_m = \theta_m^{(0)}\,\partial_u^m + \cdots + \theta_m^{(m-1)}\,\partial_u + \theta_m^{(m)},$$

where each $\theta_m^{(k)}$ is a power series in u with coefficients in the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^+)$. In particular,

$$\theta_m^{(m)} = (-1)^m \operatorname{tr} \mathcal{L}^+(u)^m + \text{ lower degree terms.}$$

For any $m \geqslant 1$ the differential operator θ_m takes the form

$$\theta_m = \theta_m^{(0)}\,\partial_u^m + \dots + \theta_m^{(m-1)}\,\partial_u + \theta_m^{(m)},$$

where each $\theta_m^{(k)}$ is a power series in u with coefficients in the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^+)$. In particular,

$$\theta_m^{(m)} = (-1)^m \operatorname{tr} \mathcal{L}^+(u)^m + \text{ lower degree terms.}$$

This follows from the relation

$$\operatorname{tr}_{1,\dots,s} P_{s-1\,s} \dots P_{1\,2}\,\mathcal{L}_1 \dots \mathcal{L}_s = \operatorname{tr} \mathcal{L}^s = \operatorname{tr}\big(2u\partial_u - \mathcal{L}^+(u)\big)^s.$$

Theorem [M.–Ragoucy 2018].

The coefficients of all power series $\theta_m^{(k)}$ generate a commutative subalgebra of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$.

Theorem [M.–Ragoucy 2018].

The coefficients of all power series $\theta_m^{(k)}$ generate a commutative subalgebra of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$.

The commuting family quantizes the well-known Hamiltonians $\operatorname{tr} L(u)^m$ of the classical trigonometric Gaudin model [O. Babelon, C.-M. Viallet 1990, T. Skrypnyk 2007].

Examples.

Examples.

The commutative subalgebra contains the coefficients of the power series $\mathrm{tr}\,\mathcal{L}^{+}(u)^{2}$,

Examples.

The commutative subalgebra contains the coefficients of the power series $\operatorname{tr} \mathcal{L}^+(u)^2$, as well as the coefficients of the power series

$$\operatorname{tr} \mathcal{L}^+(u)^3 - 2u \operatorname{tr} \mathcal{L}^+(u)\mathcal{L}^+(u)' + \sum_{i,j=1}^{N} \operatorname{sgn}(j-i)\mathcal{L}^+_{ij}(u)\mathcal{L}^+_{ji}(u).$$

Calculating classical limits

Calculating classical limits

Begin with the Bethe subalgebra of the q-Yangian $\mathrm{Y}_q(\mathfrak{gl}_N)$.

Calculating classical limits

Begin with the Bethe subalgebra of the q-Yangian $\mathrm{Y}_q(\mathfrak{gl}_N)$.

The algebra $\mathrm{Y}_q(\mathfrak{gl}_N)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad 1 \leqslant i,j \leqslant N, \qquad r = 0, 1, \dots,$$

with the conditions that $l_{ij}^{+}[0] = 0$ for $i > j$ and the elements $l_{ii}^{+}[0]$ are invertible,

Calculating classical limits

Begin with the Bethe subalgebra of the q-Yangian $\mathrm{Y}_q(\mathfrak{gl}_N)$.

The algebra $\mathrm{Y}_q(\mathfrak{gl}_N)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad 1 \leqslant i,j \leqslant N, \qquad r = 0, 1, \dots,$$

with the conditions that $l_{ij}^{+}[0] = 0$ for $i > j$ and the elements $l_{ii}^{+}[0]$

are invertible, subject to the defining relations

$$R(u/v)L_1^{+}(u)L_2^{+}(v) = L_2^{+}(v)L_1^{+}(u)R(u/v).$$

Here $L^{+}(u) = \big[\, l_{ij}^{+}(u) \big]$ and

$$l_{ij}^{+}(u) = \sum_{r=0}^{\infty} l_{ij}^{+}[-r]\, u^{r}.$$

The R-matrix is given by

$$R(x) = \sum_i e_{ii} \otimes e_{ii} + \frac{1-x}{q-q^{-1}x} \sum_{i \neq j} e_{ii} \otimes e_{jj}$$
$$+ \frac{(q-q^{-1})\,x}{q-q^{-1}x} \sum_{i>j} e_{ij} \otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x} \sum_{i<j} e_{ij} \otimes e_{ji}.$$

The R-matrix is given by

$$R(x) = \sum_i e_{ii} \otimes e_{ii} + \frac{1-x}{q-q^{-1}x} \sum_{i\neq j} e_{ii} \otimes e_{jj} + \frac{(q-q^{-1})\,x}{q-q^{-1}x} \sum_{i>j} e_{ij} \otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x} \sum_{i<j} e_{ij} \otimes e_{ji}.$$

Consider the q-permutation

$P^q \in \mathrm{End}\,(\mathbb{C}^N \otimes \mathbb{C}^N) \cong \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N$ defined by

$$P^q = \sum_i e_{ii} \otimes e_{ii} + q \sum_{i>j} e_{ij} \otimes e_{ji} + q^{-1} \sum_{i<j} e_{ij} \otimes e_{ji}.$$

The symmetric group $\mathfrak{S}_k$ acts on the tensor product space $(\mathbb{C}^N)^{\otimes k}$ by $s_a \mapsto P^q_{s_a} := P^q_{a\,a+1}$ for $a = 1, \ldots, k-1$, where s_a denotes the transposition $(a, a+1)$.

The symmetric group $\mathfrak{S}_k$ acts on the tensor product space $(\mathbb{C}^N)^{\otimes k}$ by $s_a \mapsto P^q_{s_a} := P^q_{a\,a+1}$ for $a = 1, \ldots, k-1$, where s_a denotes the transposition $(a, a+1)$.

Denote by $A^{(k)}$ the image of the normalized antisymmetrizer associated with the q-permutations:

$$A^{(k)} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot P^q_{\sigma}.$$

For each $k = 1, \dots, N$ consider the power series in u defined by

$$\mathrm{tr}_{1,\dots,k} A^{(k)} L_1^+(u) \dots L_k^+(uq^{-2k+2})$$

with the trace taken over all k copies of $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^N}_{k} \otimes \mathrm{Y}_q(\mathfrak{gl}_N)[[u]].$$

It is well-known that the coefficients of all power series generate a commutative subalgebra $\mathcal{B}_N$ of $\mathrm{Y}_q(\mathfrak{gl}_N)$.

Another family of generators of this subalgebra can be obtained from the Newton identities

[A. Chervov, G. Falqui, V. Rubtsov, A. Silantyev 2014].

Another family of generators of this subalgebra can be obtained

from the Newton identities

[A. Chervov, G. Falqui, V. Rubtsov, A. Silantyev 2014].

They imply that the coefficients of all power series

$$\mathrm{tr}_{1,\dots,k}\, P^{q}_{(k,k-1,\dots,1)} L^{+}_{1}(u)\dots L^{+}_{k}(uq^{-2k+2}), \qquad k = 1, 2, \dots$$

belong to $\mathcal{B}_N$.

Another family of generators of this subalgebra can be obtained from the Newton identities

[A. Chervov, G. Falqui, V. Rubtsov, A. Silantyev 2014].

They imply that the coefficients of all power series

$$\mathrm{tr}_{1,\dots,k}\, P^{q}_{(k,k-1,\dots,1)} L^{+}_1(u)\dots L^{+}_k(uq^{-2k+2}), \qquad k=1,2,\dots$$

belong to $\mathcal{B}_N$.

Introduce the operator δ such that $\delta\, g(u) = g(uq^{-2})\,\delta$. Adjoining this element to the algebra $\mathrm{Y}_q(\mathfrak{gl}_N)[[u]]$, set $M = L^{+}(u)\,\delta$.

For each $m \geqslant 1$ consider the expression

$$\mathcal{M}_m = \frac{1}{(q-1)^m} \left(1 - (M_m)^{\rightarrow}\right) \left(P_{m-1\,m} - P^q_{m-1\,m}(M_{m-1})^{\rightarrow}\right) \dots \left(P_{12} - P^q_{12}(M_1)^{\rightarrow}\right) 1,$$

For each $m \geqslant 1$ consider the expression

$$\mathcal{M}_m = \frac{1}{(q-1)^m} \left(1 - (M_m)^{\rightarrow}\right) \left(P_{m-1\,m} - P^q_{m-1\,m} (M_{m-1})^{\rightarrow}\right) \dots \left(P_{12} - P^q_{12} (M_1)^{\rightarrow}\right) 1,$$

where the arrow indicates right multiplication:

$$\left(P_{a\,a+1} - P^q_{a\,a+1} \big(M_a\big)^{\rightarrow}\right) X := P_{a\,a+1}\, X - P^q_{a\,a+1}\, X\, M_a.$$

For each $m \geqslant 1$ consider the expression

$$\mathcal{M}_m = \frac{1}{(q-1)^m} \left(1 - (M_m)^{\rightarrow}\right) \Big(P_{m-1\,m} - P^{q}_{m-1\,m}(M_{m-1})^{\rightarrow}\Big) \\ \dots \Big(P_{12} - P^{q}_{12}(M_1)^{\rightarrow}\Big)\,1,$$

where the arrow indicates right multiplication:

$$\Big(P_{a\,a+1} - P^{q}_{a\,a+1}\big(M_a\big)^{\rightarrow}\Big)\,X := P_{a\,a+1}\,X - P^{q}_{a\,a+1}\,X\,M_a.$$

Take trace over all m copies of $\mathrm{End}\,\mathbb{C}^N$:

$$\mathrm{tr}_{1,\dots,m}\,\mathcal{M}_m \in \mathrm{Y}_q(\mathfrak{gl}_N)[[u]][\delta].$$

Lemma. All coefficients of the polynomial $\mathrm{tr}_{1,\dots,m}\,\mathcal{M}_m$ belong to the algebra $\mathcal{B}_N[[u]]$.

Lemma. All coefficients of the polynomial $\mathrm{tr}_{1,\dots,m}\,\mathcal{M}_m$ belong to the algebra $\mathcal{B}_N[[u]]$.

Calculate the classical limits $q \to 1$ of these elements.

Lemma. All coefficients of the polynomial $\mathrm{tr}_{1,\dots,m}\,\mathcal{M}_m$ belong to the algebra $\mathcal{B}_N[[u]]$.

Calculate the classical limits $q \to 1$ of these elements. They will form a commuting family of elements of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}^{+})$.

Write

$$\delta = 1 - 2(q-1)\,u\partial_u + \dots.$$

We have

$$L^+(u) = 1 + (q-1)\,\mathcal{L}^+(u) + \dots$$

and

$$1 - M = (q-1)\,\mathcal{L} + \dots$$

with $\mathcal{L} = 2u\partial_u - \mathcal{L}^+(u)$.

Furthermore,

$$P - P^q = (q-1)\mathcal{T} + \dots,$$

Furthermore,

$$P - P^q = (q-1)\mathcal{T} + \dots,$$

where

$$\mathcal{T} = \sum_{i,j=1}^{N} \operatorname{sgn}(j-i)\, e_{ij} \otimes e_{ji}.$$

Furthermore,

$$P - P^q = (q-1)\mathcal{T} + \dots,$$

where

$$\mathcal{T} = \sum_{i,j=1}^{N} \operatorname{sgn}(j-i)\, e_{ij} \otimes e_{ji}.$$

The coefficients θ_m are then found by

$$\theta_m = \operatorname{tr}_{1,\dots,m} \overline{\mathcal{M}}_m,$$

Furthermore,

$$P - P^q = (q-1)\mathcal{T} + \dots,$$

where

$$\mathcal{T} = \sum_{i,j=1}^{N} \operatorname{sgn}(j-i)\, e_{ij} \otimes e_{ji}.$$

The coefficients θ_m are then found by

$$\theta_m = \operatorname{tr}_{1,\dots,m} \overline{\mathcal{M}}_m,$$

where $\overline{\mathcal{M}}_m$ is the classical limit of the polynomial $\mathcal{M}_m$.

Invariants of the vacuum module

Invariants of the vacuum module

Consider the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ in its RLL presentation.

Invariants of the vacuum module

Consider the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ in its RLL presentation. We will need the normalized R-matrix

$$\overline{R}(x) = f(x)R(x),$$

Invariants of the vacuum module

Consider the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ in its RLL presentation. We will need the normalized R-matrix

$$\overline{R}(x) = f(x)R(x),$$

where, as before,

$$R(x) = \sum_i e_{ii} \otimes e_{ii} + \frac{1-x}{q-q^{-1}x} \sum_{i \neq j} e_{ii} \otimes e_{jj} + \frac{(q-q^{-1})\,x}{q-q^{-1}x} \sum_{i>j} e_{ij} \otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x} \sum_{i<j} e_{ij} \otimes e_{ji}$$

and

$$f(x) = 1 + \sum_{k=1}^{\infty} f_k(q)\, x^k$$

and

$$f(x) = 1 + \sum_{k=1}^{\infty} f_k(q) x^k$$

is a formal power series in x whose coefficients $f_k(q)$ are rational functions in q uniquely determined by the relation

$$f(xq^{2N}) = f(x) \frac{(1 - xq^2)(1 - xq^{2N-2})}{(1 - x)(1 - xq^{2N})}.$$

The quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad l_{ij}^{-}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant N, \qquad r = 0, 1, \dots,$$

and the invertible central element q^c, subject to the defining relations

The quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad l_{ij}^{-}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant N, \qquad r = 0,1,\dots,$$

and the invertible central element q^c, subject to the defining relations

$$l_{ji}^{+}[0] = l_{ij}^{-}[0] = 0 \qquad \text{for} \qquad 1 \leqslant i < j \leqslant N,$$

$$l_{ii}^{+}[0]\, l_{ii}^{-}[0] = l_{ii}^{-}[0]\, l_{ii}^{+}[0] = 1 \qquad \text{for} \qquad i = 1,\dots,N,$$

The quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad l_{ij}^{-}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant N, \qquad r = 0,1,\dots,$$

and the invertible central element q^c, subject to the defining relations

$$l_{ji}^{+}[0] = l_{ij}^{-}[0] = 0 \qquad \text{for} \qquad 1 \leqslant i < j \leqslant N,$$

$$l_{ii}^{+}[0]\, l_{ii}^{-}[0] = l_{ii}^{-}[0]\, l_{ii}^{+}[0] = 1 \qquad \text{for} \qquad i = 1,\dots,N,$$

and

$$R(u/v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R(u/v),$$

$$\overline{R}(uq^{-c}/v)L_1^{+}(u)L_2^{-}(v) = L_2^{-}(v)L_1^{+}(u)\overline{R}(uq^{c}/v).$$

The vacuum module at the critical level $c = -N$ over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is the universal module $V_q(\mathfrak{gl}_N)$ generated by a nonzero vector $\mathbf{1}$ subject to the conditions

$$L^-(u)\mathbf{1} = I\mathbf{1}, \qquad q^c\mathbf{1} = q^{-N}\mathbf{1},$$

where I denotes the identity matrix.

The vacuum module at the critical level $c = -N$ over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is the universal module $V_q(\mathfrak{gl}_N)$ generated by a nonzero vector $\mathbf{1}$ subject to the conditions

$$L^{-}(u)\mathbf{1} = I\mathbf{1}, \qquad q^{c}\mathbf{1} = q^{-N}\mathbf{1},$$

where I denotes the identity matrix.

As a vector space, $V_q(\mathfrak{gl}_N)$ can be identified with the subalgebra $\mathrm{Y}_q(\mathfrak{gl}_N)$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ generated by the coefficients of all series $l^{+}_{ij}(u)$ subject to the additional relations $l^{+}_{ii}[0] = 1$.

The subspace of invariants of $V_q(\mathfrak{gl}_N)$ is defined by

$$\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N) = \{v \in V_q(\mathfrak{gl}_N) \mid L^-(u)v = Iv\}.$$

The subspace of invariants of $V_q(\mathfrak{gl}_N)$ is defined by

$$\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N) = \{v \in V_q(\mathfrak{gl}_N) \mid L^-(u)v = Iv\}.$$

Identify $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N)$ with a subspace of $\mathrm{Y}_q(\mathfrak{gl}_N)$.

The subspace of invariants of $V_q(\mathfrak{gl}_N)$ is defined by

$$\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N) = \{v \in V_q(\mathfrak{gl}_N) \mid L^-(u)v = Iv\}.$$

Identify $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_N)$ with a subspace of $\mathrm{Y}_q(\mathfrak{gl}_N)$.

This subspace is closed under the multiplication in the quantum affine algebra and can be regarded as a subalgebra of $\mathrm{Y}_q(\mathfrak{gl}_N)$.

Define differential operators $\vartheta_m \in \mathrm{U}(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$ by

$$\sum_{m=1}^{\infty} \vartheta_m y^m = \sum_{s=1}^{\infty} y^s \operatorname{tr}_{1,\dots,s} T_{s-1\,s}(y) \dots T_{1\,2}(y) \overline{\mathcal{L}}_1 \dots \overline{\mathcal{L}}_s,$$

Define differential operators $\vartheta_m \in \mathrm{U}(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$ by

$$\sum_{m=1}^{\infty} \vartheta_m y^m = \sum_{s=1}^{\infty} y^s \operatorname{tr}_{1,\dots,s} T_{s-1\, s}(y) \dots T_{1\,2}(y)\, \overline{\mathcal{L}}_1 \dots \overline{\mathcal{L}}_s,$$

where $\overline{\mathcal{L}} = 2u\partial_u - \rho - \mathcal{L}^+(u)$ and ρ is the diagonal matrix

$$\rho = \operatorname{diag}\big[N-1, N-3, \dots, -N+1\big].$$

Define differential operators $\vartheta_m \in \mathrm{U}(\widehat{\mathfrak{g}}^+)[[u, \partial_u]]$ by

$$\sum_{m=1}^{\infty} \vartheta_m y^m = \sum_{s=1}^{\infty} y^s \operatorname{tr}_{1,\dots,s} T_{s-1\,s}(y) \dots T_{1\,2}(y)\, \overline{\mathcal{L}}_1 \dots \overline{\mathcal{L}}_s,$$

where $\overline{\mathcal{L}} = 2u\partial_u - \rho - \mathcal{L}^+(u)$ and ρ is the diagonal matrix

$$\rho = \operatorname{diag}\big[N-1, N-3, \dots, -N+1\big].$$

The differential operator ϑ_m takes the form

$$\vartheta_m = \vartheta_m^{(0)}\, \partial_u^m + \dots + \vartheta_m^{(m-1)}\, \partial_u + \vartheta_m^{(m)},$$

where each $\vartheta_m^{(k)}$ is a power series in $\mathrm{U}(\widehat{\mathfrak{g}}^+)[[u]]$.

Theorem [M.–Ragoucy 2018].

The coefficients of the power series $\vartheta_m^{(k)}$ generate a commutative subalgebra of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$.

Theorem [M.–Ragoucy 2018].

The coefficients of the power series $\vartheta_m^{(k)}$ generate a commutative subalgebra of $\mathrm{U}(\widehat{\mathfrak{g}}^+)$.

Moreover, these coefficients belong to the algebra of invariants $\mathfrak{z}_{\mathrm{tr}}(\widehat{\mathfrak{gl}}_N)$ of the vacuum module $V_q(\mathfrak{gl}_N)$.