RATIONAL POINTS AND COXETER GROUP ACTIONS ON THE COHOMOLOGY OF TORIC VARIETIES

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Abstract. We derive a simple formula for the action of a finite crystallographic Coxeter group on the cohomology of its associated complex toric variety, using the method of counting rational points over finite fields, and the Hodge structure of the cohomology. Various applications are given, including the determination of the graded multiplicity of the reflection representation.

1. Introduction and statement of main result

Let V be an vector space of finite dimension n over $\mathbb{R}$. Let Φ be a root system in V, and let W be the associated Coxeter group, which is generated by the reflections in hyperplanes orthogonal to the roots; we take W to be finite and crystallographic, and write $\langle -, - \rangle$ for a W-invariant positive definite bilinear form on V. Assume chosen a simple system $\Pi \subseteq \Phi$, which forms a basis of V. Let $L := \mathbb{Z}\Phi$ be the root lattice, and $M := \{v \in V \mid \langle v, \alpha \rangle \in \mathbb{Z} \ \forall \ \alpha \in L\}$ be the corresponding weight lattice.

As explained in [F], there is a fan $\Delta = \Delta_W$ of convex polynedral cones in M, and hence a "toric variety" associated with this data. This is a smooth complex projective variety, which we shall denote by $\mathcal{T}_W$. This variety, and hence its cohomology, carries a natural action of the group W. In this work we shall determine this action, in the sense that we shall give an explicit formula for the equivariant Poincaré polynomial

$$P_W(t, w) := \sum_{i \geq 0} \mathrm{Trace}(w, H^i(\mathcal{T}_W, \mathbb{C}))t^i \in \mathbb{C}[t],$$

for each element $w \in W$.

Equivalently, if $R(W)$ denotes the complex character ring of W, we shall determine the element

$$P_W(t) := \sum_i H^i(\mathcal{T}_W, \mathbb{C})t^i \in R(W)[t]$$

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by means of its value on elements $w \in W$. This question arises, among other places, in the study (cf. [DP, DPGM]) of compactifications of reductive groups and the cohomology of complete symmetric varieties.

This problem was addressed by different methods by Procesi in [P]. He made use of the fact that $\mathcal{T}_W$ may be described in terms of repeated blowups, and the cohomology of the blowup of a space along a subspace is straightforward to compute. Procesi's result is well suited to the recursive determination of $P_W(t)$.

Stembridge, in [St], studied the same problem indirectly, using a result of Danilov [Da] to identify the cohomology ring with a certain commutative algebra. His result [St, Corollary 1.6] is similar to our Theorem 1.1 below, but retains a recursive flavour. The main thrust of [St] is the identification of the total cohomology with a permutation representation of W. In their work [DoLu, Theorem 2.1] Dolgachev and Lunts also prove the same formula as Stembridge, using the T-equivariant cohomology of the toric variety $\mathcal{T}_W$. In the Appendix below, we prove that the Dolgachev-Lunts-Stembridge formula is equivalent to ours.

In this work we use the method of counting rational points over finite fields, combined with the Hodge structure of the cohomology, developed in [Le2, DL, KL1, KL2]. This results in a quick and direct derivation of a closed formula for $P_W(w,t)$, which is quite easy to evaluate in many cases. Some consequences of the general formula are discussed in §§3,4; included among these are the fact that the alternating representation of W does not occur in $H^*(\mathcal{T}_W,\mathbb{C})$, and a formula giving the graded multiplicity of the reflection representation of W in the cohomology ring (see §4.1).

The case $w=1$ of our formula (1.2) below was proved by Fulton in [F, §4.5], for smooth complete toric varieties $\mathcal{T}$ which includes our $\mathcal{T}_W$; the weight filtration of the cohomology $H^*(\mathcal{T},\mathbb{C})$ is also determined for these $\mathcal{T}$. Our work may be regarded as an equivariant generalisation of the results in [F, *loc. cit.]* for the particular varieties $\mathcal{T}_W$.

We proceed now to state our basic formula.

For each subset $J \subseteq \Pi$, let W_J be the corresponding parabolic subgroup generated by the reflections in the hyperplanes orthogonal to the roots in J, and V_J the linear span of J.

Theorem 1.1. *Let Φ, Π, W and $\mathcal{T}_W$ be as above. Then $H^i(\mathcal{T}_W,\mathbb{C})=0$ if i is odd. The even dimensional cohomology is described as follows. For each $J \subseteq \Pi$, let $\gamma_J(t)$ be the $\mathbb{C}[t]$-valued class function on W_J given by $\gamma_{J,t}(w) := \det_{V_J}(t^2-w)$, where this is interpreted as 1 if $J=\emptyset$. Then*

$$P_W(t) = \sum_{J \subseteq \Pi} \mathrm{Ind}_{W_J}^W(\gamma_{J,t}). \tag{1.2}$$

This may be reformulated as follows.

Corollary 1.3. *Maintain the above notation. For each subset $J \subseteq \Pi$, let $\rho_{J,i}$ be the i^{th} exterior power of the (reflection) representation of W_J on V_J ($i = 1, \dots, |J|$). Then*

$$P_W(t) = \sum_{i=0}^{n} (-t^2)^i \sum_{\substack{J \subseteq \Pi \\ |J| \geq i}} (-1)^{|J|} \operatorname{Ind}_{W_J}^{W} \rho_{J,|J|-i}. \tag{1.4}$$

Proof. If $w \in W_J$ has eigenvalues $\lambda_1, \dots, \lambda_{|J|}$ on V_J, then $\det_{V_J}(t^2 - w) = \prod_{j=1}^{|J|}(t^2 - \lambda_j)$. It follows that

$$\gamma_{J,t}(w) = \sum_{i=0}^{|J|} t^{2(|J|-i)} (-1)^i \rho_{J,i}(w).$$

The assertion is now immediate from Theorem 1.1. □

Theorem 1.1 may be restated as the assertion that $H^{2i+1}(\mathcal{T}_W, \mathbb{C}) = 0$ for all i, which of course is well known (cf. [F, Prop., p.92], or (2.1)(iii) below), while as W-module,

$$H^{2i}(\mathcal{T}_W, \mathbb{C}) \cong (-1)^i \sum_{\substack{J \subseteq \Pi \\ |J| \geq i}} (-1)^{|J|} \operatorname{Ind}_{W_J}^{W} \rho_{J,|J|-i}. \tag{1.5}$$

2. Proof of the main theorem

Our basic tools will be the Hodge structure of $H^*(\mathcal{T}_W)$, and the counting of rational points over finite fields (cf. [F, p. 94] and [KL1, KL2, Le3]). The following result is well known.

Lemma 2.1. *(i) Let $Z = Z(\Delta)$ be the toric variety associated with a fan Δ. If d_k is the number of k-dimensional polyhedral cones in Δ ($k = 1, \dots, n = \dim Z$), then the (non-equivariant) compactly supported weight polynomial (for the definition see* [DL, (1.5)]*) is given by*

$$W_c(Z, t) = \sum_{k=0}^{n} d_k (t^2 - 1)^{n-k}.$$

(ii) [F, p. 94] *The number of points of the $\mathbb{Z}$-scheme Z over $\mathbb{F}_q$ is*

$$|Z(\mathbb{F}_q)| = \sum_{k=0}^{n} d_k (q-1)^{n-k} := S(q).$$

(iii)[1] *If Δ is simplicial and complete, in particular if Z is non-singular and projective, then Z has only even cohomology. Moreover $H^{2j}(Z,\mathbb{C})$ is a pure Hodge structure of type (j,j). Thus Z is mixed Tate in the sense of* [KL2].

Proof. The statements (i) and (ii) may be found in [F, pp. 94,104] and in [DL, (2.8),(3.3),§5].

If Δ is simplicial and complete, then the compact supports weight polynomial $W_c(Z,1) = \dim H_c^*(Z,\mathbb{C}) = \sum_j \dim H_c^j(Z,\mathbb{C})$, (see [F, pp. 93,94]). Thus, writing $S(q)$ for the polynomial which gives the number of $\mathbb{F}_q$-points of Z, we have $S(1) = d_0 = \dim H_c^*(Z,\mathbb{C})$. All the assertions of (iii) now follow immediately from [KL2, Proposition 3.3(2)]. $\square$

For any variety (i.e. reduced scheme of finite type) X defined over the finite field $\mathbb{F}_q$, denote by F the endomorphism of $X \otimes \overline{\mathbb{F}_q} := X(q)$ obtained by raising local coordinates to the q^{th} power. The action induced by F on ℓ-adic cohomology is defined as follows. There is a natural action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on the ℓ-adic cohomology spaces $H_c^j(X \otimes \overline{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$. The action induced on $H_c^j(X \otimes \overline{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ by the *inverse* of the arithmetic (q-power) Frobenius automorphism in $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ will also be denoted by F. With this convention, we have the well known fixed point formula of Grothendieck:

$$|X^F| = \sum_{j=0}^{2\dim X} (-1)^j \,\mathrm{Trace}(F, H_c^j(X(q), \overline{\mathbb{Q}}_\ell)). \tag{2.2}$$

Proposition 2.3. *For any element $w \in W$, the cardinality $|\mathcal{T}_W^{wF}|$ is a polynomial $S(q,w)$ in q, and we have*

$$P_W(t,w) = S(t^2,w).$$

Proof. The automorphism w of $\mathcal{T}_W$ clearly commutes with the geometric Frobenius endomorphism described above. It follows that w and F induce commuting endomorphisms on $H_c^j(X(q), \overline{\mathbb{Q}}_\ell)$. Hence from Grothendieck's fixed point formula (2.2) we have

$$\begin{aligned} |\mathcal{T}_W^{wF}| &= \sum_{j=0}^{2n} (-1)^j \,\mathrm{Trace}(wF, H_c^j(\mathcal{T}_W(q), \overline{\mathbb{Q}}_\ell)) \\ &= \sum_{j=0}^{2n} \mathrm{Trace}(w, H^{2j}(\mathcal{T}_W(q), \overline{\mathbb{Q}}_\ell)) q^j \text{ by Poincaré duality and 2.1(iii)} \\ &= \sum_{j=0}^{2n} \mathrm{Trace}(w, H^{2j}(\mathcal{T}_W, \mathbb{C})) q^j \text{ for almost all } q, \text{ by [KL1, (1.2)].} \end{aligned}$$

[1] See Remark 2.8 below

The Proposition is now immediate. □

Rather than applying Proposition 2.3 directly, we shall make use of the fact that there is an action of the torus $T \cong (\mathbb{C}^\times)^n$ on $\mathcal{T}_W$ which partitions $\mathcal{T}_W$ into the (finite) union of its orbits, which are locally closed subvarieties, each isomorphic to a torus.

The following result, (see [Le3, Theorem 2.5]), is designed to handle this type of situation. For any complex algebraic variety with a G-action, where G is a finite group, $W^G_{c,X}(t)$ denotes the compactly supported equivariant weight polynomial

$$W^G_{c,X}(t) = \sum_m \sum_j (-1)^j \operatorname{Gr}^W_m H^j_c(X, \mathbb{C}) t^m,$$

regarded as an element of $R(G)[t]$, where $R(G)$ is the Grothendieck ring of complex representations of G and Gr^W_m denotes the m^{th} graded component of the weight filtration of H^j_c.

Proposition 2.4. *(cf.*[DL, Le3, McM]*) Let X be a complex algebraic variety with a G-action, where G is a finite group. Suppose X is a finite disjoint union $X = \coprod_{i \in \mathcal{I}} X_i$ of locally closed subvarieties X_i which are permuted by G. Then*

$$W^G_{c,X}(t) = \sum_{\iota \in \mathcal{I}/G} \operatorname{Ind}^G_{G_i} W^{G_i}_{c,X_i}(t), \tag{2.4.1}$$

where the sum is over the G-orbits ι in $\mathcal{I}$, i is any element of ι, and G_i is the isotropy group of i in G.

We are now in a position to give the

Proof of Theorem 1.1. In case $X = \mathcal{T}_W$ and $G = W$, let Γ be the set of polyhedral cones of the fan defined by the root system Φ. This is also described as the set of closures of the regions into which V is partitioned by the reflecting hyperplanes of W. As explained in [F, Chapter 3], the torus $T = T_\Lambda \cong (\mathbb{C}^\times)^n$ acts on $\mathcal{T}_W$. For each cone $\tau \in \Gamma$, there is a distinguished point $x_\tau \in \mathcal{T}_W$, and the orbit $Z(\tau) := T \cdot x_\tau$ is isomorphic to a torus of dimension equal to $n - \dim \tau$. Moreover $\mathcal{T}_W$ is the disjoint union of the tori $Z(\tau)$. To describe the W-action, we require the following details.

The cones $\tau \in \Gamma$ are in bijection with the cosets wW_J ($w \in W, J \subseteq \Pi$) of the standard parabolic subgroups W_J of W. We have $\dim Z(\tau) = n - \dim \tau$, and wW_J is a face of $w'W_{J'}$ if $wW_J \supseteq w'W_{J'}$. If $\tau(wW_J)$ denotes the cone corresponding to wW_J and $Z(wW_J)$ denotes the corresponding T-orbit in $\mathcal{T}_W$, then $\dim \tau(wW_J) = n - |J|$, so $\dim Z(wW_J) = |J|$, and the character group of $Z(wW_J)$ is the lattice $\mathbb{Z}\Phi_{w(J)}$, where Φ_K is the sub-root system of Φ spanned by K. Thus the cone $\tau = \{0\}$ corresponds to W, and $Z(\{0\}) = Z(W)$ is the dense orbit $T = (\mathbb{C}^\times)^n$ in $\mathcal{T}_W$. Similarly the $|W|$ chambers of V each correspond to a torus of dimension 0, i.e. a point in $\mathcal{T}_W$.

The action of W is described as follows. The element $g \in W$ takes $Z(\tau)$ to $Z(g\tau)$, i.e. $Z(wW_J)$ to $Z(gwW_J)$. The set Γ/W of orbits of W on Γ is therefore in bijection with the subsets J of Π. If $\mathcal{O}_J$ is the orbit corresponding to J, then we may (and do) select $\tau(W_J) \in \mathcal{O}_J$ as the representative element of the orbit. Note that the set of representatives $\{\tau(W_J) \mid J \subseteq \Pi\}$ is precisely the set of facets of the fundamental chamber of the W-action on V which corresponds to the simple system Π. Since the isotropy group of $\tau(W_J)$ is W_J, we have the following immediate consequence of Proposition 2.4.

$$W^W_{c,\mathcal{T}_W}(t) = \sum_{J \subseteq \Pi} \operatorname{Ind}^W_{W_J} W^{W_J}_{c,Z(W_J)}(t) \tag{2.5}$$

We are therefore reduced to computing

$$W^{W_J}_{c,Z(W_J)}(t,w) = \sum_m \sum_j (-1)^j \operatorname{Trace}(w, \operatorname{Gr}^W_m H^j_c(Z(W_J),\mathbb{C}))t^m$$

for $w \in W_J$. For this, observe first that $Z(W_J)$ is a torus of dimension $|J|$, and therefore is minimally pure [DL, §3]. Thus $H^j_c(Z(W_J),\mathbb{C})$ is pure of weight $2j - 2|J|$. Hence by [Le3, (2.6)], we have

$$W^{W_J}_{c,Z(W_J)}(t,w) = |Z(W_J)^{wF}|_{q \mapsto t^2} = S_{Z(W_J)}(t^2,w), \tag{2.6}$$

where $S_{Z(W_J)}(q,w)$ is the polynomial in q which gives the number of points of $Z(W_J)$ fixed by wF for almost all q.

But wF acts on the character group $\mathbb{Z}\Phi_J$ of $Z(W_J)$ as qw. It follows (see, e.g. [Ca, 3.2.3]) that $S_{Z(W_J)}(q,w) = |\det_{V_J}(qw-1)| = |\det_{V_J}(q-w)|$. Moreover since those eigenvalues of w which are not ± 1 come in conjugate pairs $e^{\pm i\theta}$ and $(q-e^{i\theta})(q-e^{-i\theta}) = q^2 - 2q\cos\theta + 1 \geq (q-1)^2 \geq 0$, we see that $S_{Z(W_J)}(q,w) = \det_{V_J}(q-w)$. Combining this with (2.6) and (2.5) we obtain

$$W^W_{c,\mathcal{T}_W}(t) = \sum_{J \subseteq \Pi} \operatorname{Ind}^W_{W_J} \gamma_J(t), \tag{2.7}$$

where, as in §1 above, γ_J is the class function on W_J which takes the value $\det_{V_J}(t^2 - w)$ on $w \in W_J$.

But by Lemma 2.1 (iii) and Poincaré duality, $H_c^{2j}(\mathcal{T}_W,\mathbb{C})$ is a pure Hodge structure of weight $2j$, while $H_c^{2j+1}(\mathcal{T}_W,\mathbb{C})=0$ for all j. It follows that

$$\begin{aligned} W^W_{c,\mathcal{T}_W}(t) &= \sum_m \sum_j (-1)^j \operatorname{Gr}^W_m H_c^j(\mathcal{T}_W,\mathbb{C}))t^m \\ &= \sum_j H_c^{2j}(\mathcal{T}_W,\mathbb{C})t^{2j} \text{ by Lemma 2.1 (iii) and Poincaré duality} \\ &= \sum_j H^{2j}(\mathcal{T}_W,\mathbb{C})t^{2j} \text{ by Poincaré duality} \\ &= P_{\mathcal{T}_W}(t). \end{aligned}$$

This completes the proof of Theorem 1.1. □

Remark 2.8. The proof of Theorem 1.1 above amounts to the computation of the polynomial $S(q,w)$ of Proposition 2.3, with the induced character formula being a convenient way to organise the computation. Explicitly, we have proved that

$$S(q,w)=\sum_{J\subseteq\Pi}\operatorname{Ind}_{W_J}^W(\det{}_{V_J}(q-w)),$$

where $w\mapsto\det_{V_J}(q-w)$ is to be thought of as a class function on W_J (when $J=\emptyset$, this function is identically 1). In this sense, our main result is a generalisation, of course applicable only to the varieties $\mathcal{T}_W$, of the formula in [F, p. 94], which is the case $w=1$ of our formula.

It follows from this formula (which is proved independently of any assertions concerning the cohomology) that $S(1,1)=|W|$. Moreover by (2.6), the weight polynomial $W_{c,\mathcal{T}_W}(t,1)=S(q,1)_{q\mapsto t^2}=S(t^2,1)$. But since $\mathcal{T}_W$ is smooth and projective, it follows from the results of [DL] or from [F, (1), p. 92] that $W_{c,\mathcal{T}_W}(t,1)$ coincides with the Poincaré polynomial of $\mathcal{T}_W$. This shows immediately (as is pointed out in [F, p.92]) that its odd cohomology vanishes and that $\sum_j\dim H^j(\mathcal{T},\mathbb{C})=|W|=S(1,1)$. Moreover it follows from [KL2, Proposition 3.3(2)] that $\mathcal{T}_W$ is mixed Tate. See [DL] for a general discussion of weight polynomials along the lines of [F, pp. 92–95].

Alternatively, it follows from the non-singular projective nature of $\mathcal{T}_W$ that [KL2, (3.7.1)] holds, i.e. that the eigenvalues of Frobenius on $H^i(\mathcal{T}_W,\overline{\mathbb{Q}}_\ell)$ all have absolute value $q^{\frac{i}{2}}$. The arguments on [KL2, p. 212] then show that all the above facts for $\mathcal{T}_W$ follow from the polynomial nature of $|\mathcal{T}_W(\mathbb{F}_q)|$.

Thus the case $w=1$ of our formula (which is due to Fulton [F]) suffices to determine the Hodge structure of the cohomology, and this in turn permits the application of our counting argument to the determination of the graded character.

3. Some applications

In this section we point out some consequences of the results above. We begin by noting that it suffices to consider irreducible root systems.

Proposition 3.1. *Suppose Φ is reducible. Then $\Phi = \Phi_1 \amalg \Phi_2$, where the Φ_i are mutually orthogonal, and if $V_i = \mathbb{C}\Phi_i$ ($i = 1, 2$) then $V = V_1 \oplus V_2$, and $W = W_1 \times W_2$, where W_i is the Coxeter group with root system Φ_i in V_i.*

With notation as in Theorem 1.1, we have for $w = (w_1, w_2) \in W$

$$P_W(t, w) = P_{W_1}(t, w_1)P_{W_2}(t, w_2).$$

Equivalently, if p_1, p_2 are functions on W_1 and W_2 respectively, define $p = p_1p_2$ to be the function on $W = W_1 \times W_2$ given by $p(w_1, w_2) = p_1(w_1)p_2(w_2)$ (where $w_i \in W_i$). Then we have the following equation in $R(W)[t]$.

$$P_W(t) = P_{W_1}(t)P_{W_2}(t).$$

Proof. This is a simple consequence of the character formula provided by Theorem 1.1. □

Remark 3.2. Proposition 3.1 may also be deduced using the Künneth theorem from the following general fact.

Proposition 3.3. *(cf.* [F, pp 19–20]*) Let N_1 and N_2 be lattices in the real vector spaces V_1 and V_2. Let Δ_1 and Δ_2 be fans of rational convex polyhedral cones in V_1, V_2 respectively, and let $\mathcal{T}_1, \mathcal{T}_2$ be the corresponding toric varieties. Define the fan $\Delta_1 \oplus \Delta_2$ in $V_1 \oplus V_2$ as that which contains the cones $\sigma_1 \oplus \sigma_2$, where $\sigma_i \in \Delta_i$. Let $\mathcal{T}_{\Delta_1 \oplus \Delta_2}$ be the corresponding toric variety.*

Then $\mathcal{T}_{\Delta_1 \oplus \Delta_2} \simeq \mathcal{T}_1 \times \mathcal{T}_2$.

The proof of (3.3) reduces easily to the affine case, where it is straightforward. As an easy consequence, we have

Corollary 3.4. *With notation as in the statement of Proposition 3.1, we have $\mathcal{T}_W \simeq \mathcal{T}_{W_1} \times \mathcal{T}_{W_2}$.*

Applying the Künneth theorem to compute the cohomology of $\mathcal{T}_W$ using 3.4, we obtain 3.1.

Theorem 3.5. *Let W be a finite crystallographic Coxeter group, and $\mathcal{T}_W$ be the corresponding toric variety. Then in the notation above:*

(i) *(*[F, p. 94]*) The Poincaré polynomial of $\mathcal{T}_W$ is given by*

$$P_W(1, t) = \sum_j \dim H^j(\mathcal{T}_W, \mathbb{C})t^j = \sum_{J \subseteq \Pi} [W : W_J](t^2 - 1)^{|J|}.$$

(ii) *We have $(P_W(t), 1_W)_W = (1+t^2)^n$, where $(-, -)_W$ denotes inner product of class functions, and $P_W(t)$ is the class function given by $P_W(t)(w) = \sum_i \mathrm{Trace}(w, H^i(\mathcal{T}_W, \mathbb{C}))t^i$.*

(iii) *The alternating character of W does not occur in $H^i(\mathcal{T}_W,\mathbb{C})$ for any i.*

Proof. The statement (i) is simply the case $w=1$ of Theorem 1.1.

To see (ii), observe that by Frobenius reciprocity, it follows from 1.3 that

$$(P_W(t),1)_W=\sum_{i=0}^{n}t^{2i}\sum_{\substack{J\subseteq\Pi\\|J|\geq i}}(-1)^{|J|-i}(\rho_{J,|J|-i},1)_{W_J}.$$

But by [Bou, Exercice 3(a), p. 127], $(\rho_{J,|J|-i},1)_{W_J}=0$ unless $i=|J|$, in which case it is 1. Hence

$$(P_W(t),1)_W=\sum_{i=0}^{n}t^{2i}\sum_{\substack{J\subseteq\Pi\\|J|=i}}1=(1+t^2)^n,$$

which is the statement (ii).

Finally, in order to compute $(P_W(t),\varepsilon_W)_W$, note that the computation above shows that we need to know $(\rho_{J,k},\varepsilon_J)_{W_J}$ for each $J\subseteq\Pi$ and $k=0,1,\dots,|J|$, where ε_J is the alternating character of W_J. For this, we note that for any k, $\rho_{J,|J|-k}\cong\varepsilon_J\rho_{J,k}$. Hence by the argument above, $(\rho_{J,k},\varepsilon_J)_{W_J}=0$ unless $k=|J|$, and is 1 in that case. Hence again applying 1.3, it follows that

$$(P_W(t),\varepsilon)_W=\sum_{i=0}^{n}t^{2i}\sum_{\substack{J\subseteq\Pi\\|J|\geq i}}(-1)^{|J|-i}(\rho_{J,|J|-i},\varepsilon_J)_{W_J}=\sum_{J\subseteq\Pi}(-1)^{|J|}=0,$$

as asserted in (iii). □

Remark 3.6. Note that in view of Theorem 3.5(i), the polynomial

$$\sum_{J\subseteq\Pi}[W:W_J](t^2-1)^{|J|}$$

has positive coefficients, a fact which is not entirely obvious.

Proposition 3.7. *We have*

(i) *The character of W on the total cohomology ring is given by*

$$P_W(1)=\sum_{J\subseteq\Pi}\mathrm{Ind}_{W_J}^{W}(\gamma_{J,1}),\tag{3.8}$$

where $\gamma_{J,1}(w)=\det_{V_J}(1-w)$ for $w\in W_J$. It is a non-negative integer for any $w\in W$ (see [St, Proposition 1.7]*).*

(ii) *If w is a Coxeter element of W then $P_W(t,w)=\prod_{j=1}^{n}(t^2-\exp(\frac{2\pi i m_j}{h}))$, where h is the Coxeter number of W and $m_1,\dots,m_n$ are its exponents.*

(iii) *If w is any elliptic element of W, $P_W(t,w)=\prod_{j=1}^{n}(t^2-\lambda_j)$, where the λ_j are the eigenvalues of w on V.*

Proof. The first part of (i) follows immediately by putting $t = 1$ in Theorem 1.1. Further, the argument in the proof of Theorem 1.1 above shows that $\det_{V_J}(q - w) \geq 0$ for any real number $q \geq 1$, whence the positivity assertion (which is due to Stembridge).

Since w has no non-zero fixed points in V, w has no conjugates in W_J for $J \neq \Pi$. Thus by (1.1), $P_{\mathcal{T}_W}(t, w) = \det_V(t - w)$. But the eigenvalues of w on V are precisely $\{\exp(\frac{2\pi i m_j}{h}) \mid j = 1, \ldots, n\}$, and the statement (ii) is immediate. The proof of (iii) is the same. $\square$

In the special case when Φ is of type A_n, so that $W \cong \mathrm{Sym}_{n+1}$, we can be more explicit about the polynomials $P_W(t, w)$.

Proposition 3.9. *Let W be the Coxeter group of type A_n, so that $W \cong Sym_{n+1}$. Then*

(i) *If w is a Coxeter element of W, then $P_W(t, w) = 1 + t^2 + t^4 + \cdots + t^{2n}$.*
(ii) *The character of W on the total cohomology ring is given by $P_W(1, w) = (\sum_i m_i)! \prod_i i^{m_i}$ if w has cycle type (i^{m_i}), i.e. m_i cycles of length i for $i = 1, 2, \ldots$.*

Proof. The first statement is a special case of (3.7)(ii).

For the second, we apply (3.7)(i), noting that $\gamma_{J,1}$ is supported on the Coxeter class of W_J. Thus in order to apply Frobenius' formula for evaluation of induced characters, we note that to evaluate the right side of (3.7)(i) at w, only those J with associated partition (i^{m_i}) contribute. The actual evaluation is easy $\square$

Remark 3.10. Combining the statements (3.5(iii)) and (3.9(ii)), we obtain

$$\sum_{\lambda = (i^{m_i})} \frac{(\sum_i m_i)!}{\prod_i m_i!} = 2^{n-1},$$

where the sum is over the partitions λ of n

Remark 3.11. The varieties $\mathcal{T}_W$ are clearly defined over $\mathbb{R}$, and one may therefore speak of the space $\mathcal{T}_W(\mathbb{R})$ of real points of $\mathcal{T}_W$. The methods of [KL2, §5] may be used to investigate these spaces. As a very simple example we cite type A_1, where $\mathcal{T}_W = \mathbb{P}^1(\mathbb{C})$ and $\mathcal{T}_W(\mathbb{R}) = \mathbb{P}^1(\mathbb{R})$. In this case we have in the above notation (with $P_Y(t)$ denoting the usual Poincaré polynomial of a topological space),

$$P_{\mathcal{T}_W(\mathbb{R})}(t) = 1 + t = |\mathcal{T}_W(\overline{\mathbb{F}_q})^F|_{q \mapsto t}.$$

This example leads naturally to the question of how the Poincaré polynomials of the real varieties $\mathcal{T}_W(\mathbb{R})$ (both equivariant and otherwise) are related to the corresponding polynomials for the complex or finite field cases.

We conclude this section by giving the values of the polynomials $P_W(t, w)$ when Φ is the root system of type B_3. This is quickly calculated by hand using the results above. Recall that the conjugacy classes of the Weyl groups of type B_n are characterised by their "cycle type" $\lambda_1^{\pm}, \dots, \lambda_p^{\pm}$, where $\sum_j \lambda_j = n$. For example $-\operatorname{Id}_V$ is of type $1^-, \dots, 1^-$. There are ten conjugacy classes in $W(B_3)$, and the values of $P_W(t, w)$ are given in the table below.

Conjugacy class (w)	$P_W(t, w)$
$(1, 1, 1)$	$t^6 + 23t^4 + 23t^2 + 1$
$(1, 2)$	$t^6 + 7t^4 + 7t^2 + 1$
$(1^-, 1, 1)$	$t^6 + 7t^4 + 7t^2 + 1$
(3)	$t^6 + 2t^4 + 2t^2 + 1$
$1^-, 2)$	$t^6 + 3t^4 + 3t^2 + 1$
$(1, 2^-)$	$t^6 + t^4 + t^2 + 1$
$(1^-, 1^-, 1)$	$t^6 + 3t^4 + 3t^2 + 1$
(3^-)	$t^6 + 1$
$(1^-, 2^-)$	$t^6 + t^4 + t^2 + 1$
$1^-, 1^-, 1^-)$	$t^6 + 3t^4 + 3t^2 + 1$

4. The reflection representation

In this section we shall apply our main theorem to determine the multiplicity of the reflection representation $\rho = \rho_W$ of W in each cohomology space $H^{2i}(\mathcal{T}_W, \mathbb{C})$. We start with some basic facts concerning the reflection representation.

4.1. **The reflection representation.** Let K be a simple system for a reflection group H in $V = \mathbb{R}^n$. Suppose $K = \amalg_{i=1}^c K_i$ is the decomposition of K into irreducible components. Then correspondingly, $H = H_1 \times \dots \times H_c$, and $H_i = H_{K_i}$ acts irreducibly on V_{K_i}, the linear span of K_i, through its reflection representation ρ_i. Moreover if ρ_K is the reflection representation of H, its decomposition into irreducible components is given by

$$\rho_K = \oplus_{i=1}^c 1_{H_1} \otimes \dots \otimes 1_{H_{i-1}} \otimes \rho_i \otimes 1_{H_{i+1}} \otimes \dots \otimes 1_{H_c}. \tag{4.1}$$

Suppose Π is as in Theorem 1.1 and let $J \subseteq \Pi$. Then the restriction to W_J of the reflection representation ρ of W is given by

$$\operatorname{Res}_{W_J}^W \rho = \rho_J \oplus |\Pi \setminus J| 1_{W_J}, \tag{4.2}$$

where ρ_J is the reflection representation of W_J (on V_J).

Next, recall that if V_1 and V_2 are vector spaces, there is a canonical isomorphism of graded vector spaces $\Lambda(V_1 \oplus V_2) \xrightarrow{\simeq} \Lambda(V_1) \otimes \Lambda(V_2)$; i.e., for each index k, we have $\Lambda^k(V_1 \oplus V_2) \cong \oplus_{i+j=k} \Lambda^i(V_1) \otimes \Lambda^j(V_2)$. It follows from (4.1) that the decomposition of $\Lambda^k \rho_K$ into irreducibles is given by

$$\Lambda^k\rho_K = \oplus_{i_1+\dots+i_c=k}\Lambda^{i_1}\rho_1\otimes\Lambda^{i_2}\rho_2\otimes\dots\otimes\Lambda^{i_c}\rho_c. \tag{4.3}$$

Note that since the representations $\Lambda^i\rho_j$ are irreducible, this implies that $\Lambda^k\rho_K$ is multiplicity free.

4.2. **A combinatorial result about trees.** Our multiplicity formula will involve the Dynkin diagram of Φ, and to evaluate it explicitly, the following discussion will be useful. The author thanks Anthony Henderson for pointing out the degree of generality in which Proposition 4.6 below holds.

Let Θ be a tree, that is, a finite connected undirected graph with no circuits. Write $n = |\Theta|$, and for $0 \le k \le n$ define $c(\Theta, k)$ by

$$c(\Theta,k) = \sum_{\substack{J\subseteq\Theta\\ |J|=k}} c(J), \tag{4.4}$$

where $c(J)$ is the number of connected components of the subgraph (forest) spanned by J. Putting the $c(\Theta, k)$ into a generating polynomial, we define

$$c_\Theta(t) := \sum_{k=0}^{n} c(\Theta,k)t^{n-k} \in \mathbb{Z}[t]. \tag{4.5}$$

We shall prove

Proposition 4.6. *Let Θ be any tree with n vertices. Then*

$$c_\Theta(t) = (1+t)^{n-2}(1+nt).$$

Proof. Note first that for any tree, the number of vertices is one more than the number of edges. Since any subset J of Θ spans a forest (disjoint union of trees), it follows that $c(J)$ is the difference between $k = |J|$ and the number $e(J)$ of edges of J. Further, each edge of Θ occurs in precisely $\binom{n-2}{k-2}$ subsets J. It follows that

$$\begin{aligned} c(\Theta,k) &= \sum_{\substack{J\subseteq\Theta\\ |J|=k}} c(J)\\ &= \sum_{\substack{J\subseteq\Theta\\ |J|=k}} (k-e(J))\\ &= k\binom{n}{k} - (n-1)\binom{n-2}{k-2}\\ &= (n-k+1)\binom{n-1}{k-1}. \end{aligned}$$

Hence

$$\begin{aligned} c_\Theta(t) &= \sum_{k=0}^{n} c(\Theta,k)t^{n-k} \\ &= \sum_{k=1}^{n}(n-k+1)\binom{n-1}{k-1}t^{n-k} \\ &= \frac{d}{dt}\sum_{k=1}^{n}\binom{n-1}{k-1}t^{n-k+1} \\ &= \frac{d}{dt}\left(t(1+t)^{n-1}\right) \\ &= (1+t)^{n-2}(1+nt), \end{aligned}$$

as stated. □

Definition 4.7. *Define the polynomial $u_n(t)$ as the value of $c_\Theta(t)$ for any tree Θ with n vertices. That is,*

$$u_n(t) := (1+t)^{n-2}(1+nt).$$

4.3. **The multiplicity theorem.** In order to discuss our result, it is convenient to define the polynomial $\mathcal{N}_\Phi(t)$ which is associated with the root system Φ.

Definition 4.8. *Let Φ be a root system and let $\Pi \subset \Phi$ be a simple system in Φ. For each subset $J \subseteq \Pi$ denote by $c(J)$ the number of connected components of J (or of the root system Φ_J spanned by J). For each integer $i \geq 0$ write*

$$\nu_\Phi(i) = \sum_{\substack{J\subseteq\Pi \\ |J|=i+1}} c(J).$$

Then $\mathcal{N}_\Phi(t) := \sum_{i\geq 0}\nu_\Phi(i)t^i$.

Lemma 4.9. *If Φ is an irreducible root system of rank n, then $\mathcal{N}_\Phi(t) = (1+t)^{n-2}(n+t)$.*

Proof. Let Θ be the Dynkin diagram of Φ. Then evidently for $i = 0,1,\ldots,n-1$, $\nu_\Phi(i) = c(\Theta, i+1)$. It follows easily that

$$\mathcal{N}_\Phi(t) = t^{n-1}u_n(t^{-1}) = (1+t)^{n-2}(n+t).$$

□

Theorem 4.10. *Let Φ be any irreducible root system of rank n ($n \geq 2$). Then $\sum_{i=0}^{n}(H^{2i}(\mathcal{T}_W,\mathbb{C}),\rho_W)_W t^i = (n-1)t(1+t)^{n-2}$.*

A straightforward consequence of Theorem 4.10 is

Corollary 4.11. *Let Φ be any root system of rank n, and denote by W and $c(\Phi)$ respectively, the corresponding Weyl group and the number of irreducible components of Φ. Then*

$$\sum_{i=0}^{n}(H^{2i}(\mathcal{T}_W,\mathbb{C}),\rho_W)t^i=(n-c(\Phi))t(1+t)^{n-2}. \tag{4.12}$$

Proof of Corollary 4.11. Writing $c=c(\Phi)$, and using notation analogous to that at the beginning of this section, we have

$$\rho_W=\oplus_{i=1}^{c}1_{W_1}\otimes\cdots\otimes\rho_i\otimes\cdots\otimes 1_{W_c}.$$

Since $H^*(\mathcal{T}_W,\mathbb{C})\cong\otimes_{i=1}^{c}H^*(\mathcal{T}_{W_i},\mathbb{C})$, it follows from Theorem 3.5(iii) and Theorem 4.10 above that

$$\sum_{i=0}^{n}(H^{2i}(\mathcal{T}_W,\mathbb{C}),\rho_W)t^i=\sum_{j=1}^{c}\prod_{\substack{i=1\\ i\neq j}}^{c}(1+t)^{n_i}(n_j-1)t(1+t)^{n_j-2},$$

where n_i is the rank of the irreducible component Φ_i of Φ. The required statement follows easily. □

Proof of Theorem 4.10. Our starting point is the formula (1.5) which describes $H^{2i}(\mathcal{T}_W,\mathbb{C})$ as a W-module.

$$H^{2i}(\mathcal{T}_W,\mathbb{C})\cong\sum_{\substack{J\subseteq\Pi\\ |J|\geq i}}(-1)^{|J|-i}\operatorname{Ind}_{W_J}^{W}(\Lambda^{|J|-i}\rho_J),$$

where ρ_J is the reflection representation of W_J.

By Frobenius reciprocity, it follows that

$$\kappa_i:=(H^{2i}(\mathcal{T}_W,\mathbb{C}),\rho)_W=\sum_{\substack{J\subseteq\Pi\\ |J|\geq i}}(-1)^{|J|-i}(\operatorname{Res}_{W_J}^{W}\rho,\Lambda^{|J|-i}\rho_J)_{W_J}.$$

We therefore turn our attention to the computation of the κ_i.

Now $\kappa_i=\sum_{\substack{J\subseteq\Pi\\ |J|\geq i}}(-1)^{|J|-i}\kappa_i(J)$, where $\kappa_i(J)=(\operatorname{Res}_{W_J}^{W}\rho,\Lambda^{|J|-i}\rho_J)_{W_J}$. Further, by (4.2), we have

$$\kappa_i(J)=(\rho_J,\Lambda^{|J|-i}\rho_J)_{W_J}+|\Pi\setminus J|(1_{W_J},\Lambda^{|J|-i}\rho_J)_{W_J}.$$

We have seen that $(1_{W_J},\Lambda^{|J|-i}\rho_J)_{W_J}=0$ unless $|J|=i$, in which case the multiplicity is 1. To compute $(\rho_J,\Lambda^{|J|-i}\rho_J)_{W_J}$, write $J=J_1\amalg\cdots\amalg J_{c(J)}$ for the decomposition of J into connected components (cf. 4.1), and let $k=|J|-i$. Then from (4.1) and (4.3) we see that $(\rho_J,\Lambda^k\rho_J)_{W_J}=0$ unless $k=1$, and when $k=1$, $(\rho_J,\rho_J)_{W_J}=c(J)$.

Hence

$$\kappa_i(J) = \begin{cases} |\Pi \setminus J| \text{ if } |J| = i \\ c(J) \text{ if } |J| = i+1 \\ 0 \text{ otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} \sum_{i=0}^{n} (H^{2i}(\mathcal{T}_W, \mathbb{C}), \rho_W) t^i &= \sum_{i=0}^{n} \kappa_i t^i \\ &= \sum_{i=0}^{n} \sum_{\substack{J \subseteq \Pi \\ |J| \geq i}} (-1)^{|J|-i} \kappa_i(J) t^i \\ &= \sum_{i=0}^{n} \left(\sum_{\substack{J \subseteq \Pi \\ |J| = i}} (n-i) - \sum_{\substack{J \subseteq \Pi \\ |J| = i+1}} c(J) \right) t^i \\ &= \sum_{i=0}^{n} \left(\binom{n}{i} (n-i) - \nu_\Phi(i) \right) t^i \\ &= n(1+t)^{n-1} - \mathcal{N}_\Phi(t). \end{aligned}$$

Finally, it follows from Lemma 4.9 that $\mathcal{N}_\Phi(t) = (1+t)^{n-2}(n+t)$. Substituting into the expression above, we obtain the Theorem. $\square$

Appendix A. Equivalence to the Dolgachev-Lunts-Stembridge formula

In this Appendix we shall show how the character formula of Dogachev, Lunts and Stembridge can be derived from our Theorem 1.1 and vice versa.

To do this, we shall evaluate our formula (1.2) at an element $w \in W$, and compare with the formula in [St, Cor. 1.6]. We start by noting that given an element $w \in W$, we may apply Frobenius' formula for induced characters to the formula (1.2) to obtain the following expression for $P_W(t, w) := \sum_{i \geq 0} \operatorname{Trace}(w, H^i(\mathcal{T}_W, \mathbb{C})) t^i \in \mathbb{C}[t]$.

$$P_W(t, w) = \sum_{\substack{xW_J \\ x^{-1}wx \in W_J}} \det{}_{V_J}(t^2 - x^{-1}wx), \tag{A.1}$$

where the sum is over all cosets xW_J of parabolic subgroups W_J ($J \subseteq \Pi$) which are fixed by w, and V_J is the span of the simple roots in J.

Next we translate the formula in [St, Cor. 1.6] into the notation of the current work. Let $\Delta = \Delta_W$ be the fan in V which corresponds to the root system Φ.

Then in the language of the proof of Theorem 1.1 above, Δ is the union of the cones $\tau(xW_J)$ over all cosets xW_J. Fix $w \in W$ and define

$$Q_W(t,w) = P_{\Delta^w}(t)(1-t^2)^{-\dim V^w} \det{}_V(1-wt^2), \tag{A.2}$$

where $V^w = \ker(w-1)$ is the fixed point subspace of w, and $P_{\Delta^w}(t)$ is the Poincaré polynomial of the toric variety $\mathcal{T}(\Delta^w)$ corresponding to the fan Δ^w obtained by intersecting the cones of Δ with V^w.

Then [St, Cor. 1.6] asserts that $Q_W(t,w) = P_W(t,w)$. The equivalence of this statement to our Theorem 1.1 will follow from

Proposition A.3. *Let $w \in W$, and define $R_W(t,w)$ to be the right side of the equation (A.1). Then $R_W(t,w) = Q_W(t,w)$.*

Proof. First, note that the cones of Δ^w are precisely those cones of Δ which are fixed by w; this is because w fixes a cone τ (setwise) if and only w fixes τ pointwise. That is, in the language above, $\Delta^w = \{\tau(xW_J) \in \Delta \mid x^{-1}wx \in W_J\}$.

It follows, using [F, p. 94] and the fact that $\dim \tau(xW_J) = n - |J|$, that

$$\begin{aligned} P_{\Delta^w}(t) &= \sum_{\tau \in \Delta^w} (t^2-1)^{\dim V^w - \dim \tau} \\ &= \sum_{\substack{xW_J \\ x^{-1}wx \in W_J}} (t^2-1)^{\dim V^w - n + |J|}. \end{aligned} \tag{A.4}$$

Substituting the expression (A.4) into (A.2) and simplifying, we obtain

$$Q(t,w) = (-1)^{\dim V^w} \det{}_V(1-wt^2) \sum_{\substack{xW_J \\ x^{-1}wx \in W_J}} (t^2-1)^{-n+|J|}. \tag{A.5}$$

Now if $x^{-1}wx \in W_J$, then $x^{-1}wx$ fixes $V_J^\perp$ pointwise, so that $V^w \supseteq V_J^\perp$. Hence for such xW_J, we have

$$\begin{aligned} \det{}_V(1-wt^2) &= \det{}_V(1-x^{-1}wxt^2) \\ &= \det{}_{V_J}(1-x^{-1}wxt^2) \det{}_{V_J^\perp}(1-x^{-1}wxt^2) \\ &= \det{}_{V_J}(1-x^{-1}wxt^2)(1-t^2)^{n-|J|}. \end{aligned} \tag{A.6}$$

Now substitute this last expression into (A.5), to obtain

(A.7)

$$\begin{aligned}Q(t,w) =&(-1)^{\dim V^w}\sum_{\substack{xW_J\\ x^{-1}wx\in W_J}}(t^2-1)^{-n+|J|}\det{}_{V_J}(1-x^{-1}wxt^2)(1-t^2)^{n-|J|}\\ =&(-1)^{\dim V^w}\sum_{\substack{xW_J\\ x^{-1}wx\in W_J}}(-1)^{n-|J|}\det{}_{V_J}(1-x^{-1}wxt^2)\\ =&\sum_{\substack{xW_J\\ x^{-1}wx\in W_J}}(-1)^{\dim V^w+n}\det{}_{V_J}(x^{-1}wxt^2-1).\end{aligned}$$

Finally, since $x^{-1}wx$ has eigenvalues (on V, and therefore V_J) which come in complex conjugate pairs or are equal to ± 1, it follows that $\det{}_V(x^{-1}wx) = (-1)^{n+\dim V^w} = \det{}_{V_J}(x^{-1}wx)$, since $x^{-1}wx$ acts trivially on $V_J^\perp$. It follows from the last line of (A.7) that

$$Q(t,w) = \sum_{\substack{xW_J\\ x^{-1}wx\in W_J}}\det{}_{V_J}(t^2 - x^{-1}wx) = R(t,w),$$

and the proof of the proposition is complete. $\square$

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