DELETION-CONTRACTION TRIANGLES FOR HAUSEL-PROUDFOOT VARIETIES

ZSUZSANNA DANCSO, MICHAEL MCBREEN, AND VIVEK SHENDE

ABSTRACT. To a graph, Hausel and Proudfoot associate two complex manifolds, $\mathfrak{B}$ and $\mathfrak{D}$, which behave, respectively like moduli of local systems on a Riemann surface, and moduli of Higgs bundles. For instance, $\mathfrak{B}$ is a moduli space of microlocal sheaves, which generalize local systems, and $\mathfrak{D}$ carries the structure of a complex integrable system.

We show the Euler characteristics of these varieties count spanning subtrees of the graph, and the point-count over a finite field for $\mathfrak{B}$ is a generating polynomial for spanning subgraphs. This polynomial satisfies a deletion-contraction relation, which we lift to a deletion-contraction exact triangle for the cohomology of $\mathfrak{B}$. There is a corresponding triangle for $\mathfrak{D}$.

Finally, we prove $\mathfrak{B}$ and $\mathfrak{D}$ are diffeomorphic, that the diffeomorphism carries the weight filtration on the cohomology of $\mathfrak{B}$ to the perverse Leray filtration on the cohomology of $\mathfrak{D}$, and that all these structures are compatible with the deletion-contraction triangles.

Contents

1. Introduction

$\mathfrak{B}$, the locus $\mathbb{C}^2 \setminus \{1 + xy = 0\}$, the first nontrivial multiplicative quiver variety, the moduli of microlocal sheaves on a singular Lagrangian torus.

$\mathfrak{D}$, a neighborhood of the nodal elliptic curve in its versal deformation, the simplest degeneration in a complex integrable system, a local model for 4-dimensional hyperkähler geometry.

Each of the above spaces is the progenitor of a family, with one member for each Γ a connected multigraph with loops. The initial examples are those associated to the graph . These families were introduced by Hausel and Proudfoot [HP], where they observed that $\mathfrak{B}$ and $\mathfrak{D}$ are analogous to moduli of local systems and the moduli of Higgs bundles on an algebraic curve, respectively, and conjectured the existence of diffeomorphisms $\mathfrak{B}(\Gamma) \cong \mathfrak{D}(\Gamma)$, analogous to the nonabelian Hodge correspondence.

That correspondence [S1, S2, S3] relates three perspectives on nonabelian Lie-group valued cohomology: locally constant sheaves (Betti), bundles with connection (de Rham), and Higgs bundles (Dolbeault). We are most interested in the case where the underlying variety is an algebraic curve C, and in the (non-complex-analytic!) diffeomorphism between the moduli $\mathcal{M}_B(C, n)$ of rank n locally constant simple sheaves, i.e. simple representations $\pi_1(C) \to GL_n(C)$, and the moduli $\mathcal{M}_{\overline{\partial}}(C, n)$ of stable rank n Higgs bundles. The Higgs bundle moduli carries Hitchin's integrable system, $H \colon \mathcal{M}_{\overline{\partial}}(C, n) \to \mathbb{A}$, where $\mathbb{A}$ parameterizes n-multisections of T^*C ('spectral curves') [H1, H2]. The fiber over the point corresponding to a smooth spectral curve Σ is its Jacobian $J(\Sigma)$.

We believe $\mathfrak{B}(\Gamma)$ and $\mathfrak{D}(\Gamma)$ are in some sense *microlocal* versions of the nonabelian cohomology spaces. By microlocal, we mean as always 'locally in the cotangent bundle', i.e. locally around the spectral curve $\Sigma \subset T^*C$, and correspondingly locally around the corresponding Hitchin fiber, itself a multisection of a cotangent bundle $T^*Bun_{GL_n}(C)$. We view the relation between them as a hint of a yet unknown microlocal nonabelian Hodge correspondence.

Consider a smooth spectral curve. A neighborhood of $J(\Sigma)$ inside $\mathcal{M}_{\overline{\partial}}(C, n)$ will be diffeomorphic (and in fact symplectomorphic) to a neighborhood of the zero section of $T^*J(\Sigma) = \mathcal{M}_{\overline{\partial}}(\Sigma, 1)$, which in turn is – by the abelian case of the nonabelian Hodge correspondence – diffeomorphic to $\mathcal{M}_B(\Sigma, 1)$. That is, there is a diffeomorphism between a neighborhood of the Dolbeault data for Σ and the moduli space of Betti data on Σ.

We turn to singular spectral curves. Assuming Σ is a reduced, possibly reducible, curve, the Hitchin fiber is a compactification of its Jacobian; we denote it $\overline{J}(\Sigma)$. We will be interested in the cohomology this singular fiber. Denoting by $\widetilde{\Sigma}$ the normalization of the curve, it is known that $H^*(\overline{J}(\Sigma)) \cong H^*(J(\widetilde{\Sigma})) \otimes D(\Sigma)$ for some graded vector space $D(\Sigma)$ depending only on the singularities of Σ.[1]

Here we focus on the simplest case, when Σ has only nodes. Let Γ_Σ be the dual graph: it has vertices for the irreducible components of Σ, and edges for the nodes. We will show that $\mathfrak{D}(\Gamma_\Sigma)$ captures much of the topology of the Hitchin system around $[\Sigma]$. The space $\mathfrak{D}(\Gamma_\Sigma)$ is smooth; $\mathfrak{D}(\Gamma_\Sigma) \times J(\widetilde{\Sigma})$ has the same dimension as $\mathcal{M}_{\overline{\partial}}(C, n)$, and it follows its construction that $\mathfrak{D}(\Gamma_\Sigma)$ carries the structure of an integrable system. The data defining $\mathfrak{D}(\Gamma)$ did not depend on complex structure parameters, so it cannot be expected that the central fiber of $\mathfrak{D}(\Gamma)$ analytically related to the Hitchin fiber $\overline{J}(\Sigma)$. In fact, even when Σ has rational components, the corresponding central fibers need not be homeomorphic, and even if they are, the corresponding integrable systems need not be fiberwise homeomorphic. Nevertheless, we will see $H^*(\mathfrak{D}(\Gamma_\Sigma)) \cong D(\Sigma)$, in fact compatibly with the perverse Leray filtration (see Remark 8.25). In this sense, $\mathfrak{D}(\Gamma_\Sigma)$ is a model (or replacement) for the local topology in $\mathcal{M}_{\overline{\partial}}(C, n)$ around $\overline{J}(\Sigma)$.

There is also a sense in which $\mathfrak{B}(\Gamma_\Sigma)$ captures 'Betti information near Σ'. More precisely, one can view Σ as a (singular) Lagrangian and study the moduli space $\mathcal{M}_B(\Sigma, 1)$ of rank one microlocal sheaves on Σ.[2] Were Σ smooth, this would be the space of rank one local systems we encountered above. In the nodal case, moduli of microlocal sheaves is shown in [BK] to match certain multiplicative Nakajima varieties [CBS, Y]; comparing the results there to the definitions here, it is immediate that there is an algebraic isomorphism

$$\mathcal{M}_B(\Sigma, 1) \cong \mathcal{M}_B(\widetilde{\Sigma}, 1) \times \mathfrak{B}(\Gamma_\Sigma)$$

In short, recent developments have clarified that $\mathfrak{B}(\Gamma)$ and $\mathfrak{D}(\Gamma)$ in some sense model the non-abelian cohomology spaces 'near' a nodal spectral curve with dual graph Γ. Given the above, one might understand the following result (conjectured by Hausel and Proudfoot) as being in the direction of a microlocal version of Simpson's correspondence:

Theorem 1.1. *(9.6, 9.10) For any graph Γ, there is a canonical homotopy equivalence induced by a non-canonical diffeomorphism $\mathfrak{D}(\Gamma) \cong \mathfrak{B}(\Gamma)$.*

Finite field specializations of the character varieties $\mathcal{M}_B(C)$, and more generally twisted versions corresponding to Higgs bundles of nonzero degree, were studied in [HRV]. In particular, a formula for their point-count was given explicitly and shown to be a certain polynomial in the size q of the finite field. This was done to probe the mixed Hodge structure on the cohomology of the

[1]When Σ not irreducible, the compactification of the Jacobian depends on the choice of a stability condition. However, it follows from [MSV] that $H^*(\overline{J}(\Sigma))$ is in fact independent of a generic such choice, and genericity is known to follow from smoothness of the total space of the Hitchin fibration.

[2]Equivalently [GPS3, Sec. 6.2], of the wrapped Fukaya category of a completion of a neigborhood of Σ. From this point of view, one sees an embedding $\mathcal{M}_B(\Sigma, 1) \to \mathcal{M}_B(C, n)$ is induced by pullback of pseudo-perfect modules under a non-exact Viterbo restriction, modulo convergence issues. The case of smooth spectral curve is [GMN2]. What is by no means clear is if or why $\overline{J}(\Sigma)$ lies in the image, let alone why it should be a deformation retract thereof. We will not use or discuss this further here.

character variety. We recall the cohomology of any algebraic variety X carries two filtrations; a decreasing 'Hodge' filtration and an increasing 'weight' filtration [Del1, Del2, Del3]. We are interested here in the latter; its i'th step on the j'th cohomology group is denoted $W_i H^j(X)$, and the associated graded spaces are denoted by $gr_i^W H^j(X)$. One records these dimensions in the mixed Poincaré polynomial:

$$P^X(q,t) = \sum_{i,j} q^i t^j \dim gr_i^W H^j(X)$$

Under specializing $q \to 1$ one recovers the usual Poincaré polynomial. Making the analogous construction with compactly supported cohomology $P_c^X(q,t)$ and specializing $t \to -1$, one recovers a 'motivic' quantity. For smooth X, one has the Poincaré duality $P_c^X(q,t) = (qt^2)^{\dim X} P^X(q^{-1}, t^{-1})$. As explained in Katz's appendix to [HRV], whenever the finite field count is a polynomial in q, then in fact this count is $P_c^X(q,-1)$. A stronger statement about X would be that its class in the Grothendieck ring of varieties is $P_c^X(\mathbb{L}, -1)$, where $\mathbb{L}$ is the class of the affine line. It is presently unknown whether this holds for the character varieties.

For the space $\mathfrak{B}(\Gamma)$, we calculate the class in the Grothendieck ring. To do so we first study how this class changes under certain graph transformations. Recall that for an edge e in a graph Γ, one writes $\Gamma \setminus e$ for the graph obtained by deleting the edge, and Γ/e for the graph obtained by contracting the edge. An edge is said to be a loop if it connects a vertex to itself, and a bridge if deleting it disconnects the graph.

It is elementary that $[\mathfrak{B}(\bigcirc\!\!\!\bullet)] = (\mathbb{L}^2 - \mathbb{L} + 1)$. We show in Remark 5.14 below that, in the Grothendieck ring of varieties,

$$[\mathfrak{B}(\Gamma)] = \begin{cases} [\mathfrak{B}(\Gamma/e)] & e \text{ is a bridge} \\ [\mathfrak{B}(\bigcirc\!\!\!\bullet)][\mathfrak{B}(\Gamma \setminus e)] & e \text{ is a loop} \\ \mathbb{L}[\mathfrak{B}(\Gamma \setminus e)] + [\mathfrak{B}(\Gamma/e)] & \text{otherwise} \end{cases} \tag{1}$$

These formulas would evidently allow in principle the recursive calculation of $[\mathfrak{B}(\Gamma)]$, and imply it is a polynomial in the affine line. In fact there is a sort of universal solution to such recursions, given by the Tutte polynomial. For a textbook treatment, see [Bol, Chap. 10]; the relevant universality statement is [Bol, Chap. 10, Thm. 2]. From it we extract a closed-form formula. Let $Span(\Gamma)$ be the set of connected spanning subgraphs, and write b_1 for the first Betti number. We have

$$[\mathfrak{B}(\Gamma)] = \sum_{\Gamma' \in Span(\Gamma)} (\mathbb{L}-1)^{2b_1(\Gamma')} \mathbb{L}^{b_1(\Gamma) - b_1(\Gamma')} \tag{2}$$

In particular, the Euler characteristic of $\mathfrak{B}(\Gamma)$ is the number of spanning subtrees of Γ.

Returning to [HRV], recall that the weight polynomial of the character varieties, together with the complete description of the cohomology for $GL(2)$, led those authors to conjecture a formula for the full mixed Hodge polynomial. The formula moreover suggested certain curious properties of the cohomology.

These properties found a conjectural conceptual explanation in the remarkable "$P = W$" conjecture of [dCHM]. Given any map of algebraic varieties $f\colon X \to A$, there is a filtration on $\mathrm{H}^\bullet(X) \cong \mathrm{H}^\bullet(A, Rf_*\mathbb{C}_X)$ arising from truncation of $Rf_*\mathbb{C}_X$ in the (middle) perverse t-structure on A; this is termed the perverse Leray filtration. The $P = W$ conjecture asserts that Simpson's correspondence, the weight filtration on the character variety goes to the *perverse Leray* filtration associated to Hitchin's integrable system on the moduli of Higgs bundles.

This $P = W$ conjecture was established in [dCHM] in the $GL(2)$ case, and very recently for any rank on a genus 2 curve [dCMS]. One of its original motivations, the 'curious Hard Lefschetz' conjecture, is now established [Mel]. Some additional special cases have been verified [SZ, Szi], and some tests of structural predictions verified [dCM]. A certain limit of the conjecture appears to be related to a comparison of limiting behavior of the Hitchin fibration with the geometry of the boundary complex of the character variety [S5]. Relationships between perverse and weight filtrations have also been found in other settings of hyperkähler geometry [dCHM2, Har, HLSY]. In particular, the 4-real-dimensional examples of the spaces under investigation here were studied in [Z]. A similar sounding (but at present not directly related) statement has been found in homological mirror symmetry [HKP]. The original conjecture remains wide open in the general case. It is unclear what is the natural setting or generality for this conjecture.

We will show that there is a sense in which $P = W$ holds even 'microlocally'. In this setting, the P side of $P = W$ is a-priori meaningful – we may use the perverse t-structure on the base of the Hitchin integrable system to give a filtration on the cohomology of fibers – but the W side is not. We give it a meaning for nodal Σ, using the space $\mathfrak{B}(\Gamma_\Sigma)$ which is an algebraic variety, hence carries a weight filtration.

We do not have even a conjecture for the mixed Hodge polynomial of $\mathfrak{B}(\Gamma)$, or of the analogous perverse Poincaré polynomial of $\mathfrak{D}(\Gamma)$. Neither do we know generators for the cohomology ring, much less relations. Nevertheless, we can show:

Theorem 1.2. *(9.19) The homotopy equivalence* $\mathfrak{D}(\Gamma) \hookrightarrow \mathfrak{B}(\Gamma)$ *carries the the weight filtration on* $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma))$ *to (twice) the perverse Leray filtration on* $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma))$.

The proof of this result is intertwined with the construction of deletion-contraction triangles. By the end we will have shown:

Theorem 1.3. *(5.15, 6.43, 9.20, 9.34) For any edge e which is neither a loop nor a bridge, there are deletion-contraction long-exact sequences, intertwined by pullback along $\mathfrak{D}(\Gamma) \hookrightarrow \mathfrak{B}(\Gamma)$.*

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{H}^{\bullet-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q})(-1) \arrow[r] \arrow[d] & \mathrm{H}^{\bullet}(\mathfrak{B}(\Gamma), \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{H}^{\bullet}(\mathfrak{B}(\Gamma/e), \mathbb{Q}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma \setminus e), \mathbb{Q})\{-1\} \arrow[r] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma), \mathbb{Q}) \arrow[r] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e), \mathbb{Q}) \arrow[r] & {}
\end{tikzcd}$$

The sequences are strictly compatible with the weight and perverse Leray filtrations, respectively. The (-1) and $\{-1\}$ indicate shifts of these filtrations.

The existence of the intertwined long exact sequences is nontrivial, but in some sense it is proven by pure thought, using the excision triangle on the top, the nearby-vanishing triangle at the bottom, and geometric arguments for commutativity of the diagram. One would like to conclude compatibility with filtrations by induction on the size of the graph. This does not immediately work, for two reasons. The first: we do not know a pure thought argument that the Dolbeault sequence is *strictly* compatible with the perverse Leray filtration; in fact, we will only learn this at the very end of the paper. The second: even had we known this, there is the following difficulty: consider two short exact sequences of filtered vector spaces, maps strictly compatible with the filtration. Suppose given an isomorphism of the underlying short exact sequences, which respects the filtration save on the middle term. Must it respect the filtrations on the middle term? Alas, no.

To deal with these difficulties we introduce yet a third filtration, which is defined only in terms of the deletion maps.

Definition 1.4. (Deletion filtration) Let Graph° be the category whose objects are connected oriented graphs and whose morphisms are inclusions whose complement contains no self-edge. Let $A : \mathrm{Graph}^\circ \to Ab$ be a covariant functor to the category with objects graded abelian groups, and morphisms maps of abelian groups such that $A^\bullet(\Gamma')[2|\Gamma'|] \to A^\bullet(\Gamma)[2|\Gamma|]$ is graded. We define:

$$D_{i-k}A^i(\Gamma) = \mathrm{Span}(\{\mathrm{image}(A(\Gamma')) \mid |\Gamma \setminus \Gamma'| = k\})$$

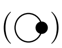
It is immediate from the definition that $D_\bullet$ is (not necessarily strictly) preserved by all maps $A(\Gamma \subset \Gamma') : A(\Gamma')\{|\Gamma'|\} \to A(\Gamma)\{|\Gamma|\}$ where $\{\cdot\}$ indicates a shift of the filtration. It is also evident that it is the minimal such filtration, subject to the normalization $0 = D_{i-1}(A^i(\Gamma)) \subset D_i(A^i(\Gamma)) = A^i(\Gamma)$ when Γ has only loops and bridges.

Once we have shown the deletion maps act identically on the cohomology of the $\mathfrak{B}(\Gamma)$ and $\mathfrak{D}(\Gamma)$, it follows that this filtration must agree on the $\mathfrak{B}$ and $\mathfrak{D}$ sides. Thus it remains to show the deletion filtration agrees with the weight and perverse filtrations. This proves to be rather involved; our argument depends on introducing a combinatorial model in which the third filtration is manifest, and then arguing on each side that this combinatorial model can be realized by some (rather different on the two sides) geometric construction.

1.1. **Outline.** We begin in Section 2 by recalling from [MSV] the combinatorial description of a certain complex $\Upsilon(\Gamma)$ associated to any graph. This complex will turn out to have geometric interpretations both as the cohomology of $\mathfrak{B}(\Gamma)$, and of $\mathfrak{D}(\Gamma)$. Nevertheless in Section 2 we confine ourselves to a purely combinatorial discussion. We construct explicitly the deletion-contraction filtration exact sequence, and note some of its properties. In particular, we observe that the deletion-contraction sequences themselves induce a filtration on the cohomology. A key point about $\Upsilon(\Gamma)$ is that the resulting filtration is easy to describe.

In Section 3 we adapt the formalism of moment maps and symplectic reduction to situations when no symplectic structure is present. Symplectic reduction applies in the situation of a group $\mathbb{G}$ acting on a symplectic manifold with moment map $\mu\colon \mathbf{X} \to \mathfrak{g}^*$ (which, together with the symplectic structure, encodes the group action). Here we consider arbitrary spaces $\mathbf{X}$ with an action of a group $\mathbb{G}$ preserving a map $\mu\colon \mathbf{X} \to \mathbb{A}$ to an abelian group $\mathbb{A}$; we call such things $(\mathbb{G}, \mathbb{A})$-spaces. The map μ in no way encodes the group action.

Nevertheless, given a $(\mathbb{G}, \mathbb{A})$-space $\mathbf{X}$, we can define its reduction $\mathbf{X} /\!\!/_{\eta \in \mathbb{A}} \mathbb{G} := \mu^{-1}(\eta)/\mathbb{G}$. Given two $(\mathbb{G}, \mathbb{A})$-spaces $\mathbf{X}, \mathbf{Y}$, we can form a product $(\mathbb{G}, \mathbb{A})$-space $\mathbf{X} \bullet \mathbf{Y}$. Similarly, we can form the quotient $\mathbf{X} \star \mathbf{Y} = \mathbf{X} \bullet \mathbf{Y} /\!\!/ \mathbb{G}$. This construction is 'functorial', meaning that a map $\mathbf{Y}' \to \mathbf{Y}$ induces a map $\mathbf{X} \star \mathbf{Y}' \to \mathbf{X} \star \mathbf{Y}$.

In Section 4 we build spaces from graphs. From any $(\mathbb{G}, \mathbb{A})$-space $\mathbf{X}$ together with a graph Γ and an element $\eta_v \in \mathbb{A}$ for each vertex of Γ, we construct a space $\mathbf{X}(\Gamma, \eta)$ in Section 4.1. Always $\mathbf{X} = \mathbf{X}($$)$.

Given an edge e in Γ, we form new graphs Γ/e and $\Gamma \setminus e$ by contracting (resp. deleting) e. Our main tools for studying $\mathbf{X}(\Gamma)$ are the two relations of the form $\mathbf{X}(\Gamma/e) \star \mathbf{X} = \mathbf{X}(\Gamma)$ and $\mathbf{X}(\Gamma/e) \star \mathbf{point} = \mathbf{X}(\Gamma \setminus e)$.

In Section 5 we turn our attention to the spaces $\mathfrak{B}(\Gamma)$. They are built from the basic space $\mathfrak{B} = \mathbb{C}^2 \setminus \{xy + 1 = 0\}$. Using functoriality of the $\star$ product, we turn properties of $\mathfrak{B}$ into properties of $\mathfrak{B}(\Gamma)$. In Section 5.4, we use this to obtain the *Betti deletion-contraction sequence*

$$\to \mathrm{H}^{\bullet-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q}(-1)) \to \mathrm{H}^{\bullet}(\mathfrak{B}(\Gamma), \mathbb{Q}) \to \mathrm{H}^{\bullet}(\mathfrak{B}(\Gamma/e), \mathbb{Q}) \to$$

The key geometric construction is an embedding of a line bundle over $\mathfrak{B}(\Gamma \setminus e)$ into $\mathfrak{B}(\Gamma)$, with complement $\mathfrak{B}(\Gamma/e)$. The resulting long exact sequence of a pair is our deletion-contraction sequence. The same geometry immediately implies Equation 1 above.

The deletion maps equip the cohomology of $\mathfrak{B}(\Gamma)$ with a deletion filtration. The deletion maps are induced by maps of algebraic varieties, hence respect the weight filtration; minimality of the deletion filtration implies it is bounded by the weight filtration. In fact, they are equal; to prove this we construct an explicit complex of differential forms, which on the one hand is sensitive to the weight filtration, and on the other, can be identified with the complex $\Upsilon(\Gamma)$, compatibly with deletion-contraction. The deletion filtration is explicit on $\Upsilon(\Gamma)$, allowing us to conclude.

In Section 6, we turn to the Dolbeault space $\mathfrak{D}(\Gamma)$. The special case $\mathfrak{D} = \mathfrak{D}(\bigcirc\!\!\bullet)$ is the Tate curve, and the more general spaces are degenerating families of abelian varieties built as subquotients of powers of $\mathfrak{D}(\bigcirc\!\!\bullet)$. We study these spaces in families; in particular, there is a family of spaces over the disk $\mathbb{D}^1$ whose general fiber is $\mathfrak{D}(\Gamma/e)$, whose special fiber is homotopic to $\mathfrak{D}(\Gamma)$, and whose singular locus is homotopic to $\mathfrak{D}(\Gamma \setminus e)$. The nearby-vanishing triangle gives rise, ultimately, to the deletion-contraction sequence in this setting.

As with the moduli of Higgs bundles, the spaces $\mathfrak{D}(\Gamma)$ have the structure of complex analytic integrable systems. We explore this structure further in Section 7, in particular describing the fibers and characterizing the monodromy. We need these results to show the deletion maps preserve the perverse filtration, hence that the deletion filtration is bounded by the perverse filtration. Additionally, borrowing a calculation of [MSV], we show that $\Upsilon(\Gamma)$ also computes the cohomology of the spaces $\mathfrak{D}(\Gamma)$, compatibly with the perverse filtrations.

Finally in Section 9, we begin comparing $\mathfrak{B}$ and $\mathfrak{D}$. First we construct a smooth embedding and homotopy equivalence between the basic spaces, $\mathfrak{D} \subset \mathfrak{B}$. Due to the similarity of the constructions of these spaces, this induces a similar inclusion $\mathfrak{D}(\Gamma) \subset \mathfrak{B}(\Gamma)$, thus proving Theorem 1.1 (9.6).

We show in Section 9.2 that the deletion maps are intertwined by $\mathfrak{D}(\Gamma) \to \mathfrak{B}(\Gamma)$. The key geometric input is a relation between the subspace used in the long exact sequence of a pair (on the Betti side) and the vanishing thimble for the degenerating family (on the Dolbeault side). It follows immediately that the Betti and Dolbeault deletion filtrations are identified. In particular, dimensions of the associated graded pieces of the Dolbeault deletion filtration equal those of the Υ-filtration. Then since Dolbeault deletion filtration is bounded by the perverse Leray filtration, but both these have associated graded dimensions matching that of the Υ filtration, we conclude that in fact these filtrations must be equal. Having identified the deletion filtrations with the weight and perverse Leray filtrations on the respective sides, we deduce Theorem 1.2 (9.19). Some further geometric considerations give the full intertwining of Theorem 1.3.

1.2. **Some additional remarks.**

Remark 1.5. In the original [HP], the spaces were defined with an arbitrary integer matrix in place of the adjacency matrix of the graph. Deletion-contraction relations have a well known generalization to this matroidal setting, and in fact all the results of the paper generalize as well, with identical proofs, save that orbifold singularities may appear in the spaces (c.f. Remark 4.14).

Remark 1.6. While we construct an embedding $\mathfrak{D}(\Gamma) \subset \mathfrak{B}(\Gamma)$, we do not know a category of which the former space is a moduli space, much less a functor between categories inducing this map. It would be preferable to have such a category and functor. Relatedly, we have described how the spaces $\mathfrak{D}(\Gamma)$ and $\mathfrak{B}(\Gamma)$ are cohomologically related to Hitchin fibers, but not given maps of spaces, much less of categories.

Is there a microlocal nonabelian Hodge theory?

Remark 1.7. The relationship between microlocal sheaves on a spectral cover and the neighborhood of the corresponding Hitchin fiber should hold in some greater generality. In particular, at least for spectral curves the links of whose singularities are torus knots, a similar statement can be tortured out of the identification in [STWZ] of moduli of Stokes data as moduli of sheaves microsupported along a Legendrian, plus the nonabelian Hodge correspondence in the presence of irregular singularities [BB]. As explained in the introduction of [STZ], a comparison of the numerics of that article with those of [OS, GORS] reveals a faint shadow of a 'P=W' phenomenon here as well.

Remark 1.8. Recall from [SW] and subsequent developments that if one considers the $N = 2$ super Yang-Mills for $U(n)$ with g adjoint matter fields, then then the vacua in $\mathbb{R}^4$ form the base of a Hitchin system corresponding to the moduli of Higgs bundles over a base curve of genus g. At low energy, in a vacuum where corresponding to a spectral curve with (for convenience) rational components, the theory is described by an abelian gauge theory with gauge fields corresponding to the components of the spectral curve, and bifundamentals or adjoints corresponding to the nodes. That is, it corresponds to the dual graph of the curve. We expect there should be some physical account of why the cohomology of the corresponding multiplicative quiver variety is identical to the cohomology of the Hitchin fiber, and more optimistically, why this should identify weight and perverse Leray filtrations (as we have mathematically proven is the case).

There is a string-theoretic account of why the perverse filtration on the cohomology of the Higgs moduli space should lead to the bigraded numbers guessed by [HRV] for the weight filtration on the character variety (see [CDP1, CDP2, CDDP, DDP, Dia, CDDNP]). It is not immediately clear how this relates to the above notions, but it would be interesting to make such a connection. In particular, unlike [HRV], we have not been able to compute (or guess) the mixed Poincaré polynomials of the multiplicative hypertoric varieties.

Remark 1.9. The embedding $\mathfrak{D}(\Gamma) \subset \mathfrak{B}(\Gamma)$ has the flavor of a hyperkähler rotation. In particular it carries the central fiber of the integrable system $\mathfrak{D}(\Gamma)$ to a non-holomorphic Lagrangian subvariety of $\mathfrak{B}(\Gamma)$, which should be the Lagrangian skeleton of an appropriate Weinstein structure. This fact, which we do not prove here, suggests a way to calculate the Fukaya category of $\mathfrak{B}(\Gamma)$, using the approach of [K, N, GPS1, GPS2, GPS3, GS]. This idea is explored in [GMcW], building on calculations of [McW].

Remark 1.10. From the geometry underlying the proof of Equation 2, one can extract a stratification of $\mathfrak{B}(\Gamma)$ by products of algebraic tori and linear spaces. Beginning with the work of Deodhar [Deo], such stratifications have been found frequently in representation theoretic contexts. [STZ, Prop. 6.31] hints how such stratifications can be given modular interpretations: the spaces are moduli of objects in the Fukaya category of a symplectic 4-manifold; and the strata each parameterize objects coming from a given immersed Lagrangian.

The present case is another example. We have recalled from [BK] that $\mathfrak{B}(\Gamma)$ is a moduli space of microlocal sheaves on a singular real surface $L = \bigcup L_i$, where the L_i are the smooth irreducible components. In this context it is most natural to view L as the Lagrangian skeleton of the symplectic plumbing W of the T^*L_i. By e.g. [GPS3, Cor. 6.3] we may trade microlocal sheaves on L for the wrapped Fukaya category of W. Now a spanning subgraph $\Gamma' \subset \Gamma$ determines an immersed Lagrangian: smooth the singularities of the skeleton corresponding to the edges Γ', and leave the nodes in $\Gamma \setminus \Gamma'$. A rank one local system on this Lagrangian, together with some extra data at the nodes, determines an object in the Fukaya category. The space of such choices is $(\mathbb{L}-1)^{2b_1(\Gamma')}\mathbb{L}^{|\Gamma|-|\Gamma'|}$. It is also possible to give a similar description in terms of the microlocal picture of [BK].

Remark 1.11. A shadow of Theorem 1.2 can be seen by comparing Equation 2 above to Theorem 1.1 of [MSV], after specializing $\mathbb{L} \to 1$ in the latter. We do not know what parameter should be introduced in our formula to recover the $\mathbb{L}$ of [MSV]; this corresponds to asking how to characterize the filtration on the *Betti* moduli space which corresponds to the weight filtration on the *Dolbeault* moduli space. This question does not arise in the setting of the original P=W conjecture, because in that situation, the cohomology of the Dolbeault space is pure, i.e. the weight filtration arises from the cohomological grading. In the present case, the (central fiber of the) Dolbeault space does not have pure cohomology.

Remark 1.12. The deletion-contraction relation enjoys various connections with the skein relation of knot theory; it may be expected that deletion-contraction exact sequences enjoy similar connections with the skein exact sequences in knot homology theories such as [Kho]. Indeed, this is true by construction in various extant categorifications of the Tutte polynomial and its specializations [ER, HR, HR2, Sto, Sto2], though we do not know how these constructions relate to $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma))$.

In this context we recall the relation between knot invariants and the perverse polynomial of Hitchin fibers [ObS, Mau]; and its conjectural lift to the cohomological level [ORS, GORS].

Acknowledgements. We thank the following people for helpful conversations: Valery Alexeev, Phil Engel, Michael Groechenig, Tamás Hausel, Raju Krishnamoorthy, Luca Migliorini, Nick Proudfoot, Michael Thaddeus.

Z. D. was supported by NSF grant no. 0932078 000 while in residence at the Mathematical Sciences Research Institute in Berkeley, California during Fall 2014, and by an Australian Research Council DECRA grant DE170101128 from 2017. M. M. was supported by the Advanced Grant "Arithmetic and Physics of Higgs moduli spaces" No. 320593 of the European Research Council.

while at the École Polytechnique Fédérale de Lausanne, and by the Natural Sciences and Engineering Research Council while at the University of Toronto. Part of this work was completed while M. M. was at the Massachusetts Institute of Technology and the Hausdorff Institute of Mathematics. V. S. was supported by the NSF grants DMS-1406871 and CAREER DMS-1654545, and a Sloan fellowship.

2. Combinatorial model

In this section, we will give a purely combinatorial model for the cohomology of our $\mathfrak{B}(\Gamma)$ or $\mathfrak{D}(\Gamma)$, equipped with the appropriate filtration. The model was originally introduced in [MSV] to describe the perverse filtration on the cohomology of the compactified Jacobian of a nodal curve, which as we have mentioned above is closely related to $\mathfrak{D}(\Gamma)$. We write our complexes over an arbitrary commutative ring R, which in the remainder of this article will always be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$.

Definition 2.1. We write $\mathbb{H}(\Gamma \setminus J, R) := \mathrm{H}_1(\Gamma \setminus J, R) \oplus \mathrm{H}^1(\Gamma \setminus J, R)$. Let

$$\Upsilon^{2k,l}(\Gamma, R) := \bigoplus_{|J|=k} \bigwedge^{l} \mathbb{H}(\Gamma \setminus J, R), \quad \Upsilon^{2k+1,l}(\Gamma, R) := 0.$$

We will often suppress the choice of R. Let $\Upsilon^{\bullet}(\Gamma) := \bigoplus_{2k+l=\bullet} \Upsilon^{2k,l}(\Gamma)$. We now define a differential $\Upsilon^{\bullet}(\Gamma) \to \Upsilon^{\bullet+1}(\Gamma)$.

Let e be an edge, viewed as a class in $\mathrm{H}^1(\Gamma, R)$. We define a linear function

$$\langle e, - \rangle : \mathbb{H}(\Gamma, R) \to R$$

by setting $\langle e, f \rangle = 0$ for $f \in \mathrm{H}^1(\Gamma, R)$, and setting $\langle e, \gamma \rangle$ to be the usual pairing for $\gamma \in \mathrm{H}_1(\Gamma, \mathbb{R})$. Consider the map $e\langle e, - \rangle \colon \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ which takes x to $e\langle e, x \rangle$. Extend, via the Leibniz rule, to a linear map $e\langle e, - \rangle \colon \bigwedge^l \mathbb{H}(\Gamma) \to \bigwedge^l \mathbb{H}(\Gamma)$.

Lemma 2.2. *The image of* $e\langle e, - \rangle$ *is the subspace*

$$e \wedge \bigwedge^{l-1} \Big(\mathrm{H}_1(\Gamma \setminus e) \oplus \mathrm{H}^1(\Gamma) \Big). \tag{3}$$

Proof. Note that $\mathrm{H}_1(\Gamma \setminus e) \oplus \mathrm{H}^1(\Gamma) = \ker(e\langle e, - \rangle)$. Choose any splitting of $\mathbb{H}(\Gamma)$ into $\ker(e\langle e, - \rangle) \oplus \mathbb{F}$ where $\mathbb{F}$ is rank one; then $e\langle e, - \rangle$ restricts to an isomorphism $\mathbb{F} \cong Re$. We have $\bigwedge^l \mathbb{H}(\Gamma) = \Big(\mathbb{F} \otimes \bigwedge^{l-1} \ker(e\langle e, - \rangle) \Big) \oplus \bigwedge^l \ker(e\langle e, - \rangle)$. The map $e\langle e, - \rangle$ takes the left-hand summand isomorphically onto 3 and kills the right-hand summand. □

When e is not a bridge, we have an identification $e \wedge \bigwedge^{l-1} \Big(\mathrm{H}_1(\Gamma \setminus e) \oplus \mathrm{H}^1(\Gamma) \Big) = \bigwedge^{l-1} \mathbb{H}(\Gamma \setminus e)$, and thus we obtain a map

$$d_e \colon \bigwedge^{l} \mathbb{H}(\Gamma) \to \bigwedge^{l-1} \mathbb{H}(\Gamma \setminus e).$$

$$\bigwedge^4 \mathbb{H}(\Theta)$$

$$\bigwedge^3 \mathbb{H}(\Theta) \xrightarrow{d_\Upsilon} \bigoplus_{i=1}^3 \bigwedge^2 \mathbb{H}(\Theta \setminus e_i)$$

$$\bigwedge^2 \mathbb{H}(\Theta) \xrightarrow{d_\Upsilon} \bigoplus_{i=1}^3 \bigwedge^1 \mathbb{H}(\Theta \setminus e_i) \xrightarrow{d_\Upsilon} \bigoplus_{i,j=1}^3 \bigwedge^0 \mathbb{H}(\Gamma \setminus e_i, e_j)$$

$$\bigwedge^1 \mathbb{H}(\Theta) \xrightarrow{d_\Upsilon} \bigoplus_{i=1}^3 \bigwedge^0 \mathbb{H}(\Theta \setminus e_i)$$

$$\bigwedge^0 \mathbb{H}(\Theta)$$

FIGURE 1. The complex $\Upsilon^\bullet(\Theta)$, where Θ is the graph with two vertices joined by three edges e_1, e_2, e_3. We have only indicated the groups which are not automatically zero for degree reasons. The cohomological grading increases as one moves up or to the right. The cohomology is described in Figure 2.

More explicitly,

$$d_e(x_1 \wedge x_2 \wedge ... \wedge x_l) = \sum_{i=1}^{l} (-1)^{i-1} \langle e, x_i \rangle x_1 \wedge x_2 \wedge ... \wedge \widehat{x}_i \wedge ... \wedge x_l.$$

Definition 2.3. Let $d_\Upsilon \colon \Upsilon^{2k,l}(\Gamma) \to \Upsilon^{2k+2,l-1}(\Gamma)$ be the linear map whose restriction to $\bigwedge^l \mathbb{H}(\Gamma \setminus J)$ is the direct sum over all non-bridge edges e in $\Gamma \setminus J$ of $d_e \colon \bigwedge^l \mathbb{H}(\Gamma \setminus J) \to \bigwedge^{l-1} \mathbb{H}(\Gamma \setminus J \setminus e)$.

Lemma 2.4. *The map d_Υ makes $\Upsilon^\bullet(\Gamma)$ into a complex, i.e. $d_\Upsilon^2 = 0$.*

Proof. It is easy to see that $d_e^2 = 0$. We must check that additionally, $d_{e_1} d_{e_2} = -d_{e_2} d_{e_1}$. The sign arises when passing from $e\langle e, - \rangle$ to d_e, which involves reordering the factors of a wedge product so that the factor e comes out in front. Indeed, we may write the image of $d_{e_2}(x_0 \wedge ... \wedge x_N)$ under $e_1\langle e_1, - \rangle$ as a sum of terms $(-1)^j x_0 \wedge ... \wedge e_1\langle e_1, x_i \rangle \wedge ... \wedge \widehat{x}_j \wedge ... \wedge x_N$ with $i < j$ and $(-1)^j x_0 \wedge ... \wedge \widehat{x}_j \wedge ... \wedge e_1 \langle e_1, x_i \rangle \wedge ... \wedge x_N$ with $i > j$. Here the hat indicates that a factor has been omitted. Then $d_{e_1} d_{e_2}(x_0 \wedge ... \wedge x_N)$ is the sum of terms $(-1)^{i+j} x_0 \wedge ... \wedge \widehat{x}_i \wedge ... \wedge \widehat{x}_j \wedge ... \wedge x_N$ with $i < j$ and $(-1)^{i+j-1} x_0 \wedge ... \wedge \widehat{x}_j \wedge ... \wedge \widehat{x}_i \wedge ... \wedge x_N$ with $i > j$. Exchanging e_1 and e_2 exchanges the signs. □

By construction, the differential d_Υ takes $\Upsilon^{2k,l}(\Gamma)$ to $\Upsilon^{2k+2,l-1}(\Gamma)$, and thus preserves the subspace $\Upsilon_m(\Gamma) := \oplus_{a+2b=m} \Upsilon^{a,b}(\Gamma)$. We thus have $\mathrm{H}^i(\Upsilon(\Gamma)) = \oplus_m \mathrm{H}^i(\Upsilon_m(\Gamma))$.

	0	1	2
4	R		
3	R^2	R	
2	R^2	0	R
1	R^2	R	
0	R		

FIGURE 2. The cohomology of $\Upsilon^\bullet(\Theta)$. The Υ-grading is indicated on the left, while the number $|J|$ of deleted edges is indicated on the top row. The cohomological grading is the sum of these numbers.

Definition 2.5. We call this extra grading on cohomology the Υ-grading, so that $\mathrm{H}^i(\Upsilon_m(\Gamma))$ has Υ-degree m.

Fix an edge $e \in \Gamma$ which is neither a loop nor a bridge. We can identify $\Upsilon^{\bullet-2}(\Gamma \setminus e)$ with the subcomplex of $\Upsilon^\bullet(\Gamma)$ consisting of summands $\bigwedge^l \mathbb{H}(\Gamma \setminus J)$ with $e \in J$. If we ignore the differential, the quotient complex is given by the summands $\bigwedge^l \mathbb{H}(\Gamma \setminus J)$ with $e \notin J$. The homotopy equivalence $\Gamma \setminus J \to (\Gamma \setminus J)/e = (\Gamma/e) \setminus J$ identifies each such summand with $\bigwedge^l \mathbb{H}((\Gamma/e) \setminus J)$; the quotient complex therefore has the same underlying graded vector space as $\Upsilon^\bullet(\Gamma/e)$. The differentials also match, and thus we have

$$0 \to \Upsilon^{\bullet-2}(\Gamma \setminus e) \to \Upsilon^\bullet(\Gamma) \to \Upsilon^\bullet(\Gamma/e) \to 0 \tag{4}$$

Definition 2.6. The resulting long exact sequence

$$\to \mathrm{H}^{\bullet-2}(\Upsilon(\Gamma \setminus e)) \xrightarrow{a_e^\Upsilon} \mathrm{H}^\bullet(\Upsilon(\Gamma)) \xrightarrow{b_e^\Upsilon} \mathrm{H}^\bullet(\Upsilon(\Gamma/e)) \xrightarrow{c_e^\Upsilon} \tag{5}$$

is the Υ-deletion contraction sequence.

With a view to applying Definition 1.4, we consider the following more general situation. Suppose Γ' is a connected subgraph of Γ whose complement contains no self-edges. Then we likewise have a subcomplex $\Upsilon^{\bullet-2|\Gamma\setminus\Gamma'|} \to \Upsilon^\bullet(\Gamma)$, given by all summands $\bigwedge^l \mathbb{H}(\Gamma \setminus J)$ with $\Gamma' \supset \Gamma \setminus J$. The induced map on cohomology can be written as the composition, in any order, of the maps a_e^Υ for $e \in \Gamma \setminus \Gamma'$. In particular, the compositions in different orders are all equal.

We can thus make the following special case of Definition 1.4.

Definition 2.7. The Υ-deletion filtration is the filtration defined by Definition 1.4, where the covariant functor A takes Γ to $\mathrm{H}^\bullet(\Upsilon(\Gamma))$ and takes $\Gamma' \to \Gamma$ to the composition of a_e^Υ (in any order) for $e \in \Gamma \setminus \Gamma'$.

By construction, the maps b_e^Υ and c_e^Υ respect the Υ-grading, whereas the map a_e^Υ increases the grading by one. Hence the kth step $D_k\mathrm{H}^i(\Upsilon(\Gamma))$ lies in the subspace of Υ-degree $\geq i-k$. The reverse inclusion is clear, and thus $D_k\mathrm{H}^i(\Upsilon(\Gamma)) = \oplus_{m \leq 2i-k}\mathrm{H}^i(\Upsilon_m(\Gamma))$.

Corollary 2.8. *The Υ-deletion filtration is induced by the Υ-grading on* $\mathrm{H}^\bullet(\Upsilon(\Gamma))$.

Viewed as a sequence of filtered vector spaces, the maps of a graded sequence strictly preserve the filtrations. Thus the Υ deletion-contraction sequence strictly preserves the Υ-filtration.

3. Non-moment maps

Given a symplectic manifold (M, ω), a function $h : M \to \mathbb{R}$ determines a vector field by dualizing dh using ω; such vector fields are termed Hamiltonian. That is, there is a natural map $\omega^{\#} d : \mathrm{Functions}(M) \to \mathrm{Sections}(TM)$.

The action of a Lie group G on M is given infinitesimally by a map from the Lie algebra of G to vector fields on M, i.e. a section of $\mathfrak{g}^* \otimes TM$. One can ask this section is obtained from some map $\mu : M \to \mathfrak{g}^*$ by composition with $\mathfrak{g}^* \otimes \omega^{\#} d$. Such a μ is necessarily G-equivariant with respect to the co-adjoint action on $\mathfrak{g}^*$. In this case the action is called "Hamiltonian" and the map μ is termed the moment map.

When $\mathfrak{g}$ is abelian, any translation of μ by an element of $\mathfrak{g}^*$ is also a moment map. Thus it is sometimes more natural to view μ as a map to a $\mathfrak{g}^*$-torsor.

An important notion in this context is the Hamiltonian reduction. The pre-image of a (co-adjoint) orbit in $\mathfrak{g}^*$ is G-invariant, so we may form its quotient by G. For an orbit O, this is denoted $M/\!/_O\, G := \mu^{-1}(O)/G$. We will be exclusively interested in the abelian case, where every point in $\mathfrak{g}^*$ is an orbit.

Much of the literature on character varieties and related moduli spaces involves Hamiltonian reduction, quasi-Hamiltonian reduction and hyperkähler reduction. The spaces we will consider in this paper also have such Hamiltonian structures. The constructions we perform with them will require and retain such structures, but often at intermediate stages will not precisely be (quasi-)Hamiltonian or hyperkähler due to the group action being too small or the target of the moment map too large. E.g., we may have a subgroup $H \subset G$ and be interested in $\mu^{-1}(O)/H$, which, if G and H are abelian, retains a Hamiltonian action of G/H. In order to keep track of some such information, we introduce the following notation.

3.1. $(\mathbb{G}, \mathbb{A})$**-spaces.** Fix abelian groups $\mathbb{G}$ and $\underline{\mathbb{A}}$. Let $\mathbb{A}$ be a torsor over $\underline{\mathbb{A}}$ (often, but not always, we will take $\mathbb{A} = \underline{\mathbb{A}}$ without further comment).

Remark 3.1. For us, $\mathbb{G}$ and $\mathbb{A}$ will always be connected abelian Lie groups. We will use additive notation for an abstract $\mathbb{A}$, but when we will use multiplicative notation when, for instance, $\mathbb{A} = \mathbb{C}^*$.

Definition 3.2. A $(\mathbb{G}, \mathbb{A})$-space will mean a topological space $\mathbf{X}$ carrying an action of $\mathbb{G}$ and a $\mathbb{G}$-invariant map $\mu_{\mathbf{X}} \colon \mathbf{X} \to \mathbb{A}$. A $(\mathbb{G}, \mathbb{A})$-map will mean a $\mathbb{G}$-equivariant $f \colon \mathbf{X} \to \mathbf{Y}$, such that $\mu_{\mathbf{Y}} = \mu_{\mathbf{Y}'} \circ f$.

Example 3.3. We write $\mathbf{0} = \mathbf{0}_{\mathbb{G},\mathbb{A}}$ for the $(\mathbb{G}, \mathbb{A})$-space given by a point carrying the trivial $\mathbb{G}$ action and whose image under the map μ has image $0 \in \mathbb{A}$.

Example 3.4. If $\mathbf{X}$ is a space with a $\mathbb{G}$-action, we write $[\mathbf{X} \times \mathbb{A}]$ for the $(\mathbb{G}, \mathbb{A})$-space whose underlying space is $\mathbf{X} \times \mathbb{A}$, on which $\mathbb{G}$ acts by multiplication on the first factor and trivially on the second, equipped with the map $\mu\colon \mathbf{X} \times \mathbb{A} \to \mathbb{A}$ via the second projection.

In particular, we will often consider $[\mathbb{G} \times \mathbb{A}]$ where $\mathbb{G}$ acts by translation on the first factor.

Example 3.5. Any $\mathbb{G}$-stable subset of a $(\mathbb{G}, \mathbb{A})$ space inherits a natural $(\mathbb{G}, \mathbb{A})$-structure. In particular, given a $(\mathbb{G}, \mathbb{A})$-space $\mathbf{X}$ and any subset $O \subset \mathbb{A}$, $\mu_{\mathbf{X}}^{-1}(O) \subset \mathbf{X}$ carries a natural $(\mathbb{G}, \mathbb{A})$-structure.

The definition of $(\mathbb{G}, \mathbb{A})$-space makes sense for various notions of space with a group action. For instance:

Definition 3.6. Let $\mathbb{G}, \mathbb{A}$ be affine algebraic groups over $\mathbb{C}$. A $(\mathbb{G}, \mathbb{A})$-variety $\mathbf{X}$ is an algebraic variety such that the $\mathbb{G}$-action and map $\mu_{\mathbf{X}}$ are algebraic.

We now connect these properties to the yet-to-appear spaces $\mathfrak{B}$ and $\mathfrak{D}$. Evidently the following statements will not make sense until referring backwards from the relevant sections. The reader can skip them on a first reading.

Example 3.7. Lemma 5.2 amounts to the assertion that $\mu_{\mathfrak{B}}^{\mathbb{C}^*} : \mathfrak{B} \to \mathbb{C}^*$, together with action of $\mathbb{C}^*$ on $\mathfrak{B}$, endows the space $\mathfrak{B}$ with the structure of a $(\mathbb{C}^*, \mathbb{C}^*)$-manifold.

Example 3.8. From Lemma 5.3 it can be deduced that $\mu_{\mathfrak{B}}^{\mathbb{C}^*} \times \mu_{\mathfrak{B}}^{\mathbb{R}}$ endows $\mathfrak{B}$ with the structure of a $(\mathbb{U}_1, \mathbb{C}^* \times \mathbb{R})$-manifold.

Example 3.9. Lemma 6.2 asserts that the map $\mu_{\mathfrak{D}}^{\mathbb{U}_1} \times q$ endows $\mathfrak{D}$ with a $(\mathbb{U}_1, \mathbb{U}_1 \times \mathbb{C})$-structure.

Example 3.10. Lemma 6.2 gives a $(\mathbb{U}_1, \mathbb{C}^* \times \mathbb{R})$ map from $\mathfrak{D} \to \mathfrak{B}$. Note this map commutes with the inclusion of $\mathbf{0}_{(\mathbb{U}_1, \mathbb{C}^* \times \mathbb{R})}$ as the unique critical point of μ.

Definition 3.11. Let $\mathbf{X}$ be a $(\mathbb{G}, \mathbb{A})$-space. For $\zeta \in \mathbb{A}$ such that $\mathbb{G}$ acts freely on $\mu^{-1}(\zeta)$, we define

$$\mathbf{X} /\!\!/_{\zeta} \mathbb{G} := \mu_X^{-1}(\zeta)/\mathbb{G}.$$

If we wish to emphasize $\mathbb{A}$, we write $\mathbf{X} /\!\!/_{\zeta \in \mathbb{A}} \mathbb{G}$.

Example 3.12. The spaces $\mathfrak{B}$ and $\mathfrak{D}$ have free $\mathbb{G}$ actions on the fibres of any of the above-given μ, except over 1 or 1×0. The corresponding quotients are all points.

Let $\mathbb{A}$ (resp. $\mathbb{A}'$) be torsors over $\underline{\mathbb{A}}$ (resp. $\underline{\mathbb{A}}'$). Let $\tau\colon \mathbb{A}' \to \mathbb{A}$ be a map compatible with a group homomorphism $\underline{\mathbb{A}}' \to \underline{\mathbb{A}}$. Via τ, we can view any $(\mathbb{G}, \mathbb{A}')$-space $\mathbf{X}$ as a $(\mathbb{G}, \mathbb{A})$-space.

Lemma 3.13. *Let $\eta \in \mathbb{A}'$. There is a Cartesian square*

$$\begin{tikzcd} \mathbf{X} /\!\!/_{\eta} \mathbb{G} \arrow[r] \arrow[d] & \mathbf{X} /\!\!/_{\tau(\eta)} \mathbb{G} \arrow[d] \\ \eta \arrow[r] & \tau^{-1}\tau(\eta) \end{tikzcd} \tag{6}$$

where the horizontal arrows are embeddings and $\tau^{-1}\tau(\eta)$ is a torsor over the kernel of $\underline{\mathbb{A}}' \to \underline{\mathbb{A}}$.

Let $\rho\colon \mathbb{G} \to \mathbb{G}'$ be an injective homomorphism. Via ρ, we can view any $(\mathbb{G}', \mathbb{A})$-space as a $(\mathbb{G}, \mathbb{A})$-space.

Lemma 3.14. *We have a surjection $\mathbf{X} /\!\!/_\eta \mathbb{G} \to \mathbf{X} /\!\!/_\eta \mathbb{G}'$.*

We will often work in the setting of Kähler and hyperkähler manifolds. Although the metrics play a rather minor role in the proof of our main theorems, we believe they are an important feature of our spaces, and in view of potential further applications we will keep track of them as they naturally arise. Recall that a Kähler manifold is a complex manifold $\mathbf{X}$ together with a metric g such that the complex structure J is flat with respect to the associated Levi-Civita connection. The Kähler form $\omega = g(-, J-)$ defines a symplectic structure on $\mathbf{X}$.

Many of our spaces are multiplicative analogues of symplectic reductions, starting from a Kähler manifold with a group action and a multiplicative moment map. With honest moment maps, the existence of a Kähler metric on the reduction is [HKLR, Theorem 3.1]; we will require a "multiplicative" version, but for $\mathbb{G} = \mathbb{U}_1^k$ this is no more difficult. We thus only discuss Kähler $(\mathbb{G}, \mathbb{A})$-spaces in this setting.

Definition 3.15. A $(\mathbb{U}_1^k, \mathbb{U}_1^k)$-Kähler manifold is a $(\mathbb{U}_1^k, \mathbb{U}_1^k)$ space with a Kähler structure such that $\mathbb{U}_1^k$ preserves the Kähler form and $\mu_{\mathbf{X}}\colon \mathbf{X} \to \mathbb{U}_1^k$ is a multiplicative moment map for the action, i.e such that on any simply connected open subset of $\mathbf{X}$, any lift of $\mu_{\mathbf{X}}$ from $\mathbb{U}_1^k = \mathbb{R}^k/\mathbb{Z}^k$ to $\mathbb{R}^k$ is an ordinary moment map for the action of $\mathbb{U}_1^k$.

A $(\mathbb{U}_1^k, \mathbb{U}_1^k \times \mathbb{C}^k)$-hyperkähler manifold is a $(\mathbb{U}_1^k, \mathbb{U}_1^k \times \mathbb{C}^k)$-manifold with a hyperkähler structure such that $\mathbb{U}_1^k$ preserves the hyperkähler form and $\mu_{\mathbf{X}}\colon \mathbf{X} \to \mathbb{U}_1^k \times \mathbb{C}^k$ is a multiplicative hyperkähler moment map for the action, i.e such that on any simply connected open subset of $\mathbf{X}$, any lift of $\mu_{\mathbf{X}}$ from $\mathbb{U}_1^k \times \mathbb{C}^k = \mathbb{R}^k/\mathbb{Z}^k \times \mathbb{C}^k$ to $\mathbb{R}^k \times \mathbb{C}^k$ is an (ordinary) hyperkähler moment map for the action of $\mathbb{U}_1^k$.

We have an example $\mu_{\mathfrak{D}}^{\mathbb{U}_1}\colon \mathfrak{D} \to \mathbb{U}_1$ of such a multiplicative moment map in Lemma 6.2.

Lemma 3.16. *If $\mathbf{X}$ is $(\mathbb{G}, \mathbb{A})$-Kähler and $\mathbb{G}$ acts freely on $\mu_{\mathbf{X}}^{-1}(\zeta)$ (and hence ζ is a regular value of $\mu_{\mathbf{X}}$), then $\mathbf{X} /\!\!/_\zeta \mathbb{G}$ is a smooth Kähler manifold. The same holds in the hyperkähler setting.*

Proof. The proof in [HKLR, Theorem 3.1] goes through unchanged when the moment map is valued in $\mathbb{R}^k/\mathbb{Z}^k$. □

3.2. Convolution of $(\mathbb{G}, \mathbb{A})$-spaces.

Definition 3.17. Let $\mathbf{X}, \mathbf{Y}$ be $(\mathbb{G}, \mathbb{A})$-spaces, and suppose we are given an isomorphism $\mathbb{A} \cong \underline{\mathbb{A}}$. We define a new $(\mathbb{G}, \mathbb{A})$-space called $\mathbf{X} \bullet \mathbf{Y}$ whose underlying space is $\mathbf{X} \times \mathbf{Y}$ by taking $\mathbb{G}$ to act diagonally and $\mu_{\mathbf{X}\bullet\mathbf{Y}} := \mu_{\mathbf{X}} + \mu_{\mathbf{Y}}\colon \mathbf{X} \times \mathbf{Y} \to \mathbb{A}$.

Definition 3.18. Given $(\mathbb{G},\mathbb{A})$-spaces $\mathbf{X}$ and $\mathbf{Y}$, we write $\mathbf{X} \star_{\mathbb{G},\mathbb{A},\zeta} \mathbf{Y} := (\mathbf{X} \bullet \mathbf{Y}) /\!\!/_{\zeta} \mathbb{G}$.

If the choice of $\mathbb{G}, \mathbb{A}$ or ζ is clear from context, we may omit them. In particular, we will uniformly employ the abbreviations $\star_{\mathbb{G}_m} := \star_{\mathbb{G}_m,\mathbb{G}_m}$, $\star_{\mathbb{U}_1} := \star_{\mathbb{U}_1,\mathbb{U}_1}$ and $\star_{\mathbb{U}_1\times\mathbb{C}} := \star_{\mathbb{U}_1,\mathbb{U}_1\times\mathbb{C}}$.

Lemma 3.19. *Given $(\mathbb{G},\mathbb{A})$-spaces $\mathbf{X}$ and $\mathbf{Y}$, the action of $\mathbb{G}$ on $\mathbf{X}$ and the map $\mu_{\mathbf{X}}$ descend to $\mathbf{X} \star_{\zeta} \mathbf{Y}$, giving it the structure of a $(\mathbb{G},\mathbb{A})$-space.*

Remark 3.20. In our usage, the notation $\mathbf{X} \star_{\zeta} \mathbf{Y}$ implicitly asserts that ζ is a regular value for the map $\mu_{\mathbf{X}} + \mu_{\mathbf{Y}} \colon \mathbf{X} \times \mathbf{Y} \to \mathbb{A}$ and that $\mathbb{G}$ acts freely on $(\mu_{\mathbf{X}} + \mu_{\mathbf{Y}})^{-1}(\zeta)$.

Remark 3.21. There is an $(\mathbb{G},\mathbb{A})$-space isomorphism $\mathbf{X} \bullet \mathbf{Y} \cong \mathbf{Y} \bullet \mathbf{X}$, and an isomorphism of spaces between $\mathbf{X} \star_{\zeta} \mathbf{Y}$ and $\mathbf{Y} \star_{\zeta} \mathbf{X}$. However the $(\mathbb{G},\mathbb{A})$-space structures on these two are related by negating the action of $\mathbb{G}$ and adjusting the moment map by $\mu \mapsto \zeta - \mu$.

Example 3.22. Recall that $\mathbf{0} = \mathbf{0}_{(\mathbb{G},\mathbb{A})}$ denotes the point with trivial $\mathbb{G}$ action and $\mu = 0 \in \mathbb{A}$. For $\zeta \in \mathbb{A}$, we have

$$\mathbf{Y} \star_{\zeta} \mathbf{0} = \mathbf{Y} /\!\!/_{\zeta} \mathbb{G} = \mathbf{0} \star_{\zeta} \mathbf{Y}$$

Lemma 3.19 asserts that the resulting space should acquire a $(\mathbb{G},\mathbb{A})$-structure. Note the resulting $\mathbb{G}$ action is trivial, and the map μ is the constant map with value ζ on the left, or 0 on the right.

Lemma 3.23. *A $(\mathbb{G},\mathbb{A})$-map $f \colon \mathbf{Y} \to \mathbf{Y}'$ induces a $(\mathbb{G},\mathbb{A})$-map*

$$f_{\mathbf{X}} \colon \mathbf{X} \star \mathbf{Y} \to \mathbf{X} \star \mathbf{Y}'$$

Proof. If $x \times y \in \mathbf{X} \times \mathbf{Y}$ lies in $\mu^{-1}_{\mathbf{X}\dot{\mathbf{Y}}}(\zeta)$, then $x \times f(y)$ lies in $\mu^{-1}_{\mathbf{X}\dot{\mathbf{Y}}'}(\zeta)$ since f preserves the moment map. The resulting map $id \times f \colon \mu^{-1}_{\mathbf{X}\dot{\mathbf{Y}}}(\zeta) \to \mu^{-1}_{\mathbf{X}\dot{\mathbf{Y}}'}(\zeta)$ is $\mathbb{G}$-equivariant, and thus descends to the quotients. $\square$

Remark 3.24. Suppose $\mathbf{Y}$ and $\mathbf{Y}'$ and $\mathbf{X}$ have exhaustions by compact $\mathbb{G}$-equivariant subsets, compatible with the map f. Then $f_{\mathbf{X}}$ is compatible with the exhaustions of $\mathbf{X} \star \mathbf{Y}$ and $\mathbf{X} \star \mathbf{Y}'$ obtained from taking cartesian products of compact sets. This will be needed in the proof of Proposition 6.32.

Lemma 3.25. *If $f \colon \mathbf{Y} \to \mathbf{Y}'$ is injective (resp surjective), then $f_{\mathbf{X}} \colon \mathbf{X} \star \mathbf{Y} \to \mathbf{X} \star \mathbf{Y}'$ is injective (resp surjective).*

Proof. If f is injective (resp surjective), then so is $f \times id \colon \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}'$. Injectivity is clearly preserved by restriction to $\mu^{-1}_{\mathbf{X}\bullet\mathbf{Y}}(\eta)$ and $\mu^{-1}_{\mathbf{X}\bullet\mathbf{Y}'}(\eta)$. To see that surjectivity is also preserved, note that if $\mu(x, y') = \eta$, then for any preimage y of y', $\mu(x, y) = \eta$. We finally need to show that passing to $\mathbb{G}$-quotients preserves injectivity and surjectivity. But by assumption, $\mathbb{G}$ acts freely on both sides, from which the conclusion directly follows. $\square$

Definition 3.26. Given an action of $\mathbb{G}$ on $\mathbf{X}$ and a subset $O \subset \mathbb{A}$, define a $(\mathbb{G},\mathbb{A})$-space $\mathbf{X} \times O$ by letting $\mathbb{G}$ act on the left factor and $\mu_{\mathbf{X}\times O}$ project onto the right factor.

Lemma 3.27. *For any $\zeta, \zeta' \in \mathbb{A}$ and $\mathbf{Y}$ a $(\mathbb{G}, \mathbb{A})$ space, we have a canonical isomorphism of $(\mathbb{G}, \mathbb{A})$ spaces:*

$$\mathbf{Y} \star_\zeta (\mathbb{G} \times \zeta') = \mu_{\mathbf{Y}}^{-1}(\zeta - \zeta')$$

where the left-hand space inherits its $(\mathbb{G}, \mathbb{A})$-structure from $\mathbf{Y}$.

More generally:

Lemma 3.28. *Let $\mathbf{Y}$ be a $(\mathbb{G}, \mathbb{A})$-space, and let $\mathbf{X}$ be a $(\mathbb{G}, \mathbb{A})$-space such that $\mu_{\mathbf{X}} \colon \mathbf{X} \to \mathbb{A}$ is a $\mathbb{G}$-bundle over its image $O \subset \mathbb{A}$. Then any trivialization of the bundle determines an isomorphism of $(\mathbb{G}, \mathbb{A})$-spaces*

$$\mathbf{X} \star_\zeta \mathbf{Y} \cong \mu_{\mathbf{Y}}^{-1}(\zeta - O).$$

Proof. Let $s \colon O \to \mathbf{X}$ be a section of $\mu_{\mathbf{X}}$. Then s determines an isomorphism of $(\mathbb{G}, \mathbb{A})$ spaces $\mathbf{X} \cong \mathbb{G} \times O$. Apply lemma 3.27 pointwise over O. □

In particular:

Lemma 3.29. *For any $\zeta \in \mathbb{A}$ and $\mathbf{Y}$ a $(\mathbb{G}, \mathbb{A})$-space, we have a canonical isomorphism of $(\mathbb{G}, \mathbb{A})$-spaces*

$$\mathbf{Y} \star_\zeta [\mathbb{G} \times \mathbb{A}] = \mathbf{Y}.$$

Lemma 3.30. *Let $\mathbb{G}$ act freely on T, and define $T \times \zeta' \in \mathcal{C}$ as above. Then $\mathbf{Y} \star (T \times \zeta')$ is naturally a $\mu_{\mathbf{Y}}^{-1}(\zeta - \zeta')$-bundle over $T/\mathbb{G}$ with structure group $\mathbb{G}$.*

Lemma 3.31. *Let $\mathbb{G}$ act freely on T, and let $O \subset \mathbb{A}$. Define $T \times O \in \mathcal{C}$ as above. Then $\mathbf{Y} \star (T \times O)$ is naturally a $\mu_{\mathbf{Y}}^{-1}(\zeta - O)$-bundle over $T/\mathbb{G}$ with structure group $\mathbb{G}$.*

We will need the following technical lemmas. They assert that if a $(\mathbb{G}, \mathbb{A})$-space $\mathbf{X}$ can be factorized in a certain way, then we can also factorize the quotient $\mathbf{X} /\!\!/_\zeta \mathbb{G}$.

Lemma 3.32. *Suppose $\mathbf{X}_1$ is a $(\mathbb{G}_1, \mathbb{A}_1)$ space and $\mathbf{X}_2$ is a space. Then $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$ is naturally also a $(\mathbb{G}_1, \mathbb{A}_1)$-space, and $\mathbf{X} /\!\!/_\zeta \mathbb{G}_1 = (\mathbf{X}_1 /\!\!/_\zeta \mathbb{G}_1) \times \mathbf{X}_2$.*

Lemma 3.33. *Retaining the setting of the previous lemma, suppose both $\mathbf{X}_2$ and $\mathbf{X}_1$ are also $\mathbb{G}_2, \mathbb{A}_2$-spaces and the $\mathbb{G}_2$ action on $\mathbf{X}$ commutes with the $\mathbb{G}_1$ action. Then*

$$(\mathbf{X}_1 \times \mathbf{X}_2) /\!\!/_{\zeta_1 \times \zeta_2} \mathbb{G}_1 \times \mathbb{G}_2 = (\mathbf{X}_1 /\!\!/_{\zeta_1} \mathbb{G}_1 \bullet \mathbf{X}_2) /\!\!/_{\zeta_2} \mathbb{G}_2.$$

We can reformulate this last lemma as follows.

Lemma 3.34. *Let $\mathbf{X}$ be a $(\mathbb{G}, \mathbb{A})$-space. Suppose $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$, $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$ and $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2$ where $\mathbb{G}_1$ acts trivially on $\mathbf{X}_2$ and the $\mathbb{A}_1$-map factors through $\mathbf{X}_1 \times \mathbf{X}_2 \to \mathbf{X}_1$. For $\zeta \in \mathbb{A}$, write ζ_1, ζ_2 for the projections to $\mathbb{A}_1, \mathbb{A}_2$.*

Then, reserving the $\star$ notation for the $(\mathbb{G}_2, \mathbb{A}_2)$ structure, we have an equality

$$\mathbf{X} /\!\!/_\zeta \mathbb{G} = (\mathbf{X}_1 /\!\!/_{\zeta_1} \mathbb{G}_1) \star_{\zeta_2} \mathbf{X}_2. \tag{7}$$

We will use Lemma 3.34 in situations where the $(\mathbb{G}_i, \mathbb{A}_i)$-structures on $\mathbf{X}_1 \times \mathbf{X}_2$ are induced from $(\mathbb{G}'_i, \mathbb{A}'_i)$-structures by maps $\rho_i \colon \mathbb{G}_i \to \mathbb{G}'_i$ and $\tau_i \colon \mathbb{A}'_i \to \mathbb{A}_i$, as in Lemmas 3.13 and 3.14. We write the product maps $\rho \colon \mathbb{G} \to \mathbb{G}'$ and $\tau \colon \mathbb{A}' \to \mathbb{A}$.

Lemma 3.35. *In this case, we can extend Equation 7 to a commutative diagram*

$$\begin{array}{ccc} \mathbf{X} /\!\!/_{\zeta} \mathbb{G} & \longrightarrow & (\mathbf{X}_1 /\!\!/_{\zeta_1} \mathbb{G}_1) \star_{\zeta_2} \mathbf{X}_2 \\ \downarrow & & \downarrow \\ \tau^{-1}(\zeta) & \longrightarrow & \tau_1^{-1}(\zeta_1) \times \mathbb{A}_2 \end{array} \tag{8}$$

where the bottom map is the natural embedding of $\tau^{-1}(\zeta) = \tau_1^{-1}(\zeta_1) \times \tau_2^{-1}(\zeta_2)$*, and the right-hand map is the descent of the product map* $\mu_1 \times \mu_2 : \mathbf{X}_1 \times \mathbf{X}_2 \to \mathbb{A}_1 \times \mathbb{A}_2$.

4. Spaces from graphs

4.1. **Construction.** Let $\mathbf{Z}$ be a $(\mathbb{G}, \mathbb{A})$-space, with fixed isomorphism $\mathbb{A} = \underline{\mathbb{A}}$. Let Γ be a connected oriented graph, and η an assignment of an element of $\mathbb{A}$ to each vertex. From this data, we will produce a new space $\mathbf{Z}(\Gamma, \eta)$.

Remark 4.1. The spaces we consider will in fact be independent of the choice of orientation, up to a canonical isomorphism.

Let $V(\Gamma), E(\Gamma)$ be the vertex and edge sets of an oriented graph Γ. Each edge $e \in E(\Gamma)$ has a head and tail vertices $h(e), t(e) \in V(\Gamma)$. We view Γ as a CW complex. This gives identifications

$$C_0(\Gamma, \mathbb{A}) = \mathbb{A}^{V(\Gamma)} = C^0(\Gamma, \mathbb{A})$$

and

$$C_1(\Gamma, \mathbb{A}) = \mathbb{A}^{E(\Gamma)} = C^1(\Gamma, \mathbb{A}).$$

The differential $d_\Gamma \colon C_1(\Gamma, \mathbb{A}) \to C_0(\Gamma, \mathbb{A})$ is the incidence map

$$\begin{array}{rcl} d_\Gamma \colon \mathbb{A}^{E(\Gamma)} & \to & \mathbb{A}^{V(\Gamma)} \\ {[e]} & \mapsto & [h(e)] - [t(e)] \end{array}$$

Write $d_\Gamma^* \colon C^0(\Gamma, \mathbb{G}) \to C^1(\Gamma, \mathbb{G})$ for the differential on cochains. Note that $\ker d_\Gamma^* = \{g, g, ..., g\}$, i.e. the subgroup of constant functions. We write $\overline{C}^0(\Gamma, \mathbb{G})$ for the quotient by this subgroup; we will sometimes abbreviate this to

$$\overline{\mathbb{G}^{V(\Gamma)}} := \mathbb{G}^{V(\Gamma)} / \{g, g, ..., g\}.$$

Likewise, we write $\overline{C}_0(\Gamma, \mathbb{A})$ for the subgroup of chains summing to zero.

Given a $(\mathbb{G}, \mathbb{A})$-space $\mathbf{Z}$, there is an action of $C^1(\Gamma, \mathbb{G}) = \mathbb{G}^{E(\Gamma)}$ on $\mathbf{Z}^{E(\Gamma)}$, together with a map $\mathbf{Z}^{E(\Gamma)} \to \mathbb{A}^E(\Gamma) = C_1(\Gamma, \mathbb{A})$.

The maps $d_\Gamma \colon C_1(\Gamma, \mathbb{A}) \to \overline{C}_0(\Gamma, \mathbb{A})$ and $d_\Gamma^* \colon C^0(\Gamma, \mathbb{G}) \to C^1(\Gamma, \mathbb{G})$ define a $(\overline{C}^0(\Gamma, \mathbb{G}), C_0(\Gamma, \mathbb{A}))$-structure on $\mathbf{Z}^{E(\Gamma)}$, via Lemmas 3.13 and 3.14.

Definition 4.2. Let Γ be a connected oriented graph, let $\mathbf{Z}$ be a $(\mathbb{G}, \mathbb{A})$-space and let $\eta \in C_0(\Gamma, \mathbb{A}) = \mathbb{A}^{V(\Gamma)}$. We define:

$$\mathbf{Z}(\Gamma, \eta) := \mathbf{Z}^{E(\Gamma)} /\!\!/_\eta \, \overline{C}^0(\Gamma, \mathbb{G}).$$

Note that if $\eta \notin \overline{C}_0(\Gamma, \mathbb{A})$, we have $\mathbf{Z}(\Gamma, \eta) = \emptyset$. We nevertheless leave ourselves the freedom to make this unfortunate choice, as it will occasionally be the right one. Typically, the choice of η will be understood, and we simply write $\mathbf{Z}(\Gamma)$. If we wish to emphasize the dependence on $(\mathbb{G}, \mathbb{A})$, we write $\mathbf{Z}^{\mathbb{G},\mathbb{A}}(\Gamma, \eta)$. The following diagram summarizes the situation:

(9)
$$\begin{tikzcd}
\mathbf{Z}(\Gamma, \eta) \arrow[r, equal] & \mu_\Gamma^{-1}(\eta)/\overline{C}^0(\Gamma, \mathbb{G}) \arrow[d] & \mu_\Gamma^{-1}(\eta) \arrow[l] \arrow[r, hook] \arrow[d] & \mathbf{Z}^{E(\Gamma)} \arrow[d] \\
 & d_\Gamma^{-1}(\eta) \arrow[r, equal] & d_\Gamma^{-1}(\eta) \arrow[r, hook] & C_1(\Gamma, \mathbb{A})
\end{tikzcd}$$

Remark 4.3. Since $d_{\bigcirc\!\!\!\bullet} = 0$, we have $\mathbf{Z} = \mathbf{Z}(\bigcirc\!\!\!\bullet, 0)$.

Lemma 4.4. *The action of* $C^1(\Gamma, \mathbb{G})$ *on* $\mathbf{Z}^{E(\Gamma)}$ *descends to a residual action of*

$$C^1(\Gamma, \mathbb{G})/\overline{C}^0(\Gamma, \mathbb{G}) = \mathrm{H}^1(\Gamma, \mathbb{G})$$

on $\mathbf{Z}(\Gamma, \eta)$.

We have the following definition, which is in a sense dual to the above.

Definition 4.5. Let $\mathrm{H}_1(\Gamma, \mathbb{A})_\eta = d_\Gamma^{-1}(\eta)$.

When nonempty, it is the translation of $\mathrm{H}_1(\Gamma, \mathbb{A}) \subset C_1(\Gamma, \mathbb{A})$ by any d_Γ-preimage of η, and is thus a torsor over $\mathrm{H}_1(\Gamma, \mathbb{A})$. We have:

Lemma 4.6.

(10)
$$\mu_\Gamma^{-1}(\eta) = \left(\mu_{\mathbf{Z}}^{E(\Gamma)}\right)^{-1}(\mathrm{H}_1(\Gamma, \mathbb{A})_\eta).$$

Proposition 4.7. *The map* $\mu_{\mathbf{Z}}^{E(\Gamma)} \colon \mathbf{Z}^{E(\Gamma)} \to \mathbb{A}^{E(\Gamma)}$ *descends to a map* $\mu_{\mathrm{res}} \colon \mathbf{Z}(\Gamma, \eta) \to \mathrm{H}_1(\Gamma, \mathbb{A})_\eta$. *This makes* $\mathbf{Z}(\Gamma, \eta)$ *into a* $(\mathrm{H}^1(\Gamma, \mathbb{G}), \mathrm{H}_1(\Gamma, \mathbb{A})_\eta)$*-space.*

Proposition 4.8. *If* $\mu_{\mathbf{Z}}$ *is proper, then so is* μ_{res}.

Proof. Properness is preserved both by restriction to the closed set $\mu_\Gamma^{-1}(\eta)$ and by descent to the quotient by $C^0(\Gamma, \mathbb{G})$. $\square$

Lemma 4.9. *Let* $\mathbf{Z} = [\mathbb{G} \times \mathbb{A}]$. *Then* $\mathbf{Z}(\Gamma, \eta) = \mathrm{H}^1(\Gamma, \mathbb{G}) \times \mathrm{H}_1(\Gamma, \mathbb{A})$*, with the obvious* $(\mathrm{H}^1(\Gamma, \mathbb{G}), \mathrm{H}_1(\Gamma, \mathbb{A}))$*-structure.*

Proof. It is not hard to give a hands-off proof, but we will eventually need to understand this isomorphism in coordinates, so we will use coordinates here.

We have $\mathbf{Z}^{E(\Gamma)} = C^1(\Gamma, \mathbb{G}) \times C_1(\Gamma, \mathbb{A})$. Choose a collection of edges $I \subset E(\Gamma)$ forming a basis of $\mathrm{H}^1(\Gamma, \mathbb{Z})$ and a collection of cycles $I^* \subset H_1(\Gamma, \mathbb{Z})$ forming a basis of homology. We have morphisms $z_e \colon C_1(\Gamma, \mathbb{A}) \to \mathbb{A}$ for $e \in I$ and $x_\gamma \colon C^1(\Gamma, \mathbb{G}) \to \mathbb{G}$ for $\gamma \in I^*$. Both sets of functions are $C^0(\Gamma, \mathbb{G})$-invariant, and thus descend to the quotient. One checks that they define an isomorphism $\mathbf{Z}(\Gamma, \eta) = \mathrm{H}^1(\Gamma, \mathbb{G}) \times \mathrm{H}_1(\Gamma, \mathbb{A})$. $\square$

4.2. **Independence of orientation.** Pick a subset J of the edges of Γ, and let Γ' be the oriented graph obtained from Γ by switching the orientation of each edge in J.

Proposition 4.10. *Let $f : \mathbf{Z} \to \mathbf{Z}$ be an automorphism of topological spaces which intertwines the $\mathbb{G}$-action with the inverse $\mathbb{G}$ action and the $\mathbb{A}$-map with the inverse $\mathbb{A}$-map. Then f determines an isomorphism of topological spaces*

$$\mathbf{Z}(\Gamma) \to \mathbf{Z}(\Gamma').$$

If f is a map of smooth manifolds or algebraic varieties, then so is the induced map.

Proof. The map $\mathbf{Z}^{E(\Gamma)} \to \mathbf{Z}^{E(\Gamma')}$ given by f on the factors in J and by the identity everywhere else intertwines the group actions and moment maps for Γ and Γ', and thus descends to the requisite isomorphism. $\square$

4.3. **Smoothness.** In order that $\mathbf{Z}(\Gamma, \eta)$ be smooth, the moment fiber should not contain any points with large stabilizer. Below, we work conditions under which this holds.

Until the end of this section, we make the following assumptions on $\mathbf{Z}$.

Hypothesis 4.11. Assume $\mathbf{Z}$ is a smooth manifold, and $\mu_{\mathbf{Z}}$ is a smooth map which defines a $\mathbb{G}$-bundle over $\mathbb{A} \setminus 1$ (or $\mathbb{A} \setminus 0$ if using additive notation).

Definition 4.12. Given $a \in \mathbb{A}^{E(\Gamma)}$, let $S(a) \subset E(\Gamma)$ be the set of edges such that $a_e = 1$. We say η is $\mathbf{Z}$-*generic* if for all $a \in \mathrm{H}_1(\Gamma, \mathbb{A})_\eta$, the graph $\Gamma \setminus S(a)$ is connected.

We will often drop the $\mathbf{Z}$ from this notation when it is clear from context.

Proposition 4.13. *Suppose that η is $\mathbf{Z}$-generic. Then $\mathbf{Z}(\Gamma, \eta)$ is smooth.*

Proof. We must show that $\mu_\Gamma^{-1}(\eta)$ is smooth, and that the action of $\mathbb{G}^{\overline{V(\Gamma)}}$ on $\mu_\Gamma^{-1}(\eta)$ is free. We begin with the second condition.

$\overline{C}^0(\Gamma, \mathbb{G})$ acts freely on $\left(\mu_{\mathbf{Z}}^{E(\Gamma)}\right)^{-1}(a)$ if

$$\overline{C}^0(\Gamma, \mathbb{G}) \xrightarrow{d_\Gamma^*} C^1(\Gamma, \mathbb{G}) \to C^1(\Gamma \setminus S(a), \mathbb{G}) \tag{11}$$

is injective. Here $\Gamma \setminus S(a)$ is the graph obtained by deleting the edges $S(a)$, and the second map is the natural projection. The composition is the differential $d^*_{\Gamma \setminus S(a)}$. Thus, the kernel is trivial exactly when $\Gamma \setminus S(a)$ is connected.

We use a similar argument to show that the map $\mu_\Gamma\colon \mathbf{Z}^{E(\Gamma)} \to \overline{C}_0(\Gamma,\mathbb{A})$ is a submersion. We can factor $d\mu_\Gamma$ as $d(d_\Gamma)\circ d(\mu^{E(\Gamma)})$. For any $z \in (\mu_{\mathbf{Z}}^{E(\Gamma)})^{-1}(a)$, the image of the differential $d\mu^{E(\Gamma)}$ contains the tangent space of $\mathbb{A}^{E(\Gamma)\setminus S(a)} = C_1(\Gamma\setminus S(a),\mathbb{A})$. Dualizing the injective composition 11 (and switching notation from $\mathbb{G}$ to $\mathbb{A}$ for our abelian group), we obtain a surjective composition

$$C_1(\Gamma\setminus S(a),\mathbb{A}) \to C_1(\Gamma,\mathbb{A}) \to \overline{C}_0(\Gamma,\mathbb{A}).$$

Thus $d(d_\Gamma)$ is a surjection even when restricted to the tangent space of $\mathbb{A}^{E(\Gamma)\setminus S(a)}$. Thus μ_Γ is a submersion, and $\mu_\Gamma^{-1}(\eta)$ is smooth.

□

Remark 4.14. We may make the same construction with d_Γ replaced by a general integer matrix $\mathbb{Z}^n \to \mathbb{Z}^k$. The definition of generic η can be extended to this case, but it will only guarantee that there are at worst orbifold singularities.

4.4. **Deletion, contraction, and convolution.**

Definition 4.15. Given a graph Γ and a nonloop edge e, the graph $\Gamma\setminus e$ is defined by deleting the edge e. The graph Γ/e is defined by "contracting" e, i.e., by removing it and collapsing $h(e)$ and $t(e)$ to a single vertex $v(e)$.

In this section we will explain how $\mathbf{Z}(\Gamma,\eta)$ behaves under contraction (Lemma 4.19) and deletion (Lemma 4.20). We start with some linear algebra, to set the stage. Let $e\in\Gamma$ be a nonloop edge. Let $\eta\in C_0(\Gamma,\mathbb{A})$.

Definition 4.16. (1) Let η/e be the pushforward of η along $\Gamma\to\Gamma/e$.
(2) Let $\eta\setminus e\in C_0(\Gamma\setminus e,\mathbb{A})$ be the image of η under the isomorphism $C_0(\Gamma,\mathbb{A}) = C_0(\Gamma\setminus e,\mathbb{A})$.
(3) Let η_e be the coefficient of η at $t(e)$.

We have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_1(\Gamma,\mathbb{A})_\eta \arrow[r] & C_1(\Gamma,\mathbb{A}) \arrow[r, "d_\Gamma"] & C_0(\Gamma,\mathbb{A}) \\
\mathrm{H}_1(\Gamma\setminus e,\mathbb{A})_{\eta/e} \arrow[r] \arrow[u] & C_1(\Gamma\setminus e,\mathbb{A}) \arrow[r, "d_{\Gamma\setminus e}"] \arrow[u] & C_0(\Gamma\setminus e,\mathbb{A}) \arrow[u]
\end{tikzcd} \tag{12}$$

where the right-hand vertical is an isomorphism, and the other two verticals are inclusions along the locus $a_e = 0$.

Likewise, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}_1(\Gamma,\mathbb{A})_\eta \arrow[r] \arrow[d] & C_1(\Gamma,\mathbb{A}) \arrow[r, "d_\Gamma"] \arrow[d] & C_0(\Gamma,\mathbb{A}) \arrow[d] \\
\mathrm{H}_1(\Gamma/e,\mathbb{A})_{\eta/e} \arrow[r] & C_1(\Gamma/e,\mathbb{A}) \arrow[r, "d_{\Gamma/e}"] & C_0(\Gamma/e,\mathbb{A})
\end{tikzcd} \tag{13}$$

where the left-hand vertical map is an isomorphism and the middle vertical forgets the edge e.

Lemma 4.17. *If η is $\mathbf{Z}$-generic, then so are $\eta \setminus e$ and η/e.*

Proof. Let $a \in \mathrm{H}_1(\Gamma \setminus e, \mathbb{A})_{\eta \setminus e}$, and let $S(a) \subset E(\Gamma)$ be as in Definition 4.12. We must show that $\Gamma \setminus e \setminus S(a)$ is connected. Let a' be the image of a in $\mathrm{H}_1(\Gamma, \mathbb{C})_\eta$. Then $S(a') = S(a) \cup e$. Since η was assumed generic, $\Gamma \setminus S(a') = \Gamma \setminus e \setminus S(a)$ is connected.

Likewise, let $a \in \mathrm{H}_1(\Gamma/e, \mathbb{A})_{\eta/e}$. Let a' be the preimage of a in $\mathrm{H}_1(\Gamma, \mathbb{A})_\eta$. Then $S(a')$ equals either $S(a)$ or $S(a) \cup e$. $\Gamma \setminus S(a')$ is connected by assumption, so $(\Gamma/e) \setminus S(a)$ is also connected. $\square$

The edge e determines a cocharacter $\mathbb{G} \to \mathrm{H}^1(\Gamma, \mathbb{G})$. By the natural isomorphism $\mathrm{H}^1(\Gamma/e, \mathbb{G}) = \mathrm{H}^1(\Gamma, \mathbb{G})$, this gives a cocharacter of $\alpha_e : \mathbb{G} \to \mathrm{H}^1(\Gamma/e, \mathbb{G})$. In other words, we have the following isomorphism of short exact sequences.

$$\begin{tikzcd}
\mathrm{H}^1(\Gamma \setminus e, \mathbb{G}) & \mathrm{H}^1(\Gamma, \mathbb{G}) \arrow[l] & \mathbb{G} \arrow[l] \\
\mathrm{cok}(\alpha_\mathrm{e}) \arrow[u] & \mathrm{H}^1(\Gamma/e, \mathbb{G}) \arrow[l] \arrow[u] & \mathbb{G} \arrow[l, "\alpha_e"'] \arrow[u]
\end{tikzcd} \tag{14}$$

Similarly, the coefficient of e determines a map $\mathrm{H}_1(\Gamma, \mathbb{A})_\eta \to \mathbb{A}$, and thus a map $\beta_e : \mathrm{H}_1(\Gamma/e, \mathbb{A})_{\eta/e} \to \mathbb{A}$. We can again write this as an isomorphism of short exact sequences.

$$\begin{tikzcd}
\mathrm{H}_1(\Gamma \setminus e, \mathbb{A})_{\eta \setminus e} \arrow[r] \arrow[d] & \mathrm{H}_1(\Gamma, \mathbb{A})_\eta \arrow[r] \arrow[d] & \mathbb{A} \arrow[d] \\
\ker(\beta_\mathrm{e}) \arrow[r] & \mathrm{H}_1(\Gamma/e, \mathbb{A})_{\eta/e} \arrow[r, "\beta_e"] & \mathbb{A}
\end{tikzcd} \tag{15}$$

We now turn to the graph spaces that, roughly speaking, live above these diagrams. We start by defining a $(\mathbb{G}, \mathbb{A})$ structure on $\mathbf{Z}(\Gamma/e, \eta/e)$ which 'remembers' the contracted edge e.

Definition 4.18. By Proposition 4.7, $\mathbf{Z}(\Gamma/e, \eta/e)$ is a $(\mathrm{H}^1(\Gamma/e, \mathbb{G}), \mathrm{H}_1(\Gamma/e, \mathbb{A})_{\eta/e})$-space. Endow $\mathbf{Z}(\Gamma/e, \eta/e)$ with the $(\mathbb{G}, \mathbb{A})$ structure given by the above cocharacter $\alpha_e \colon \mathbb{G} \to \mathrm{H}^1(\Gamma/e, \mathbb{G})$ and moment map $\beta_e \colon \mathrm{H}_1(\Gamma/e, \mathbb{A})_{\eta/e} \to \mathbb{A}$, as in Lemmas 3.13 and 3.14.

Lemma 4.19. *We have a commutative diagram*

$$\begin{tikzcd}
\mathbf{Z}(\Gamma, \eta) \arrow[r] \arrow[d] & \mathbf{Z}(\Gamma/e, \eta/e) \star_{\eta_e} \mathbf{Z} \arrow[d] \\
\mathrm{H}_1(\Gamma, \mathbb{A})_\eta \arrow[r] & \mathrm{H}_1(\Gamma/e, \mathbb{A})_{\eta/e} \times \mathbb{A}
\end{tikzcd} \tag{16}$$

where the top map is an isomorphism, the right-hand map is the descent of the product map $\mathbf{Z}(\Gamma/e, \eta/e) \times \mathbf{Z} \to \mathrm{H}_1(\Gamma/e, \mathbb{A}) \times \mathbb{A}$ and the bottom map is given by pushforward along $\Gamma \to \Gamma/e$ on the first factor and the coefficient of e on the second factor.

The $(\mathbb{G}, \mathbb{A})$-structure on $\mathbf{Z}(\Gamma/e, \eta/e)$ used in the top right is the one from Definition 4.18.

Proof. We have $\mathbf{Z}(\Gamma,\eta) = \mathbf{Z}^{E(\Gamma)} /\!\!/ \overline{C}^0(\Gamma,\mathbb{G})$ and likewise, $\mathbf{Z}(\Gamma/e,\eta/e) = \mathbf{Z}^{E(\Gamma/e)} /\!\!/_{\eta/e} \overline{C}^0(\Gamma/e,\mathbb{G})$. Thus we can replace the top row by the arrow

$$\mathbf{Z}^{E(\Gamma)} /\!\!/_{\eta} \overline{C}^0(\Gamma,\mathbb{G}) \to \left(\left(\mathbf{Z}^{E(\Gamma/e)} /\!\!/_{\eta/e} \overline{C}^0(\Gamma/e,\mathbb{G})\right) \times \mathbf{Z}\right) /\!\!/_{\eta_e} \mathbb{G}.$$

Our key tool to construct this diagram is Lemma 3.35. To set the ground, we first establish the necessary splittings. We start with the factorisation

$$\mathbf{Z}^{E(\Gamma)} = \mathbf{Z}^{E(\Gamma/e)} \times \mathbf{Z}^{e}. \tag{17}$$

Consider the embedding $\overline{C}^0(\Gamma/e;\mathbb{G}) \to \overline{C}^0(\Gamma,\mathbb{G})$ given by pulling back along $\Gamma \to \Gamma/e$. The image consists of cochains with equal value at $t(e)$ and $h(e)$. The cocharacter $\mathbb{G} \to \overline{C}^0(\Gamma,\mathbb{G})$ determined by the vertex $t(e)$ gives a second subgroup. Together, they define a splitting

$$\overline{C}^0(\Gamma,\mathbb{G}) = \overline{C}^0(\Gamma/e,\mathbb{G}) \times \mathbb{G}. \tag{18}$$

The first factor acts trivially on $\mathbf{Z}^e$.

Dually, pushing forward along $\Gamma \to \Gamma/e$ defines a map $C_0(\Gamma,\mathbb{A}) \to C_0(\Gamma/e,\mathbb{A})$. Taking the coefficient of $t(e)$ defines a second map $C_0(\Gamma,\mathbb{A}) \to \mathbb{A}$. Together they define a splitting

$$C_0(\Gamma,\mathbb{A}) = C_0(\Gamma/e,\mathbb{A}) \times \mathbb{A} \tag{19}$$

which identifies η with $(\eta/e,\eta_e)$. The map $\mathbf{Z}^{E(\Gamma)} \to C_0(\Gamma/e,\mathbb{A})$ factors through $\mathbf{Z}^{E(\Gamma)} \to \mathbf{Z}^{E(\Gamma/e)}$.

We can now apply Lemma 3.35, using the splittings 17, 18, 19.

□

Let $\mathbf{0}$ be the point with the trivial $(\mathbb{G},\mathbb{A})$-structure.

Lemma 4.20. *Suppose $\Gamma \setminus e$ is connected. Then we have the commutative diagram*

$$\begin{tikzcd} \mathbf{Z}(\Gamma\setminus e,\eta\setminus e) \arrow[r] \arrow[d] & \mathbf{Z}(\Gamma/e,\eta/e) \star_{\eta_e} \mathbf{0} \arrow[d] \\ \mathrm{H}_1(\Gamma\setminus e,\mathbb{A})_{\eta\setminus e} \arrow[r] & \mathrm{H}_1(\Gamma/e,\mathbb{A})_{\eta/e} \times \mathbb{A} \end{tikzcd} \tag{20}$$

where the top row an isomorphism and the bottom row is given by the top left map of Diagram 15 on the first factor, and the zero map on the second factor.

Proof. The top left hand side is by definition $\mathbf{Z}^{E(\Gamma\setminus e)} /\!\!/_{\eta\setminus e} \overline{C}^0(\Gamma\setminus e,\mathbb{G})$. By Example 3.22, the top right hand side can also be written as $\mathbf{Z}(\Gamma/e,\eta/e) /\!\!/_{\eta_e} \mathbb{G}$, or equivalently $(\mathbf{Z}^{E(\Gamma/e)} /\!\!/_{\eta/e} \overline{C}^0(\Gamma/e,\mathbb{G})) /\!\!/_{\eta_e} \mathbb{G}$. Thus we can replace the desired top row by the arrow

$$\mathbf{Z}^{E(\Gamma\setminus e)} /\!\!/_{\eta\setminus e} \overline{C}^0(\Gamma\setminus e,\mathbb{G}) \to (\mathbf{Z}^{E(\Gamma/e)} /\!\!/_{\eta/e} \overline{C}^0(\Gamma/e,\mathbb{G})) /\!\!/_{\eta_e} \mathbb{G}.$$

We get Γ/e from $\Gamma\setminus e$ by identifying the vertices $t(e)$ and $h(e)$. Combined with the splittings 18 and 19, this gives identifications $E(\Gamma/e) = E(\Gamma\setminus e)$ and thus $\mathbf{Z}^{E(\Gamma/e)} = \mathbf{Z}^{E(\Gamma\setminus e)}$.

Moreover, $\overline{C}^0(\Gamma \setminus e, \mathbb{G}) = \overline{C}^0(\Gamma, \mathbb{G}) = \overline{C}^0(\Gamma/e, \mathbb{G}) \times \mathbb{G}$ and $C_0(\Gamma \setminus e, \mathbb{A}) = C_0(\Gamma, \mathbb{A}) = C_0(\Gamma/e, \mathbb{A}) \times \mathbb{A}$, where we have again used the splittings 18 and 19. Making these substitutions, the desired top row becomes

$$\mathbf{Z}^{E(\Gamma/e)} /\!\!/_{(\eta/e,\eta_e)} \overline{C}^0(\Gamma/e, \mathbb{G}) \times \mathbb{G} \to (\mathbf{Z}^{E(\Gamma\setminus e)} /\!\!/_{\eta/e} \overline{C}^0(\Gamma/e, \mathbb{G})) /\!\!/_{\eta_e} \mathbb{G}.$$

As in Lemma 4.19, we now obtain the requisite diagram by applying Lemma 3.35. □

Note that $\mathbf{Z}(\Gamma/e, \eta/e) \star_{\eta_e} \mathbf{0}$ is defined also when $\Gamma \setminus e$ is disconnected. We now turn to that case.

Lemma 4.21. *Let η be generic, and let e be a bridge, i.e. suppose $\Gamma \setminus e$ is disconnected. Then $\mathrm{H}_1(\Gamma \setminus e, \mathbb{A})_{\eta\setminus e}$ is empty.*

Proof. By genericity of η, $\Gamma \setminus S(a)$ is connected for all $a \in \mathrm{H}_1(\Gamma, \mathbb{A})_\eta$. On the other hand, since e is a bridge, $\Gamma \setminus S(a)$ is disconnected for all a in the subset $\mathrm{H}_1(\Gamma \setminus e, \mathbb{A})_{\eta\setminus e}$. Thus this subset must be empty. □

Corollary 4.22. *Suppose $\Gamma \setminus e$ is disconnected and η is generic. Then*

$$\mathbf{Z}(\Gamma/e, \eta/e) \star_{\eta_e} \mathbf{0} = \emptyset.$$

The same holds when $\mathbf{0}$ is replaced by any $(\mathbb{G}, \mathbb{A})$-space $\mathbf{S}$ with $\mu(\mathbf{S}) = 1$.

Proof. By construction, the left-hand side is a quotient of a preimage of $\mathrm{H}_1(\Gamma \setminus e, \mathbb{A})_{\eta\setminus e}$, which is empty by Lemma 4.21. □

Remark 4.23. Comparing with Lemma 4.20, we may extend the definion of $\mathbf{Z}(\Gamma, \eta)$ to disconnected Γ (and generic η) by setting $\mathbf{Z}(\Gamma, \eta) = \emptyset$.

Remark 4.24. Our spaces $\mathfrak{B}$ and $\mathfrak{D}$ each contain a canonical fixed point under the group action. This will allow us to connect the deletion and contraction relations above.

We often abbreviate $\mathbf{Z}(\Gamma) := \mathbf{Z}(\Gamma, \eta)$ and likewise $\mathbf{Z}(\Gamma/e) := \mathbf{Z}(\Gamma/e, \eta/e)$ and $\mathbf{Z}(\Gamma \setminus e) := \mathbf{Z}(\Gamma \setminus e, \eta \setminus e)$.

5. $\mathfrak{B}(\Gamma)$

5.1. **Construction.**

Definition 5.1. We define the space

$$\mathfrak{B} := \mathbb{C}^2 \setminus \{1 + xy = 0\}.$$

and the maps

$$\begin{array}{rcl} \mu_{\mathfrak{B}}^{\mathbb{C}^*} : \mathfrak{B} & \to & \mathbb{C}^* \\ (x,y) & \mapsto & 1+xy \\ \mu_{\mathfrak{B}}^{\mathbb{R}} : \mathfrak{B} & \to & \mathbb{R} \\ (x,y) & \mapsto & |x|^2 - |y|^2 \end{array}$$

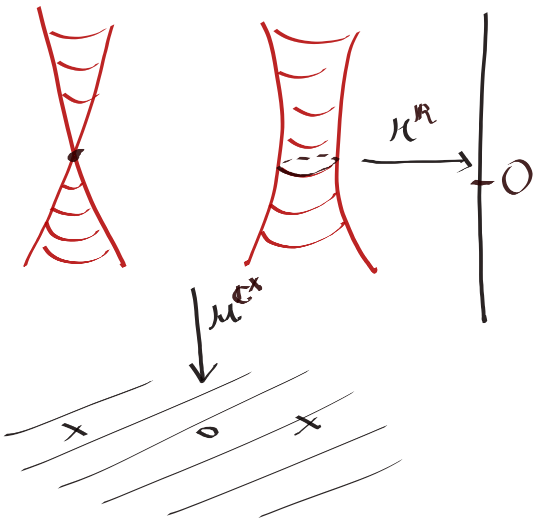


FIGURE 3. A schematic picture of $\mathfrak{B}$ and its various moment maps and their targets. Two fibers of $\mu^{\mathbb{C}^*}$ are shown, and the intersection of each fiber with $(\mu^{\mathbb{R}})^{-1}(0)$ is indicated in black.

Lemma 5.2. *The map $\mu_{\mathfrak{B}}^{\mathbb{C}^*}$ is invariant under the $\mathbb{C}^*$ action on $\mathfrak{B}$ given by $\tau \cdot (x,y) = (\tau x, \tau^{-1} y)$.*

Lemma 5.3. *The map $\mu_{\mathfrak{B}}^{\mathbb{R}}$ is invariant under the $\mathbb{U}_1 \subset \mathbb{C}^*$ action.*

Lemma 5.4. *The fibers of $\mu_{\mathfrak{B}}^{\mathbb{R}} \times \mu_{\mathfrak{B}}^{\mathbb{C}^*}$ over the complement of 0×1 are a free $\mathbb{U}_1$-orbits, thus defining a principle $\mathbb{U}_1$-bundle $\mathcal{P}_{\mathfrak{B}}$ over $\mathbb{R} \times \mathbb{C}^* \setminus 0 \times 1$.*

Lemma 5.5. *Let $\omega = Im(dx \wedge dy)$. The action of $\mathbb{U}_1 \subset \mathbb{C}^*$ preserves ω.*

Proposition 5.6. *The space $\mathfrak{B} = \mathbb{C}^2 \setminus \{xy + 1 = 0\}$ has the following properties:*

- *$\mathfrak{B}$ is a smooth algebraic $(\mathbb{G}_m, \mathbb{G}_m)$-variety, with action $(x,y) \to (\tau x, \tau^{-1} y)$ and map $\mu_{\mathfrak{B}}^{\mathbb{C}^*}(x,y) = (1+xy)$.*
- *It has a single $\mathbb{G}_m$-fixed point at $(0,0)$. Every other point has trivial stabilizer.*
- *The inclusion $\mathbf{0}_{(\mathbb{G}_m,\mathbb{G}_m)} \to (0,0) \in \mathfrak{B}$ is a morphism of $(\mathbb{G}_m, \mathbb{G}_m)$ varieties.*
- *The attracting cell at this fixed point is $\mathbf{S}_{\mathfrak{B}} := \{x = 0\}$.*
- *Note that $\mathbf{S}_{\mathfrak{B}} \cong \mathbb{A}^1$ with its natural $\mathbb{G}_m$-action.*

- *The map*

$$\begin{array}{rcl} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} & \to & [\mathbb{G}_m \times \mathbb{G}_m] \\ (x,y) & \mapsto & (x, xy+1) \end{array}$$

  is an isomorphism of $(\mathbb{G}_m, \mathbb{G}_m)$*-spaces.*
- *As a* $(\mathbb{G}_m, \mathbb{G}_m)$*-space,* $\mathfrak{B}$ *satisfies Hypothesis 4.11.*

By the last of the above properties, we can speak of $\mathfrak{B}$-generic parameters $\eta \in C_0(\Gamma, \mathbb{U}_1) \subset C_0(\Gamma, \mathbb{C}^*)$ in the sense of Definition 4.12.

Definition 5.7. Given a graph Γ and a $\mathfrak{B}$-generic η, we abbreviate

$$\mathfrak{B}(\Gamma) := \mathfrak{B}^{(\mathbb{C}^*, \mathbb{C}^*)}(\Gamma, \eta).$$

We will suppress the dependence on η for most of this paper. In coordinates, it is described as follows. Recall that $\mathfrak{B}^{E(\Gamma)}$ has coordinates x_e, y_e for $e \in E(\Gamma)$. The subset $\mu_\Gamma^{-1}(\eta) \subset \mathfrak{B}^{E(\Gamma)}$ is defined by

$$\prod_{\text{edges exiting } v} (1 + x_e y_e) \prod_{\text{edges entering } v} (1 + x_e y_e)^{-1} = \eta_v$$

for each $v \in V(\Gamma)$. Then

$$\mathfrak{B}(\Gamma) = \mu_\Gamma^{-1}(\eta) / \overline{C}^0(\Gamma, \mathbb{C}^*)$$

where the factor of $\mathbb{C}^*$ attached to v acts by $\tau x_e, \tau^{-1} y_e$ on incoming edges, and $\tau^{-1} x_e, \tau y_e$ on outgoing edges. Hence $\mathfrak{B}(\Gamma, \eta)$ is a smooth complex affine variety.

5.2. **Contraction of bridges.**

Lemma 5.8. *Let* e *be a bridge of* Γ*. Then we have a canonical isomorphism* $\mathfrak{B}(\Gamma) = \mathfrak{B}(\Gamma/e)$*.*

Proof. By Lemma 4.19, we can write $\mathfrak{B}(\Gamma) = \mathfrak{B}(\Gamma/e) \star_{\mathbb{C}^*} \mathfrak{B}$. On the other hand, $\mathfrak{B}$ is the disjoint union of $\mathbf{S}_{\mathfrak{B}}$ and $[\mathbb{C}^* \times \mathbb{C}^*]$ as explained above.

Thus $\mathfrak{B}(\Gamma/e) \star_{\mathbb{C}^*} \mathfrak{B}$ is the disjoint union of $\mathfrak{B}(\Gamma/e) \star_{\mathbb{C}^*} \mathbf{S}_{\mathfrak{B}}$ and $\mathfrak{B}(\Gamma/e) \star_{\mathbb{C}^*} [\mathbb{C}^* \times \mathbb{C}^*]$ by Lemma 3.25. The former is the empty set by Corollary 4.22. The latter equals $\mathfrak{B}(\Gamma/e)$ by Lemma 3.29. □

5.3. **Independence of orientation.** For this paper, we will work with a fixed orientation of Γ. However, the dependence on the chosen orientation is quite mild, as shown by the following.

Proposition 5.9. *If* Γ, Γ' *differ only by the choice of orientation, then there is a canonical isomorphism*

$$\mathfrak{B}(\Gamma) \to \mathfrak{B}(\Gamma)'$$

Proof. By Proposition 4.10, it is enough to find an automorphism $\mathfrak{B} \to \mathfrak{B}$ intertwining the $\mathbb{C}^*$-action and the $\mathbb{C}^*$-moment map with their inverses. This is given by $(x,y) \to (y, (1+xy)^{-1}x)$. □

Remark 5.10. This is an especially simple case of the proof of independence-of-orientation for multiplicative quiver varieties in [CBS].

5.4. **Deletion-contraction sequence.** From Proposition 5.6, there are natural inclusions and projections of $(\mathbb{G}_m, \mathbb{G}_m)$ spaces as follows:

$$\mathbf{0} \xleftarrow{\pi} \mathbf{S}_{\mathfrak{B}} \xrightarrow{I} \mathfrak{B} \xleftarrow{J} \mathbb{G}_m \times \mathbb{G}_m \tag{21}$$

As I, J give a decomposition of $\mathfrak{B}$ into closed and open subsets, we have the following exact triangle in $D^b(\mathfrak{B})$:

$$I_! I^! \mathbb{Q} \to \mathbb{Q} \to J_* J^* \mathbb{Q} \xrightarrow{[1]} \tag{22}$$

Because I is the complex codimension one closed inclusion of one smooth variety in another,

$$I_! I^! \mathbb{Q} = \mathbb{Q}_{\mathbf{S}_{\mathfrak{B}}}[-2](-1).$$

Proposition 5.11. $\mathrm{H}^\bullet(\mathfrak{B}, \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}[-1](-1) \oplus \mathbb{Q}[-2](-2)$.

Proof. Taking sections of the triangle 22 returns the excision sequence in cohomology. Its terms are as follows:

$$\mathrm{H}^\bullet(\mathfrak{B}; I_! I^! \mathbb{Q}) = \mathrm{H}^\bullet(\mathfrak{B}, \mathbb{Q}_{\mathbf{S}_{\mathfrak{B}}}[-2](-1)) = \mathrm{H}^\bullet(\mathbf{S}_{\mathfrak{B}}, \mathbb{Q})[-2](-1) = \mathbb{Q}[-2](-1)$$

$$\mathrm{H}^\bullet(\mathfrak{B}; J_* J^* \mathbb{Q}) = \mathrm{H}^\bullet(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{Q}) = (\mathbb{Q} \oplus \mathbb{Q}[-1](-1))^{\otimes 2} = \mathbb{Q} \oplus \mathbb{Q}^{\oplus 2}[-1](-1) \oplus \mathbb{Q}[-2](-2)$$

The connecting map which has a chance to be nonzero, namely $\mathrm{H}^1(\mathfrak{B}; J_* J^* \mathbb{Q}) \to \mathrm{H}^2(\mathfrak{B}; I_! I^! \mathbb{Q})$ in fact must be so: $\mathfrak{B} \sim \mathfrak{D}$ which retracts to a nodal genus one curve; thus we know the nonzero Betti numbers are $b^0 = b^1 = b^2 = 1$. ☐

Fix a smooth irreducible algebraic $(\mathbb{G}_m, \mathbb{G}_m)$-space $\mathbf{Y}$ on which $\mu_{\mathbf{Y}}$ is nonconstant, and $\zeta \in \mathbb{G}_m$ such that $\mathbb{G}_m$ acts freely on $\mu^{-1}_{\mathbf{Y} \bullet \mathfrak{B}}(\zeta) \subset \mathbf{Y} \bullet \mathfrak{B}$.

Taking $\star_{\mathbb{G}_m} := \star_{\mathbb{G}_m, \mathbb{G}_m, \zeta}$ with Equation 21 induces morphisms:

$$\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{0} \xleftarrow{\pi_{\mathbf{Y}}} \mathbf{Y} \star_{\mathbb{G}_m} \mathbf{S} \xrightarrow{I_{\mathbf{Y}}} \mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B} \xleftarrow{J_{\mathbf{Y}}} \mathbf{Y} \star_{\mathbb{G}_m} [\mathbb{G}_m \times \mathbb{G}_m]$$

Lemma 5.12. *$I_{\mathbf{Y}}$ (resp $J_{\mathbf{Y}}$) is the inclusion of a smooth divisor (resp its complement).*

Proof. By Lemma 3.25, $J_{\mathbf{Y}}$ is the complement of $I_{\mathbf{Y}}$. Let us see that $I_{\mathbf{Y}}$ is a smooth divisor. Consider the divisor $\mathbf{Y} \times \mathbf{S}_{\mathfrak{B}} \subset \mathbf{Y} \times \mathfrak{B}$. Since the function $\mu_{\mathfrak{D}}$ is nonconstant on $\mathbf{Y}$, the function $\mu_{\mathbf{Y} \bullet \mathbf{Z}}$ is nonconstant on $\mathbf{Y} \times \mathbf{S}_{\mathfrak{B}}$ and $\mathbf{Y} \times \mathfrak{B}$. Thus $\mu^{-1}_{\mathbf{Y} \bullet \mathbf{Z}}(\zeta) \cap (\mathbf{Y} \times \mathbf{S}_{\mathfrak{B}}) \subset \mu^{-1}_{\mathbf{Y} \bullet \mathbf{Z}}(\zeta)$ is a divisor. Passing to the quotient by the (free) $\mathbb{G}$ action, $\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{S}_{\mathfrak{B}}$ is a divisor in $\mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B}$. ☐

We thus obtain, as in 22, the triangle

$$\mathbb{Q}_{\mathbf{Y}\star\mathbf{S}_{\mathfrak{B}}}[-2](-1) \to \mathbb{Q} \to (J_{\mathbf{Y}})_* J_{\mathbf{Y}}^* \mathbb{Q} \xrightarrow{[1]} \tag{23}$$

Lemma 5.13. *The map $\pi_{\mathbf{Y}}$ has fiber $\mathbb{A}^1$ and a section induced from the inclusion $\mathbf{0} \to \mathbf{S}_{\mathfrak{B}}$.*

Proof. We have $\mu_{\mathfrak{D}}(\mathbf{S}_{\mathfrak{B}}) = 0$. Hence $\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{S}_{\mathfrak{B}} = \big(\mu_{\mathbf{Y}}^{-1}(\zeta) \times \mathbf{S}_{\mathfrak{B}}\big)/\mathbb{G}_m$. By assumption, $\mathbb{G}_m$ acts freely on $\mu_{\mathbf{Y}}^{-1}(\zeta)$. Hence the quotient is a bundle over $\mu_{\mathbf{Y}}^{-1}(\zeta)$ with fiber isomorphic to $\mathbf{S}_{\mathfrak{B}} \cong \mathbb{A}^1$. □

In particular, either push forward along $\pi_{\mathbf{Y}}$ or pullback along the section induces:

$$\mathrm{H}^\bullet(\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{S}, \mathbb{Q}) \cong \mathrm{H}^\bullet(\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{0}, \mathbb{Q}) \cong \mathrm{H}^\bullet(\mathbf{Y} /\!\!/_\zeta \mathbb{G}, \mathbb{Q}). \tag{24}$$

We have $\mathbf{Y} \star_{\mathbb{G}_m} [\mathbb{G}_m \times \mathbb{G}_m] = \mathbf{Y}$ by Lemma 3.29 and and $\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{0} = \mathbf{Y} /\!\!/_\zeta \mathbb{G}_m$ by Lemma 3.22.

Remark 5.14. In particular, in the Grothendieck ring of varieties, $[\mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B}] = [\mathbf{Y}] + [\mathbb{A}^1][\mathbf{Y} /\!\!/_\zeta \mathbb{G}_m]$. Taking $\mathbf{Y} = \mathfrak{B}(\Gamma/e)$ gives $[\mathfrak{B}(\Gamma)] = [\mathfrak{B}(\Gamma/e)] + [\mathbb{A}^1][\mathfrak{B}(\Gamma \setminus e)]$.

Taking cohomology of the triangle 23 and combining with the above isomorphisms, we obtain the diagram:

$$\begin{tikzcd}[column sep=small]
& \mathrm{H}^{\bullet-2}(\mathbf{Y} /\!\!/_\zeta \mathbb{G}_m, \mathbb{Q})(-1) \arrow[dr, dashed] & & \mathrm{H}^\bullet(\mathbf{Y}, \mathbb{Q}) \arrow[dr, dashed] & \\
\cdots \arrow[r] \arrow[ur, dashed] & \mathrm{H}^{\bullet-2}(\mathbf{Y} \star_{\mathbb{G}_m} \mathbf{S}, \mathbb{Q})(-1) \arrow[r] \arrow[u, "(\mathbf{0}\to\mathbf{S})^*"] & \mathrm{H}^\bullet(\mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B}, \mathbb{Q}) \arrow[r] \arrow[ur, dashed] & \mathrm{H}^\bullet(\mathbf{Y} \star_{\mathbb{G}_m} [\mathbb{G}_m \times \mathbb{G}_m], \mathbb{Q}) \arrow[r] \arrow[u, "="] & \cdots
\end{tikzcd}$$

We call the dashed long exact sequence the $\mathfrak{B}$ deletion-contraction sequence of Y. The terminology is motivated by the following special case:

Theorem 5.15. *Let Γ be a graph, e a non-loop non-bridge edge, and η chosen such that $\mathfrak{B}(\Gamma)$ is smooth. Then there is a long exact sequence*

$$\to \mathrm{H}^{\bullet-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q})(-1) \xrightarrow{a^{\mathfrak{B}}} \mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q}) \xrightarrow{b^{\mathfrak{B}}} \mathrm{H}^\bullet(\mathfrak{B}(\Gamma/e), \mathbb{Q}) \xrightarrow{c^{\mathfrak{B}}[+1]} \tag{25}$$

Moreover, the maps strictly preserve the weight filtration on each space (taking into account the Tate twist on the left-hand term).

Note that by Lemma 5.8, we can extend this statement to the case where e is a bridge, by simply defining $\mathfrak{B}(\Gamma \setminus e, \eta \setminus e) = \emptyset$.

Proof. Apply the $\mathfrak{B}$ deletion-contraction sequence to $\mathbf{Y} = \mathfrak{B}(\Gamma/e)$ and $\zeta = \eta_e$ as specified in Section 4.4. Lemmas 4.19 and 4.20 give the desired identifications of the convolutions in the sequence with the stated spaces. The statement on weight filtrations holds for the long exact sequence of any pair of smooth algebraic varieties. □

When we wish to highlight which edge of Γ is in play, we may write $a_e^{\mathfrak{B}}$ and $b_e^{\mathfrak{B}}$.

5.5. **Strata and charts.** We define some subsets of $\mathfrak{B}(\Gamma)$ with a view to applying the results of Appendix A.2. Inside $\mathfrak{B}(\Gamma)$, we have divisors $D_e, e \in E(\Gamma)$ cut out by $x_e = 0$. Their union has simple normal crossings; we denote it $D(\Gamma)$. For $J \subset E(\Gamma)$, we write $D_J = \bigcap_{j \in J} D_j$ and $D^{J^c} = \bigcup_{j \notin J} D_j$. Always $\mathbf{K}_J := D^{J_c} \cap D_J$ is a snc divisor in D_J, with complement $\mathbf{U}_J$.

We will also consider the following variant, in which the role of certain coordinates x_e is played instead by y_e.

Definition 5.16. Let $J \subset E(\Gamma)$ and fix $e' \notin J$. We define $\mathbf{K}'_J$ to be the union of divisors in D_J given by $D_e \cap D_J$, $e \notin J, e \neq e'$ and $\{y_{e'} = 0\} \cap D_J$. We define $\mathbf{U}'_J := D_J \setminus \mathbf{K}'_J$.

Thus $\mathbf{U}_J \cap \mathbf{U}'_J$ is the locus in D_J where neither $x_{e'}$ nor $y_{e'}$ vanishes. On the other hand, $\mathbf{U}_J \cup \mathbf{U}'_J$ is the locus where $x_{e'}$ and $y_{e'}$ are not both zero; its complement in D_J has codimension two. This explains the usefulness of considering $\mathbf{U}'_J$, as together with $\mathbf{U}_J$ it provides a cover of D_J up to codimension two. For instance, let $J' = J \cup e'$. Consider the divisor $D_{J'} \subset D_J$, which lies in the complement of the open set $\mathbf{U}_J \subset D_J$. We have:

Lemma 5.17. *$\mathbf{U}'_J \cap D_{J'}$ is open and dense in $D_{J'}$.*

Proof. $D_{J'}$ has codimension one in D_J, and lies in the complement of $\mathbf{U}_J$. Since $\mathbf{U}_J \cup \mathbf{U}'_J$ has codimension two, the result follows. $\square$

We may summarize the relations between these spaces in the following diagram, indicating which maps are open and dense embeddings, and which are closed embeddings of positive codimension.

$$\begin{tikzcd}
\mathbf{U}_J \arrow[r, "open"] & D_J & \mathbf{U}'_J \arrow[l, "open"'] \\
 & D_{J'} \arrow[u, "closed"] & D_{J'} \cap \mathbf{U}'_J \arrow[l, "open"'] \arrow[u, "closed"]
\end{tikzcd} \tag{26}$$

Our eventual goal in this section is to compute the cohomology of $\mathfrak{B}(\Gamma)$ by applying the techniques of Section A.2 to the above strata. This will involve sheaves of differential forms. As the space $\mathfrak{B}(\Gamma)$ is affine, we will everywhere take termwise global sections in complexes of differential forms, and discuss the resulting complexes of sections rather than complexes of sheaves.

We now outline the remainder of the argument. We first show that $\mathbf{U}_J$ and $\mathbf{U}'_J$ are vector bundles over the torus $\mathrm{H}^1(\Gamma \setminus J; \mathbb{C}^*) \times \mathrm{H}_1(\Gamma \setminus J; \mathbb{C}^*)$ (Lemma 5.18, Lemma 5.19 and Corollary 5.20). We will use this presentation of $\mathbf{U}_J$ to define a subspace of closed forms on $\mathbf{U}_J$, forming a basis of its de Rahm cohomology (Definition 5.25 and Lemma 5.28) and isomorphic to a summand of the combinatorial complex $\Upsilon^\bullet$ from Section 2. By rewriting these forms as meromorphic differentials in the toric coordinates on $\mathbf{U}'_J$ and using Diagram 26 to pass from $\mathbf{U}'_J$ to $D_{J'}$, we compute their residue along $D_{J'}$ (Proposition 5.29). We conclude that the inclusion intertwines the combinatorial differential on $\Upsilon^\bullet$ with $d_{dR} + d_{\mathrm{res}}$. We then show that the resulting map $\Upsilon^\bullet \to \Omega_{\mathfrak{B}(\Gamma), D(\Gamma)}$ is a quasi-isomorphism (Theorem 5.30).

Lemma 5.18. *There is a map $\pi_J \colon D_J \to \mathfrak{B}(\Gamma \setminus J)$ expressing D_J as a rank $|J|$ vector bundle over $\mathfrak{B}(\Gamma \setminus J)$.*

Proof. Consider the basic space $\mathfrak{B}$. We have a diagram

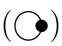
$$\begin{tikzcd} \mathfrak{B} & \{x=0\} \arrow[l] \arrow[d, "\pi"] \\ & \mathbf{n} = \{\mathbf{x} = \mathbf{y} = \mathbf{0}\} \end{tikzcd}$$

The map π makes $\{x = 0\}$ a $\mathbb{C}^*$-equivariant rank one vector bundle over $\mathbf{n}$, trivialized by the function y. Note also that the moment map $(1 + xy)$ has constant value 1 on this locus.

D_J is by construction a quotient of $\Big(\prod_{e\in J}\{x_e = 0\} \times \prod_{e\notin J} \mathfrak{B}\Big) \bigcap \mu^{-1}(\eta)$ by $\overline{C}^0(\Gamma, \mathbb{C}^*)$. The maps $\{x_e = 0\} \to \mathbf{n}$ for $e \in J$ combine to give a map

$$\Big(\prod_{e\in J}\{x_e = 0\} \times \prod_{e\notin J} \mathfrak{B}\Big) \bigcap \mu^{-1}(\eta) \to \Big(\prod_{e\in J}\mathbf{n} \times \prod_{e\notin J} \mathfrak{B}\Big) \bigcap \mu^{-1}(\eta).$$

This is a $C^1(\Gamma, \mathbb{C}^*)$-equivariant vector bundle (if we ignore the equivariant structure, it is a trivial vector bundle) over the target with fiber $\mathbb{C}^J$. Taking the quotient by $\overline{C}^0(\Gamma, \mathbb{C}^*)$ defines a $\mathrm{H}^1(\Gamma \setminus J, \mathbb{C}^*)$-equivariant vector bundle $\pi \colon D_J \to \mathfrak{B}(\Gamma \setminus J)$. □

Recall that for the basic space $\mathfrak{B} = \mathfrak{B}($$)$ where  is the graph composed of a single loop e, we have $\mathfrak{B} \setminus D_e \cong [\mathbb{C}^* \times \mathbb{C}^*]$. More generally, we have the following special case of Lemma 4.9.

Lemma 5.19.
$$\mathfrak{B}(\Gamma) \setminus D(\Gamma) \cong [\mathbb{C}^* \times \mathbb{C}^*](\Gamma, \eta) \cong \mathrm{H}^1(\Gamma, \mathbb{C}^*) \times \mathrm{H}_1(\Gamma, \mathbb{C}^*),$$
where the middle term is the graph space associated to $[\mathbb{C}^* \times \mathbb{C}^*]$, in the notation of Definition 4.2.

More precisely, recall the coordinates x_e, y_e on the prequotient $\mathfrak{B}^{E(\Gamma)}$. Choose a set of edges $\{e\} \subset E(\Gamma)$ and cycles $\{\gamma\} \subset \mathrm{H}_1(\Gamma, \mathbb{Z})$ which define bases of $\mathrm{H}^1(\Gamma, \mathbb{Z})$ and $\mathrm{H}_1(\Gamma, \mathbb{Z})$ respectively. This identifies $\mathrm{H}^1(\Gamma, \mathbb{C}^*) \times \mathrm{H}_1(\Gamma, \mathbb{C}^*)$ with $(\mathbb{C}^*)^{h^1(\Gamma)} \times (\mathbb{C}^*)^{h_1(\Gamma)}$. Then our isomorphism is realized by the $\mathbb{C}^*$-valued coordinates $x_\gamma = \prod_{e\in\Gamma} x_e^{\langle e,\gamma\rangle}$ and $z_e = x_e y_e + 1$.

Similarly, recall that $D(\Gamma)'$ is the union of $\{y_{e'} = 0\}$ with $D_e, e \neq e'$. Then by the same argument as above, we have an isomorphism

$$\mathfrak{B}(\Gamma) \setminus D(\Gamma)' \cong \mathrm{H}^1(\Gamma, \mathbb{C}^*) \times \mathrm{H}_1(\Gamma, \mathbb{C}^*),$$

with coordinates $z'_e := z_e$ and $x'_\gamma := y_{e'}^{-\langle e',\gamma\rangle} \prod_{e\neq e'} x_e^{\langle e,\gamma\rangle}$.

Corollary 5.20. *The restriction of π_J to $\mathbf{U}_J = D_J \setminus \mathbf{K}_J$ defines a $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$-equivariant vector bundle over $\mathrm{H}_1(\Gamma \setminus J, \mathbb{C}^*) \times \mathrm{H}^1(\Gamma \setminus J, \mathbb{C}^*)$.*

Proof. Combine Lemma 5.18 with Lemma 5.19 (substituting $\Gamma \setminus J$ for Γ in the latter). □

From now on, we fix a set of edges $\{e\}$ and cycles $\{\gamma\}$ forming bases of $\mathrm{H}_1(\Gamma\setminus J,\mathbb{Z})$ and $\mathrm{H}^1(\Gamma\setminus J,\mathbb{Z})$. As a result of the above construction, we have holomorphic functions x_γ and z_e on $\mathbf{U}_J$, which define coordinates on the base of the vector bundle $\pi_J\colon \mathbf{U}_J\to \mathrm{H}_1(\Gamma\setminus J,\mathbb{C}^*)\times\mathrm{H}^1(\Gamma\setminus J,\mathbb{C}^*)$. These functions arise from $\overline{C}^0(\Gamma,\mathbb{C}^*)$-invariant rational functions on $\{x_e=0\colon e\in J\}\subset\mathfrak{B}^{E(\Gamma)}$, and thus extend to rational functions on D_J.

In particular, we have the following :

Lemma 5.21. *Let $J'=J\cup e'$, and let $\gamma\in\mathrm{H}_1(\Gamma\setminus J',\mathbb{Z})$. Consider the homonymous coordinates x_γ on $\mathbf{U}_{J'}$ and x_γ on $\mathbf{U}_J$. These are both restrictions of the same rational function on D_J. Likewise, any edge e defines a holomorphic function on D_J which restricts to the functions named z_e on each stratum $\mathbf{U}_J$, $\mathbf{U}_{J'}$.*

We now describe the coordinate transform between $\mathbf{U}_J$ and $\mathbf{U}'_J$. Recall that on the basic space, we have $x_{e'}y_{e'}+1=z_{e'}$. Thus for $\gamma\in\mathrm{H}_1(\Gamma\setminus J,\mathbb{Z})$ we have

$$x_\gamma=\prod_{e\in\Gamma}x_e^{\langle e,\gamma\rangle}=\left(\frac{z_{e'}-1}{y_{e'}}\right)^{\langle e,\gamma\rangle}\prod_{e\neq e'}x_e^{\langle e,\gamma\rangle}.$$

Corollary 5.22. *The coordinates x_γ, z_e and x'_γ, z'_e on the two charts are related by*

$$x_\gamma=(z_{e'}-1)^{\langle e',\gamma\rangle}x'_\gamma,\quad z_e=z'_e. \tag{27}$$

Lemma 5.23. *Let $J\subset E(\Gamma)$ and let $e'\notin J$. The divisor $\{x_{e'}=0\}$ in $\mathbf{U}'_J$ is cut out by the function $z'_{e'}-1$.*

Proof. This is a direct consequence of the equation $x_{e'}y_{e'}+1=z_{e'}$ on the basic space. ☐

5.6. **Differential forms and $\Upsilon(\Gamma)$.** Recall from Appendix A.2 that given a space X with a normal crossings divisor $D=\bigcup D_i$, the cohomology of the space can be computed by a certain complex $\Omega^\bullet_{X,D}$, whose underlying vector space is $\bigoplus\Omega_{D_I}\langle D^{I^c}\cap D_I\rangle$, where $D_I=\cap_{i\in I}D_i$ and $D^{I^c}=\cup_{i\notin I}D_i$. In this section we study this construction for $\mathfrak{B}(\Gamma)$, using the stratifications of the previous section.

We will need to construct some differential forms on $\mathbf{U}_J$, extend them to meromorphic forms on $\mathbf{U}'_J$, and compute their residues along $x_{e'}=0$. We begin with the following general construction.

Lemma 5.24. *Let V be a complex vector space with lattice $V_\mathbb{Z}$ and dual V^*. Then any $w\in V^*$ defines a 1-form $dw\in\Omega^1(V)$, which descends to a one form (which we denote by the same symbol dw) on the torus $V/V_\mathbb{Z}$. This defines a linear map $V^*\to\Omega^1(V/V_\mathbb{Z})$, which induces isomorphisms $V^*\cong\mathrm{H}^1(V/V_\mathbb{Z},\mathbb{C})$ and $V^*_\mathbb{Z}\cong\mathrm{H}^1(V/V_\mathbb{Z},\mathbb{Z})$.*

Definition 5.25. We apply the above construction in the following special case. The space $V=\mathrm{H}_1(\Gamma\setminus J,\mathbb{C})\oplus\mathrm{H}^1(\Gamma\setminus J,\mathbb{C})$ is self dual, with lattice $V_\mathbb{Z}:=\mathrm{H}_1(\Gamma\setminus J,\mathbb{Z})\oplus\mathrm{H}^1(\Gamma\setminus J,\mathbb{Z})$ and quotient

$\mathrm{H}_1(\Gamma \setminus J, \mathbb{C}^*) \times \mathrm{H}^1(\Gamma \setminus J, \mathbb{C}^*)$. Applying the lemma, we obtain a map

$$\mathbb{H}(\Gamma \setminus J, \mathbb{C}) := \mathrm{H}_1(\Gamma \setminus J, \mathbb{C}) \oplus \mathrm{H}^1(\Gamma \setminus J, \mathbb{C}) \to \Omega^1(\mathrm{H}_1(\Gamma \setminus J, \mathbb{C}^*) \times \mathrm{H}^1(\Gamma \setminus J, \mathbb{C}^*)).$$

Composing with the pullback along the vector bundle $\pi_J \colon \mathbf{U}_J \to \mathrm{H}_1(\Gamma \setminus J, \mathbb{C}^*) \times \mathrm{H}^1(\Gamma \setminus J, \mathbb{C}^*)$, we obtain a map

$$\mathfrak{d}_J \colon \mathbb{H}(\Gamma \setminus J, \mathbb{C}) \to \Omega^1_{D_J}\langle \mathbf{K}_J \rangle$$

Lemma 5.26. *$\mathfrak{d}_J$ induces an isomorphism with degree one cohomology.*

Proof. $\Omega^1_{D_J}\langle \mathbf{K}_J \rangle$ computes the cohomology of $D_J \setminus \mathbf{K}_J = \mathbf{U}_J$, which by Lemma 5.20 is a vector bundle over a torus. Thus it is enough to check that $\mathfrak{d}$ induces an isomorphism after composition with restriction to the torus. The result follows from Lemma 5.24 applied to this torus. □

In the coordinates described above, we have

$$2\pi i \mathfrak{d}_J(e \oplus \gamma) = \frac{dx_\gamma}{x_\gamma} + \frac{dz_e}{z_e}.$$

Since these coordinates are rational functions on D_J, $\mathfrak{d}_J(e \oplus \gamma)$ defines a rational differential on D_J. Fixing $e' \in J$ and using the coordinates on $\mathbf{U}'_J$ as above, we can rewrite this as

$$2\pi i \mathfrak{d}_J(e \oplus \gamma) = \frac{dx'_\gamma}{x'_\gamma} + \langle e', \gamma \rangle \frac{d(z'_{e'} - 1)}{z'_{e'} - 1} + \frac{dz'_e}{z'_e}. \tag{28}$$

The formula shows in particular that $\mathfrak{d}_J(e \oplus \gamma)$ has logarithmic singularities along the divisor $\{x_{e'} = 0\}$, which is cut out in these coordinates by the function $z_{e'} - 1$. Since this holds for all $e' \notin J$, i.e. for all the components of $\mathbf{K}_J$, we have:

Corollary 5.27. *The image of $\mathfrak{d}_J$ lies in $D_J\langle \mathbf{K}_J \rangle$.*

We extend $\mathfrak{d}_J$ to an inclusion $\bigwedge^\bullet \mathbb{H}(\Gamma \setminus J, \mathbb{C}) \to \Omega^\bullet_{D_J}\langle \mathbf{K}_J \rangle$. Lemma 5.26 tells us that this induces an isomorphism of degree one cohomology. Since the right-hand side computes the cohomology of $\mathbf{U}_J$, which is a vector bundle over a torus, and we know that the cohomology of a torus is an exterior algebra on its degree one cohomology, we deduce:

Lemma 5.28. *The map $\mathfrak{d}_J \colon \bigwedge^\bullet \mathbb{H}(\Gamma \setminus J, \mathbb{C}) \to \Omega^\bullet_{D_J}\langle \mathbf{K}_J \rangle$ is a quasi-isomorphism.*

Proposition 5.29. *The following diagram commutes.*

$$\begin{tikzcd} \bigwedge^\bullet \mathbb{H}(\Gamma \setminus J, \mathbb{C}) \arrow[r, "\mathfrak{d}_J"] \arrow[d, "d_{e'}"] & \Omega^\bullet(D_J\langle \mathbf{K}_J \rangle) \arrow[d, "\mathrm{res}_{J \to J'}"] \\ \bigwedge^{\bullet-1} \mathbb{H}(\Gamma \setminus J', \mathbb{C}) \arrow[r, "\mathfrak{d}_{J'}"] & \Omega^{\bullet-1}(D_{J'}\langle \mathbf{K}_{J'} \rangle) \end{tikzcd} \tag{29}$$

Proof. We will prove the case $J = \emptyset$; the other cases are identical. As in the proof of Lemma 2.2, we can write $\mathbb{H}(\Gamma, \mathbb{C}) = \mathbb{F} \oplus \mathbb{K}$ where $\mathbb{K} = \mathrm{H}_1(\Gamma \setminus e, \mathbb{C}) \oplus \mathrm{H}^1(\Gamma, \mathbb{C})$ and $\mathbb{F}$ is any complementary rank-one subspace.

Let us first assume $\bullet = 1$. By Equation 28, we see that $\mathrm{res}_{J\to J'} \circ \mathfrak{d}_J$ factors through the projection to $\mathbb{F}$, and is given by the pairing $\langle e', \gamma\rangle$. This proves the result when $\bullet = 1$.

For $\bullet > 1$, we can write $\bigwedge^\bullet \mathbb{H}(\Gamma,\mathbb{C}) = \mathbb{F} \wedge \bigwedge^{\bullet-1}\mathbb{K} \oplus \bigwedge^\bullet \mathbb{K}$. Then $\mathrm{res}_{J\to J'} \circ \mathfrak{d}_J$ kills the second summand, and acts on the first by $\mathrm{res}_{J\to J'} \circ \mathfrak{d}(\sigma\wedge\tau) = \langle e', \sigma\rangle \mathfrak{d}(\tau)_{D_{J'}}$ where $\mathfrak{d}(\tau)_{D_{J'}}$ is the restriction of $\mathfrak{d}(\tau)$ to $D_{J'}$. To compute this restriction, we use Lemma 5.21, which immediately implies $\mathfrak{d}_J(\tau)_{D_{J'}} = \mathfrak{d}_{J'}(\tau)$.

Comparing with the formula for d_e yields the result. □

Theorem 5.30. *Define $\mathfrak{d}$ by taking the direct sum of the maps $\mathfrak{d}_J$ for all J. Then $\mathfrak{d}$ induces an inclusion of complexes $\Upsilon^\bullet(\Gamma,\mathbb{C}) \to \Omega^\bullet_{\mathfrak{B}(\Gamma),D}$, which is in fact a quasi-isomorphism.*

Proof. First, let us observe it is an inclusion of bigraded (by degree of wedge and size of J) vector spaces.

The de Rham differential vanishes on the image of $\mathfrak{d}$, since by construction it is composed of wedge products of closed forms. Proposition 5.29 shows that d_{res} restricts to d_Υ.

The fact that the map is a quasi-isomorphism can be seen as follows. Filter both complexes by the size of J, and consider the associated map of spectral sequences. It induces an isomorphism in cohomology on the first page, and thus on all subsequent pages.

□

Remark 5.31. In fact, the above argument shows that $\Upsilon^\bullet(\Gamma,\mathbb{C}) \to \Omega^\bullet_{\mathfrak{B}(\Gamma),D}$ is an isomorphism in the filtered derived category.

Since $\mathfrak{B}(\Gamma)$ is affine, Proposition A.5 yields a quasi-isomorphism $\Omega^\bullet_{\mathfrak{B}(\Gamma)} \to \Omega^\bullet_{\mathfrak{B}(\Gamma),D}$. Combined with the quasi-isomorphism $\mathfrak{d}\colon \Upsilon^\bullet(\Gamma,\mathbb{C}) \to \Omega^\bullet_{\mathfrak{B}(\Gamma),D}$, we have the following corollary:

Corollary 5.32. *There is a canonical isomorphism* $\mathrm{H}^\bullet(\Upsilon^\bullet(\Gamma,\mathbb{C})) \to \mathrm{H}^\bullet(\mathfrak{B}(\Gamma),\mathbb{C})$.

5.7. **The deletion filtration.** We now characterise the weight filtration in terms of the deletion-contraction sequences attached to the edges of Γ.

With a view to applying Definition 1.4, we consider the following generalisation of the deletion map $a_e^{\mathfrak{B}}$. The set-up is similar to Definition 2.7. Suppose Γ' is a connected subgraph of Γ whose complement contains no self-edges. There is a Gysin map $\mathrm{H}^{\bullet-2|\Gamma\setminus\Gamma'|}(D_{\Gamma\setminus\Gamma'},\mathbb{Q}) \to \mathrm{H}^\bullet(\mathfrak{B}(\Gamma),\mathbb{Q})$. We can identify the domain with $\mathrm{H}^{\bullet-2|\Gamma\setminus\Gamma'|}(\mathfrak{B}(\Gamma',\eta'),\mathbb{Q})$ by Lemma 5.18. When $\Gamma\setminus\Gamma' = e$, the Gysin map is by definition the map $a_e^{\mathfrak{B}}$. In general we have the following.

Lemma 5.33. *For $\Gamma' \subset \Gamma$ as above, the Gysin map equals the composition (in any order) of $a_e^{\mathfrak{B}}$ for $e \in \Gamma\setminus\Gamma'$.*

Proof. Follows from the fact that for any ordering $\{e_1,\ldots,e_n\}$ of $\Gamma\setminus\Gamma'$, each inclusion in the corresponding flag of subspaces $D_{\Gamma\setminus\Gamma'} \subset \ldots \subset D_{e_{n-1},e_n} \subset D_{e_n} \subset D_\emptyset = \mathfrak{B}(\Gamma)$ is the inclusion of a codimension 2 submanifold. □

Corollary 5.34. *The maps $a_e^{\mathfrak{B}}$ for different edges commute whenever their composition is defined.*

We can thus make the following special case of Definition 1.4.

Definition 5.35. The *Betti deletion filtration* is the increasing filtration obtained from Definition 1.4, where the covariant functor A is defined on objects Γ by $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q})$ and on morphisms by taking the inclusion $\Gamma' \to \Gamma$ to the composition of deletion maps $a_e^{\mathfrak{B}}$ (in any order) for $e \in \Gamma \setminus \Gamma'$.

The deletion map $a_e^{\mathfrak{B}}$ sends $D_r\mathrm{H}^{i-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q})$ to a subspace of $D_{r+1}\mathrm{H}^i(\mathfrak{B}(\Gamma), \mathbb{Q})$.

Proposition 5.36. $D_k\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q}) \subset W_{2k}\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q})$.

Proof. We proceed by induction on the number N of edges that are neither bridges nor loops. Suppose $N = 0$, and let Γ' be the contraction of Γ along all bridges. By Lemma 5.8, we have $\mathfrak{B}(\Gamma) = \mathfrak{B}(\Gamma')$. By assumption, Γ' is a collection of loops ending at a single vertex, and thus $\mathfrak{B}(\Gamma) \cong \mathfrak{B}(\bigcirc\!\!\!\bullet)^{h_1(\Gamma)}$. Its Hodge structure is thus a tensor power of the Hodge structure for $\mathfrak{B}(\bigcirc\!\!\!\bullet)$. By Proposition 5.11, its degree i cohomology has weight $2i$. By definition, the same is true (with $2i$ replaced by i) for the deletion filtration. This concludes the base case.

Suppose $N > 0$. Note that $D_k\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q})$ is spanned by the images under $a_e^{\mathfrak{B}}$ for various $e \in E(\Gamma)$ of $D_{k-1}\mathrm{H}^{\bullet-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q})$. Consider the sequence

$$\to \mathrm{H}^{i-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q}) \otimes \mathbb{Q}(-1) \xrightarrow{a_e^{\mathfrak{B}}} \mathrm{H}^i(\mathfrak{B}(\Gamma), \mathbb{Q}) \xrightarrow{b_e^{\mathfrak{B}}} \mathrm{H}^i(\mathfrak{B}(\Gamma/e), \mathbb{Q}) \xrightarrow{c_e^{\mathfrak{B}}}$$

By induction, we have that $D_{k-1}\mathrm{H}^{i-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q}) \subset W_{2k-2}\mathrm{H}^{i-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q})$. Taking into account the Tate twist, and noting that $a_e^{\mathfrak{B}}$ preserves the filtration, it follows that the image of $D_{k-1}\mathrm{H}^{i-2}(\mathfrak{B}(\Gamma \setminus e), \mathbb{Q})$ lies in $W_{2k}\mathrm{H}^i(\mathfrak{B}(\Gamma), \mathbb{Q})$, as desired. □

We now exploit the identification of Corollary 5.32 of the cohomology of $\mathfrak{B}(\Gamma)$ with that of our combinatorial model $\Upsilon(\Gamma)$. The point is that we already know that the deletion filtration for $\Upsilon^\bullet(\Gamma)$ is (strictly) preserved by the Υ deletion sequence, hence we may conclude the same for the deletion filtration on $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q})$. We know by Proposition 5.36 that the deletion filtration is contained in the weight filtration; now by induction and the five lemma we may conclude they are equal.

The quasi-isomorphism $\mathfrak{d}$ of Theorem 5.30, and its analogues for $\Gamma \setminus e$ and Γ/e, defines a map of short exact sequences from Sequence 4 to the short exact sequence $\Omega^{\bullet-2}_{D_e,\mathbf{K}_e} \to \Omega^\bullet_{X,D} \to \Omega_{X\setminus D_e, D\setminus D_e}$. As a result, we obtain a map of long exact sequences from Sequence 5 to

$$\to \mathrm{H}^{\bullet-2}(\Omega^\bullet_{D_e,\mathbf{K}_e}) \to \mathrm{H}^\bullet(\Omega^\bullet_{X,D}) \to \mathrm{H}^\bullet(\Omega_{X\setminus D_e, D\setminus D_e}) \xrightarrow{[+1]} \tag{30}$$

Each of the individual maps is a quasi-isomorphism, and so the two long exact sequences in cohomology are identified. (Commutativity of the relevant diagram is a matter of unravelling definitions.) By Corollary A.6, the second sequence may be identified with the deletion contraction sequence on $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{C})$ for the edge e.

Corollary 5.37. $\mathfrak{d}$ *identifies the* Υ*-deletion filtration with the Betti deletion filtration.*

Proposition 5.38. *The Betti deletion-contraction sequence is strictly compatible with the Betti deletion filtration.*

Proof. By Corollary 5.32, we can identify $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{C})$ with the cohomology $\mathrm{H}^\bullet(\Upsilon, \mathbb{C})$ of the graph-theoretic complex Υ. By Corollary 5.37, this identification matches the Betti deletion filtration with the Υ-deletion filtration, the Betti deletion-contraction sequence with the Υ-deletion-contraction sequence. Since the Υ deletion-contraction sequence strictly preserves the Υ-filtration, the same is true for the Betti deletion-contraction sequence. □

With Proposition 5.38 in hand, we can prove our main result in this section, namely that the Betti deletion filtration equals the weight filtration.

Proposition 5.39. *The weight filtration is given by doubling the Betti deletion filtration, i.e.*

$$W_{2k}\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q}) = W_{2k+1}\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q}) = D_k\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q}).$$

Proof. By Proposition 5.36, it is enough to show that the deletion and weight filtrations have isomorphic associated graded spaces. As in the proof of Proposition 5.36, we will use induction on the number N of edges which are neither bridges nor loops to reduce to the case of a graph with a single vertex and n loops. We have already seen, in the proof of Proposition 5.36, that the claim holds for such graphs.

Now let Γ be a general graph, and pick an edge $e \in \Gamma$ which is neither a bridge nor a loop. Recall that the deletion-contraction sequence for e is strictly compatible with the weight-filtration. By Proposition 5.38, the same is true of the deletion-filtration.

Write $\mathrm{gr}^{2D}(DCS_e)$ for the sequence obtained from the deletion-contraction sequence by doubling indices and taking the associated graded spaces for the deletion filtration. Likewise, write $\mathrm{gr}^W(DCS_e)$ for the sequence obtained from the deletion-contraction sequence by taking the associated graded spaces for the weight filtration. By Lemma A.8, $\mathrm{gr}^{2D}(DCS_e)$ and $\mathrm{gr}^W(DCS_e)$ are exact.

By Proposition 5.36, the identity map on cohomology induces a map $\pi\colon \mathrm{gr}^{2D}(DCS_e) \to \mathrm{gr}^W(DCS_e)$.

By induction, we may assume that the filtrations coincide for $\Gamma \setminus e$ and Γ/e, and thus π is an isomorphism on two out of every three terms. By the five-lemma, π is an isomorphism on all terms.

It follows that the associated graded spaces coincide for Γ, and thus that the filtrations coincide for Γ. □

6. $\mathfrak{D}(\Gamma)$

6.1. **Construction.** The space $\mathfrak{D}$ will be a neighborhood of the nodal rational curve with dual graph inside a family of genus one curves. Coordinates convenient to our purposes are provided by Raynaud's construction of the universal cover of the deformation of the nodal elliptic curve.[3]

Let $\mathbb{D} \subset \mathbb{C}$ be the interior of the unit disk, and $\mathbb{D}^*$ the punctured disk. Take q the coordinate on $\mathbb{D}$. One can form over $\mathbb{D}^*$ the family of genus one curves with fiber $\mathbb{C}^*/q^{\mathbb{Z}}$; it is by definition a quotient $(\mathbb{D}^* \times \mathbb{C}^*)/\mathbb{Z}$.

The monodromy of this family is such that it is natural to fill the special fiber by the rational curve with dual graph , and one wants to extend the quotient description accordingly. The picture is that one takes $\mathbb{D} \times \mathbb{P}^1$, iteratively blows up the points at the intersection of the fiber over zero and the strict transforms of the sections $\mathbb{D} \times 0$ and $\mathbb{D} \times \infty$, and then finally deletes these sections. The result has central fiber an infinite chain of $\mathbb{P}^1$.

It is now possible to extend the $\mathbb{Z}$ action, as can be verified most easily in the following coordinate description. Let $\mathbb{C}^2_n$ be a copy of $\mathbb{C}^2$ with coordinates x_n, y_n; we take one for each $n \in \mathbb{Z}$. We glue them by identifying

$$\mathbb{C}^2_n \setminus \{x_n = 0\} \leftrightarrow \mathbb{C}^2_{n+1} \setminus \{y_{n+1} = 0\}$$

by the relations

$$x_n y_n = x_{n+1} y_{n+1}$$
$$x_n = y_{n+1}^{-1}$$

We call the resulting space $\widetilde{\mathfrak{D}}$.

Theorem/Definition 6.1. *Let $\widetilde{q} : \widetilde{\mathfrak{D}} \to \mathbb{C}$ be the function defined by $\widetilde{q}(x_n, y_n) = x_n y_n$. Then the $\mathbb{Z}$ action on $\widetilde{\mathfrak{D}}$ defined by $k + (x_n, y_n) = k + (x_{n+k}, y_{n+k})$ is free and discontinuous over $q^{-1}(\mathbb{D})$.*

We write $\mathfrak{D} := \widetilde{q}^{-1}(\mathbb{D})/\mathbb{Z}$, and $q : \mathfrak{D} \to \mathbb{D}$ for the induced map on the quotient.

Proof. First we note that $\widetilde{q}$ is indeed defined on $\widetilde{\mathfrak{D}}$ since $x_n y_n = x_{n+1} y_{n+1}$. Let us check the free and discontinuousness separately for $z \in \tilde{q}^{-1}(\mathbb{D}^*) \cong \mathbb{D}^* \times \mathbb{C}^*$, and $z \in \tilde{q}^{-1}(0)$. In the former case, $n \in \mathbb{Z}$ acts by multiplication by q^n on the $\mathbb{C}^*$-factor, which is free and discontinuous if $|q| \neq 1$. In the latter case, $n \in \mathbb{Z}$ acts by translating the infinite chain $\tilde{q}^{-1}(0)$ by n steps. □

As one sees in the proof, this construction indeed fills in $\mathfrak{D}|_{\mathbb{D}^*} = (\mathbb{D}^* \times \mathbb{C}^*)/\mathbb{Z}$ with . One can also see the group actions. Let $\mathbb{C}^*$ act on $\mathbb{C}^2_n$ by $(x_n, y_n) \to (\tau x_n, \tau^{-1} y_n)$. Then these actions respect the gluings made to form $\widetilde{D}$, hence descend to $\widetilde{\mathfrak{D}}$. The map $\widetilde{q}$ evidently is equivariant for this action, and it further descends to $\mathfrak{D}$, where it acts through the quotient $\mathbb{C}^*/q^{\mathbb{Z}}$ in general fibers. A section of the map $\mathfrak{D} \to \mathbb{D}^1$ is given in coordinates by fixing $x_{2n} = y_{2n+1} = 1$.

Finally, we note this $\mathbb{C}^*$ action preserves the holomorphic symplectic form $dx_n \wedge dy_n$, which descends to holomorphic symplectic forms on $\widetilde{\mathfrak{D}}$ and $\mathfrak{D}$.

[3]The usual interest in this construction is that it makes sense in rigid analytic geometry; see e.g. [DR, Sec. VII], [Mum1, p. 135]. Here we are using a complex analytic version.

The hyperkähler geometry of the space $\mathfrak{D}$ has been studied by various authors, for instance [OV, Go, GrWi, Gr, GMN]. Statements similar to those of the following lemma may appear in those references; we learned them from two letters from Michael Thaddeus [Th] to Hausel and Proudfoot.

Lemma 6.2. *There is a* $\mathbb{U}_1$*-invariant* $\mathcal{C}^\infty$ *map* $\mu_{\mathfrak{D}}^{\mathbb{U}_1}\colon \mathfrak{D} \to \mathbb{U}_1$ *such that*

$$\pi_{\mathfrak{D}} := \mu_{\mathfrak{D}}^{\mathbb{U}_1} \times q\colon \mathfrak{D} \to \mathbb{U}_1 \times \mathbb{C}$$

endows $\mathfrak{D} \setminus \pi_{\mathfrak{D}}^{-1}(\mathbf{1} \times 0)$ *with the structure of a principal* $\mathbb{U}_1$*-bundle* $\mathcal{P}_{\mathfrak{D}}$ *over* $\mathbb{U}_1 \times \mathbb{D} \setminus \mathbf{1} \times 0$ *with Chern class* $c_1(\mathcal{P}) = 1 \in \mathrm{H}^2(\mathbb{U}_1 \times \mathbb{D} \setminus \mathbf{1} \times 0, \mathbb{Z}) = \mathbb{Z}$.

Moreover, $\mathfrak{D}$ *is endowed with a hyperkähler metric* g *such that the lift* $\widetilde{\mu}_{\mathfrak{D}}^{\mathbb{U}_1}\colon \widetilde{\mathfrak{D}} \to \mathbb{R}$ *to the universal cover is a moment map for the action of* $\mathbb{U}_1$ *with respect to* g*, and* $\widetilde{\mu}_{\mathfrak{D}}^{\mathbb{U}_1} \times \widetilde{q}\colon \widetilde{\mathfrak{D}} \to \mathbb{R} \times \mathbb{C}$ *is a hyperkähler moment map.*

In particular, $\mathfrak{D}$ *carries the structure of a* $(\mathbb{U}_1, \mathbb{U}_1 \times \mathbb{C})$*-hyperkähler space, in the sense of Definition 3.15.*

We will often abbreviate $\mu_{\mathfrak{D}}^{\mathbb{U}_1}$ to $\mu_{\mathfrak{D}}$. We may rescale the Kähler form on $\mathfrak{D}$ so that the volume of any fiber of q equals one. Because the fibers are one complex dimensional, this form is integral. This will be helpful for later arguments regarding projectivity.

Let us collect some properties of $\mathfrak{D}$.

Proposition 6.3. $\mathfrak{D}$ *is a complex manifold and a* $(\mathbb{U}_1, \mathbb{C} \times \mathbb{U}_1)$*-manifold, whose image in* $\mathbb{C}$ *in fact lies in* $\mathbb{D}^1$*. The* $\mathbb{U}_1$ *action has a single fixed point, which we call* $\mathbf{n}$*, and the* $\mathbb{U}_1$*-invariant map*

$$\pi_{\mathfrak{D}} = q \times \mu_{\mathfrak{D}}\colon \mathfrak{D} \to \mathbb{D}^1 \times \mathbb{U}_1$$

has the properties that

1. *The map* $\pi_{\mathfrak{D}}$ *defines a* $\mathbb{U}_1$*-bundle, denoted* $\mathscr{P}_{\mathfrak{D}}$*, over the complement of* 0×1.
2. *The map* q *is holomorphic and smooth away from* 0
3. *The fibre* $q^{-1}(0)$ *is a nodal rational curve.*
4. *For* $\epsilon \neq 0$*, there is a an isomorphism of* $(\mathbb{U}_1, \mathbb{U}_1)$ *manifolds* $q^{-1}(\epsilon) \cong [\mathbb{U}_1 \times \mathbb{U}_1]$.
5. *The only critical value of* $\pi_{\mathfrak{D}}$ *is* 0*; its preimage is a single point which coincides with* $\mathbf{n}$.
6. *Locally near* $\mathbf{n}$*, the* $\mathbb{U}_1$ *action is given in coordinates by* $\tau \cdot (x, y) = (\tau x, \tau^{-1} y)$.

Remark 6.4. While $\mathfrak{D}$ has the structure of a $(\mathbb{U}_1, \mathbb{C} \times \mathbb{U}_1)$-manifold via $\pi_{\mathfrak{D}}$, we will often be interested in the restricted structure of a $(\mathbb{U}_1, \mathbb{U}_1)$-manifold given by just using $\mu_{\mathfrak{D}}$.

We apply the construction of Definition 4.2 using the $(\mathbb{U}_1, \mathbb{U}_1 \times \mathbb{C})$-manifold structure on $\mathfrak{D}$.

Lemma 6.5. $\mathfrak{D}$ *satisfies Hypothesis 4.11 as both a* $(\mathbb{U}_1, \mathbb{U}_1)$*-space and as a* $(\mathbb{U}_1, \mathbb{U}_1 \times \mathbb{C})$*-space*

We can thus speak of $\mathfrak{D}$-generic parameters $\eta \in C_0(\Gamma, \mathbb{U}_1)$ or $\eta \in C_0(\Gamma, \mathbb{U}_1 \times \mathbb{C})$ in the sense of Definition 4.12.

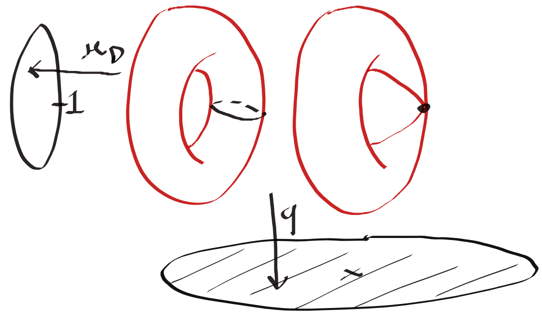


FIGURE 4. A schematic picture of $\mathfrak{D}$ and its various moment maps and their targets. Two fibers of q are shown, and the intersection of each fiber with $\mu_{\mathfrak{D}}^{-1}(1)$ is indicated in black.

Definition 6.6. Given a graph Γ and a $\mathfrak{D}$-generic $\eta \in (\mathbb{U}_1 \times 0)^{V(\Gamma)} \subset C_0(\Gamma, \mathbb{U}_1 \times \mathbb{C})$, we set

$$\mathfrak{D}(\Gamma) := \mathfrak{D}^{(\mathbb{U}_1, \mathbb{U}_1 \times \mathbb{C})}(\Gamma, \eta).$$

We can write $\mathfrak{D}(\Gamma, \eta)$ more explicitly as $\mu_\Gamma^{-1}(\eta \times 0)/\overline{C}^0(\Gamma, \mathbb{U}_1)$ where $\mu_\Gamma^{-1}(\eta \times 0) \subset \mathfrak{D}^{E(\Gamma)}$ is the subset satisfying

$$\sum_{\text{edges exiting } v} q_e - \sum_{\text{edges entering } v} q_e = 0 \tag{31}$$

$$\prod_{\text{edges exiting } v} \mu_e^{\mathbb{U}_1} \prod_{\text{edges entering } v} \left(\mu_e^{\mathbb{U}_1}\right)^{-1} = \eta_v. \tag{32}$$

Proposition 6.7. *$\mathfrak{D}(\Gamma, \eta)$ is a (non-complete) hyperkähler manifold.*

Proof. Since η was chosen to be generic, $\mathfrak{D}(\Gamma, \eta)$ is smooth. As $\mu_{\mathfrak{D}}^{\mathbb{U}_1} \times q$ is a multiplicative moment map for a hyperkähler action of $\mathbb{U}_1$ by Lemma 6.2, $\mathfrak{D}(\Gamma, \eta)$ is the hyperkähler reduction of a hyperkähler manifold. □

Proposition 6.8. *$\mathfrak{D}(\Gamma, \eta)$ is equipped with a complex analytic action of $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$ and a proper holomorphic $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$-invariant map $q_{\mathrm{res}} \colon \mathfrak{D}(\Gamma, \eta) \to \mathbb{C}^{E(\Gamma)}$ whose image is the intersection of the unit polydisk with $\mathrm{H}_1(\Gamma, \mathbb{C})$.*

Proof. Propositions 4.7 and 4.8 yield a proper map $\mathfrak{D}(\Gamma, \eta) \to \mathrm{H}_1(\Gamma, \mathbb{C} \times \mathbb{U}_1)_\eta$. Composing with the projection $\mathrm{H}_1(\Gamma, \mathbb{C} \times \mathbb{U}_1)_\eta \to \mathrm{H}_1(\Gamma, \mathbb{C})$ preserves properness. Concretely, the projection is induced by restricting $q^{E(\Gamma)}$ to the zero fiber of the moment map and descending to the quotient. The result is holomorphic and $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$-invariant since q is holomorphic and $\mathbb{C}^*$-invariant. □

We write Ω_Γ for the associated holomorphic symplectic form on $\mathfrak{D}(\Gamma)$.

Proposition 6.9. *The action of $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$ preserves Ω_Γ.*

Proof. This follows from the fact that the $\mathbb{C}^*$ action preserves the holomorphic symplectic form on $\mathfrak{D}$. □

We write $\mathfrak{V}$ for the space $\mathfrak{D}$ viewed as a $(\mathbb{U}_1, \mathbb{U}_1)$-manifold, i.e. we forget the projection to $\mathbb{D}$.

Definition 6.10. For $\mathfrak{V}$-generic η, we write $\mathfrak{V}(\Gamma, \eta) := \mathfrak{V}^{\mathbb{U}_1, \mathbb{U}_1}(\Gamma, \eta)$.

More explicitly, we have $\mathfrak{V}(\Gamma, \eta) = \mu_\Gamma^{-1}(\eta)/\overline{C}^0(\Gamma, \mathbb{U}_1)$, where $\mu_\Gamma^{-1}(\eta) \subset \mathfrak{V}^{E(\Gamma)}$ is the subset satisfying

$$\prod_{\text{edges exiting } v} \mu_e^{\mathbb{U}_1} \prod_{\text{edges entering } v} \left(\mu_e^{\mathbb{U}_1}\right)^{-1} = \eta_v. \tag{33}$$

Proposition 6.11. *$\mathfrak{V}(\Gamma, \eta)$ is a (non-complete) Kähler manifold, equipped with a complex analytic action of $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$ and a proper holomorphic $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$-invariant map $q_{\mathrm{res}} \colon \mathfrak{V}(\Gamma, \eta) \to \mathbb{C}^{E(\Gamma)}$ whose image is the unit polydisk.*

Proof. The proof is similar to that of Proposition 6.8. Since $\mu_{\mathfrak{D}}^{\mathbb{U}_1}$ is a multiplicative moment map for a Kähler action of $\mathbb{U}_1$ by Lemma 6.2, $\mathfrak{V}(\Gamma, \eta)$ is the Kähler reduction of a Kähler manifold. Since η was chosen to be generic, $\mathfrak{V}(\Gamma, \eta)$ is smooth. The quotient is thus a Kähler manifold. The complex analytic action of $\mathbb{C}^*$ on $\mathfrak{D}$ descends to an action of $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$ on $\mathfrak{V}(\Gamma, \eta)$, preserving the fibers of q_{res}. $\square$

Note η is $\mathfrak{D}$-generic iff it is $\mathfrak{V}$-generic. Thus the following diagram commutes, and is in fact a fiber product

$$\begin{tikzcd} \mathfrak{D}(\Gamma, \eta) \arrow[r] \arrow[d] & \mathfrak{V}(\Gamma, \eta) \arrow[d, "q_{\mathrm{res}}"] \\ \mathrm{H}_1(\Gamma, \mathbb{C}) \arrow[r] & \mathbb{C}^{E(\Gamma)}. \end{tikzcd} \tag{34}$$

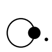
Note that neither of the vertical maps are surjective. We will also denote the left-hand horizontal map by q_{res}.

We suppress the dependence on η for much of the remainder of the article.

Remark 6.12. As with $\mathfrak{B}(\Gamma)$, we can show that the dependence on the chosen orientation is quite mild. More precisely, if Γ, Γ' differ only by the choice of orientation, then there is a canonical isomorphism of smooth manifolds

$$\mathfrak{D}(\Gamma) \to \mathfrak{D}(\Gamma)'$$

By Proposition 4.10, it is enough to find an isomorphism $\mathfrak{D} \to \mathfrak{D}$ of smooth manifolds, intertwining the $\mathbb{U}_1$-action and the $\mathbb{U}_1 \times \mathbb{C}$-moment map with their inverses. One can construct such a map by arguments similar to the proof of Lemma 9.1. We will not need this result elsewhere in the paper.

6.2. **Motive of the central fiber.** The map $q : \mathfrak{D} \to \mathbb{D}$ has fiber $q^{-1}(0)$ a nodal rational curve with dual graph . While the algebraic structure of this curve is nontrivial, its class in the Grothendieck group of varieties is simply $[\mathbb{A}^1]$. We now compute $[q_{\mathrm{res}}^{-1}(0)]$.

Proposition 6.13. *The* $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$*-fixed points of* $\mathfrak{D}(\Gamma)$ *are indexed by the spanning trees* $\Gamma' \subset \Gamma$*. The fixed point* $p_{\Gamma'}$ *is the reduction by* $\overline{C}^0(\Gamma, \mathbb{U}_1)$ *of the subspace* $\prod_{e \notin \Gamma'} \mathbf{n} \times \prod_{e \in \Gamma'} \mathfrak{D} \subset \prod_{e \in \Gamma} \mathfrak{D}$*.*

Proof. Let $p \in \mathfrak{D}(\Gamma)$, and let $\tilde{p}$ be a lift to $\mathfrak{D}^{E(\Gamma)}$. p is fixed by $\mathrm{H}^1(\Gamma, \mathbb{U}_1)$ if and only if the action of $C^1(\Gamma, \mathbb{U}_1)$ on $\tilde{p}$ preserves the $\overline{C}^0(\Gamma, \mathbb{U}_1)$-orbit of $\tilde{p}$.

Let Γ' be the unique subgraph of Γ such that $\tilde{p} \in \prod_{e \notin \Gamma'} \mathbf{n} \times \prod_{e \in \Gamma'} (\mathfrak{D} \setminus \mathbf{n}) \subset \prod_{e \in \Gamma} \mathfrak{D}$. Since we know the $\overline{C}^0(\Gamma, \mathbb{U}_1)$ orbit of $\tilde{p}$ is free by assumption, this subgraph must contain all vertices of Γ.

The action of $C^1(\Gamma, \mathbb{U}_1)$ on this subspace factors through a free action of $C^1(\Gamma', \mathbb{U}_1)$, which descends to a free action of $\mathrm{H}^1(\Gamma', \mathbb{U}_1) = C^1(\Gamma', \mathbb{U}_1)/\overline{C}^0(\Gamma', \mathbb{U}_1)$ on the quotient space. Hence p is a fixed point if and only if $\mathrm{H}^1(\Gamma', \mathbb{U}_1)$ is trivial, i.e. Γ' is a tree.

From the description of $p_{\Gamma'}$, it follows that it is also fixed by $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$. □

Corollary 6.14. *All* $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$*-fixed points of* $\mathfrak{D}(\Gamma)$ *are contained in the central fiber* $q_{\mathrm{res}}^{-1}(0)$*.*

Theorem 6.15. *In the Grothendieck group of varieties, the class of the central fiber is:*

$$[q_{\mathrm{res}}^{-1}(0)] = (\textit{\# of spanning trees of } \Gamma) \times \mathbb{A}^{h^1(\Gamma)}$$

Proof. Pick a cocharacter $\sigma \colon \mathbb{C}^* \to \mathrm{H}^1(\Gamma, \mathbb{C}^*)$ whose image is not contained in the kernel of any restriction map $\mathrm{H}^1(\Gamma, \mathbb{C}^*) \to \mathrm{H}^1(\Gamma \setminus e, \mathbb{C}^*)$.

The resulting complex analytic action of $\mathbb{C}^*$ on $\mathfrak{D}(\Gamma)$ preserves the fibers of q_{res} and has isolated fixed points $p_{\Gamma'} \in q_{\mathrm{res}}^{-1}(0)$, naturally indexed by the spanning trees $\Gamma' \subset \Gamma$.

Since the action of $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$ preserves the holomorphic symplectic form on $\mathfrak{D}(\Gamma)$, the attracting cells of the $\mathbb{C}^*$ action are lagrangian, and thus of complex dimension $1/2 \dim \mathfrak{D}(\Gamma) = h^1(\Gamma)$. Each attracting cell is therefore isomorphic to $\mathbb{A}^{h^1(\Gamma)}$. Since $q_{\mathrm{res}}^{-1}(0)$ is proper, it is the disjoint union of these attracting cells. □

Remark 6.16. It is possible to give an alternative proof of the theorem by using the explicit description below of $q_{\mathrm{res}}^{-1}(0)$ as a union of toric varieties.

Remark 6.17. The class of the general fiber has a similar description, which we will not need in this paper. Namely, let $\Gamma' \subset \Gamma$ be the (possibly disconnected) subgraph consisting of the edges $e \in E(\Gamma)$ for which $b_e \neq 0$. We can view b as a generic point in $\mathbb{D}^{E(\Gamma')}$. This defines an abelian variety $\mathfrak{V}(\Gamma')_b$; it is a product of smaller abelian varieties determined by the connected components of Γ'; the general form of the factors is described explicitly in Proposition 7.3 below. Then we have

$$[q_{\mathrm{res}}^{-1}(b)] = [\mathfrak{V}(\Gamma')_b] \times (\text{ \# of spanning trees of } \Gamma/\Gamma') \times \mathbb{A}^{h^1(\Gamma/\Gamma')}$$

Here Γ/Γ' is the contraction of Γ by Γ', where exceptionally we allow the contraction of subgraphs of genus > 0.

6.3. **Vanishing cycles for** q**.**

Let us study the nearby-vanishing exact triangle with respect to the map $q\colon \mathfrak{D} \to \mathbb{C}$. This is a triangle of sheaves on $q^{-1}(0)$:

$$\Psi_q\mathbb{Q} \to \mathbb{Q}|_{q^{-1}(0)} \to \Phi_q\mathbb{Q} \xrightarrow{[1]}$$

As is well known, in this case $\Psi_q\mathbb{Q} = \mathbb{Q}_{\mathbf{n}}[-2]$, represented by the vanishing cycle of the genus one Riemann surface $q^{-1}(\epsilon)$ as $\epsilon \to 0$. Passing to cohomology gives the sequence

$$\mathbb{Q}[-2] \to \mathbb{Q} \oplus \mathbb{Q}[-1] \oplus \mathbb{Q}[-2] \to \mathbb{Q} \oplus \mathbb{Q}^{\oplus 2}[-1] \oplus \mathbb{Q}[-2] \xrightarrow{[1]}$$

Note this is identical to the sequence appearing in the calculation of $\mathrm{H}^\bullet(\mathfrak{B}, \mathbb{Q})$ in Proposition 5.11, save that the weight grading no longer appears. We wish to express this fact in a precise and generalizable form. To this end, we recall one formulation of the nearby and vanishing cycle functors.

Given any space $\mathbf{X}$ and map $q\colon \mathbf{X} \to \mathbb{C}$, denote the inclusion of the zero fibre by $i\colon q^{-1}(0) \subset \mathbf{X}$, and consider the inclusions

$$q^{-1}(\{\mathrm{Re}(z) \leq 0\}) \xrightarrow{\mathcal{I}} \mathbf{X} \xleftarrow{\mathcal{J}} q^{-1}(\{\mathrm{Re}(z) > 0\})$$

The corresponding excision triangle $\mathcal{I}_!\mathcal{I}^!\mathbb{Q} \to \mathbb{Q} \to \mathcal{J}_*\mathcal{J}^*\mathbb{Q} \xrightarrow{[1]}$ restricts to the nearby-vanishing triangle. That is,

$$\Psi_q\mathbb{Q} = i^*\mathcal{J}_*\mathcal{J}^*\mathbb{Q}.$$
$$\Phi_q\mathbb{Q} = i^*\mathcal{I}_!\mathcal{I}^!\mathbb{Q}.$$

We now return to the case at hand. We define

$$\mathbf{S}_{\mathfrak{D}} := \pi_{\mathfrak{D}}^{-1}(\mathbb{R}^{\leq 0} \times 1)$$

and write for the inclusions $\mathbf{S}_{\mathfrak{D}} \xrightarrow{I} \mathfrak{D} \xleftarrow{J} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}$.

Lemma 6.18. $\mathrm{S}_{\mathfrak{D}}$ *is a codimension 2 submanifold of* $\mathfrak{D}$*, diffeomorphic to an open disk.*

Proof. Since $\pi_{\mathfrak{D}}$ is a circle fibration away from 0×1, the preimage $\pi_{\mathfrak{D}}^{-1}(\mathbb{R}^{<0} \times 1)$ is evidently a cylinder. It suffices to investigate the geometry near 0×1. This point is the image of the fixed point $\mathbf{n}$, where in local coordinates the circle action is $\tau \cdot (x, y) = (\tau x, \tau^{-1} y)$. It follows that the fibration is equivariantly diffeomorphic to the standard Hopf fibration $\mathbb{R}^4 \to \mathbb{R}^3$, where it can be checked in coordinates that the preimage of any smooth ray leaving the origin is a disk. . □

Remark 6.19. In Lemma 9.1 below, we use a more elaborate version of this argument to construct an embedding $\mathfrak{D} \subset \mathfrak{B}$, with respect to which $\mathbf{S}_{\mathfrak{D}} = \mathbf{S}_{\mathfrak{B}} \cap \mathfrak{D}$.

Lemma 6.20. *Let* $\epsilon > 0$ *be sufficiently small. We have a retraction of* $(\mathbb{U}_1, \mathbb{U}_1)$*-spaces* $\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \to q^{-1}(\epsilon)$.

Proof. $\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}$ is a trivial $\mathbb{U}_1$-bundle over $\mathbb{D}^1 \times \mathbb{U}_1 \setminus [0,1) \times 1$. Any retraction of $\mathbb{D}^1 \setminus [0,1)$ to ϵ can be lifted to an $(\mathbb{U}_1, \mathbb{U}_1)$-retraction of the total space of the bundle. □

Observe that we have a closed inclusion

$$\mathbf{S}_{\mathfrak{D}} = \pi_{\mathfrak{D}}^{-1}(\mathbb{R}^{\leq 0} \times 1) \subset \pi_{\mathfrak{D}}^{-1}(\{\mathrm{Re}(z) \leq 0\} \times \mathbb{U}_1) = q^{-1}(\{\mathrm{Re}(z) \leq 0\})$$

Thus the excision triangle for $\mathbf{S}_{\mathfrak{D}}$ maps to the excision triangle for $q^{-1}(\{\mathrm{Re}(z) \leq 0\})$.

$$\begin{tikzcd}
I_! I^! \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[r] \arrow[d] & J_* J^* \mathbb{Q} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\mathcal{I}_! \mathcal{I}^! \mathbb{Q} \arrow[r] & \mathbb{Q} \arrow[r] & \mathcal{J}_* \mathcal{J}^* \mathbb{Q} \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

Proposition 6.21. *The restriction of the above diagram to the nodal rational curve at $q^{-1}(0)$ is an isomorphism of triangles.*

Proof. We have $\mathbf{S}_{\mathfrak{D}} \cap q^{-1}(0) = \mathbf{n}$, i.e. the only point of intersection between $\mathbf{S}_{\mathfrak{D}}$ and the central fiber is the node. Away from the node, the map q is smooth. Thus along $q^{-1}(0) \setminus \mathbf{n}$, the diagram simply restricts to

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[r, "{[1]}"] \arrow[d] & {} \\
0 \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q} \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

with all maps given by the identity or the zero map. On the other hand, the Milnor fiber of q at $\mathbf{n}$ has cohomology supported in degrees 0 and 1, and the degree one homology is generated by any orbit of $\mathbb{U}_1$. Similarly, the degree one homology of $V \setminus V \cap \mathbf{S}_{\mathfrak{D}}$ is generated by any orbit of $\mathbb{U}_1$, since $\mathbf{S}_{\mathfrak{D}}$ is a smooth $\mathbb{U}_1$-stable divisor in the complex structure on $\mathfrak{B}^{<}$. Thus the restriction map from $V \setminus V \cap \mathbf{S}_{\mathfrak{D}}$ to the Milnor fiber induces an isomorphism on cohomology. It follows that the right-hand vertical map is an isomorphism in a neighborhood of $\mathbf{n}$. Since it is also an isomorphism away from $\mathbf{n}$, it is a global isomorphism. Since the central vertical map of the diagram is simply the identity, the left-hand map must also be an isomorphism. □

As previously claimed, we have the following corollary.

Corollary 6.22.

$$\Psi_q \mathbb{Q} = \mathbb{Q}_{\mathbf{n}}[-2]$$

Proof. We have

$$\Psi_q \mathbb{Q} = i^* \mathcal{I}_! \mathcal{I}^! \mathbb{Q} = i^* I_! I^! \mathbb{Q} = i^* \mathbb{Q}_{\mathbf{S}_{\mathfrak{D}}}[-2] = \mathbb{Q}_{\mathbf{n}}[-2.]$$

Here, the first equality is essentially the definition of of vanishing cycles. The second equality holds because of the isomorphism of exact triangles. The third equality holds because $\mathbf{S}_{\mathfrak{D}}$ is a real codimension 2 submanifold. The final equality holds because $\mathbf{S}_{\mathfrak{D}} \cap q^{-1}(0) = \mathbf{n}$. □

Remark 6.23. The locus $\mathbf{S}_{\mathfrak{D}}$ is a Lefschetz thimble for the vanishing cycle.

6.4. **Vanishing cycles and convolution.** We wish to show that the constructions of the previous section "commute with convolution" with an auxilliary factor. For this section, the convolution $\star$ will always mean the $\star_{\mathbb{U}_1,\mathbb{U}_1,\zeta}$ product, for some fixed $\zeta \in \mathbb{U}_1$.

We will study convolution with some auxilliary $(\mathbb{U}_1, \mathbb{U}_1)$-manifold $\mathbf{X}$. A complication is that in our applications, $\mathbf{X}$ will be noncompact. To prove our results we impose the following rather strong tameness condition, which we will later show is satisfied in our examples.

Hypothesis 6.24. There is an open contractible neighborhood $\mathfrak{U} \subset \mathbb{U}_1$ of $\zeta \in \mathbb{U}_1$, such that:

(1) The map $\mu_{\mathbf{X}} \colon \mathbf{X} \to \mathbb{U}_1$ is locally constant near ζ, in the sense that for some interval $\zeta \in \mathfrak{U} \subset \mathbb{U}_1$, the space $\mu_{\mathbf{X}}^{-1}(\mathfrak{U}) \subset \mathbf{X}$ is isomorphic as a $(\mathbb{U}_1, \mathbb{U}_1)$-space to $[\mathfrak{U} \times \mu_{\mathbf{X}}^{-1}(\zeta)]$.
(2) The action of $\mathbb{U}_1$ on $\mu_{\mathbf{X}}^{-1}(\zeta)$ is free.

Recall the map $q \colon \mathfrak{D} \to \mathbb{C}$. Composing with projection onto the second factor, we get a map $\tilde{q}_2 \colon \mathbf{X} \times \mathfrak{D} \to \mathbb{C}$. It is $\mathbb{U}_1 \times \mathbb{U}_1$-invariant, hence we can define the induced map

$$q_2 \colon \mathbf{X} \star \mathfrak{D} \to \mathbb{C}.$$

Note that $\tilde{q}_2^{-1}(\epsilon) = \mathbf{X} \times q^{-1}(\epsilon)$ and thus $q_2^{-1}(\epsilon) = \mathbf{X} \star q^{-1}(\epsilon)$. When $\epsilon \neq 0$, $q^{-1}(\epsilon) \cong \mathbb{U}_1 \times \mathbb{U}_1$ as a $(\mathbb{U}_1, \mathbb{U}_1)$-space, and thus $q_2^{-1}(\epsilon) \cong \mathbf{X}$.

Lemma 6.25. *The singular locus of q_2 is contained in $\mathbf{X} \star \mathbf{n}$.*

Proof. Let $z \in \mathfrak{D} \setminus \mathbf{n}$. Choose a small open neighborhood of polydisk $V_1 \times V_2$ containing $\pi_{\mathfrak{D}}(z) \in \mathbb{U}_1 \times \mathbb{D}$, not containing the point $(0, 1)$, on which the $\mathbb{U}_1$ bundle $\mathscr{P}_{\mathfrak{D}}$ can be trivialized. By Lemma 3.28, we have a diffeomorphism

$$\pi_{\mathfrak{D}}^{-1}(V_1 \times V_2) \star \mathbf{X} \cong V_1 \times \mu_{\mathbf{X}}^{-1}(V_2)$$

and under this isomorphism q_2 becomes projection onto V_1. This concludes the proof. □

Lemma 6.26. $(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}) \cap q_2^{-1}(0) = \mathbf{X} \star \mathbf{n}$.

Proof. We have $\mathbf{S}_{\mathfrak{D}} \cap q^{-1}(0) = \mathbf{n}$ and $q_2^{-1}(0) = \mathbf{X} \star q^{-1}(0)$. Hence $(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}) \cap (\mathbf{X} \star q^{-1}(0)) = \mathbf{X} \star (\mathbf{S}_{\mathfrak{D}} \cap q^{-1}(0)) = \mathbf{X} \star \mathbf{n}$. □

Lemma 6.27. *$\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}$ is a real-codimension two submanifold of $\mathbf{X} \star \mathfrak{D}$.*

Proof. Let $\mu_{\mathbf{X} \bullet \mathfrak{D}} = \mu_{\mathbf{X}} + \mu_{\mathfrak{D}} \colon \mathbf{X} \times \mathfrak{D} \to \mathbb{U}_1$. Then $\mathbf{X} \star \mathfrak{D} = \mu_{\mathbf{X} \bullet \mathfrak{D}}^{-1}(\zeta)/\mathbb{U}_1$. It will suffice to show that $\mu_{\mathbf{X} \bullet \mathfrak{D}}^{-1}(\zeta)$ and $\mathbf{X} \times \mathbf{S}_{\mathfrak{D}}$ intersect transversely in $\mathbf{X} \times \mathfrak{D}$. Thus let us check that, along the intersection, $\ker D\mu_{\mathbf{X} \bullet \mathfrak{D}} + T\mathbf{X} \oplus T\mathbf{S}_{\mathfrak{D}}$ spans the entire tangent space.

By assumption, $\mu_{\mathbf{X}}$ defines a fiber bundle over $\mathfrak{U}$, hence it is submersive. Hence $\mu_{\mathbf{X}\bullet\mathfrak{D}}\colon \mathbf{X}\times\mathfrak{D}\to\mathbb{U}_1$ is a submersion everywhere. It follows that the kernel of $D\mu_{\mathbf{X}\bullet\mathfrak{D}} = D\mu_{\mathbf{X}} + D\mu_{\mathfrak{D}}$ has codimension one. Since it does not contain $T\mathbf{X}$, it follows that $\ker D\mu_{\mathbf{X}\bullet\mathfrak{D}} + T\mathbf{X}\oplus T\mathbf{S}_{\mathfrak{D}}$ must be the entire tangent space. □

We wish to prove an analogue of Proposition 6.21. We have the complementary inclusions

$$\mathbf{X}\star\mathbf{S}_{\mathfrak{D}} \xrightarrow{I} \mathbf{X}\star\mathfrak{D} \xleftarrow{J} \mathbf{X}\star(\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}})$$

The adjunction triangle is

$$\to I_*I^!\mathbb{Q}\to\mathbb{Q}\to J_*J^*\mathbb{Q}\to \tag{35}$$

On the other hand, let $i\colon \mathbf{X}\star q_2^{-1}(0)\to\mathbf{X}\star\mathfrak{D}$ be the inclusion. We have the nearby-vanishing triangle

$$\Phi_{q_2}\mathbb{Q}\to i^*\mathbb{Q}\to\Psi_{q_2}\mathbb{Q}\to \tag{36}$$

which we may compute as the pullback under i^* of the excision triangle $\mathcal{I}_!\mathcal{I}^!\mathbb{Q}\to\mathbb{Q}\to\mathcal{J}_*\mathcal{J}^*\mathbb{Q}\xrightarrow{[1]}$ attached to the excision sequence

$$q_2^{-1}(\{\mathrm{Re}(z)\le 0\}) \xrightarrow{\mathcal{I}} \mathbf{X}\star\mathfrak{D} \xleftarrow{\mathcal{J}} q_2^{-1}(\{\mathrm{Re}(z)>0\}).$$

Proposition 6.28. *The restriction of the diagram*

$$\begin{tikzcd} I_!I^!\mathbb{Q} \arrow[r]\arrow[d] & \mathbb{Q} \arrow[r]\arrow[d] & J_*J^*\mathbb{Q} \arrow[r, "{[1]}"]\arrow[d] & {} \\ \mathcal{I}_!\mathcal{I}^!\mathbb{Q} \arrow[r] & \mathbb{Q} \arrow[r] & \mathcal{J}_*\mathcal{J}^*\mathbb{Q} \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

to $q_2^{-1}(0)$ is an isomorphism.

We will split Proposition 6.28 into two parts, which we prove separately:

Proposition 6.29. *The restriction of Diagram 6.28 to $q_2^{-1}(0)\setminus(\mathbf{X}\star\mathbf{n})$ is an isomorphism.*

Proof. By Lemma 6.26, $q_2^{-1}(0)\setminus(\mathbf{X}\star\mathbf{n}) = q_2^{-1}(0)\setminus q_2^{-1}(0)\cap(\mathbf{X}\star\mathbf{S}_{\mathfrak{D}})$. By definition, this locus avoids the image of I (and thus is contained in the image of J). It follows that the top line of 6.28 restricts to $0\to\mathbb{Q}\to\mathbb{Q}\xrightarrow{[1]}$.

On the other hand, according to Lemma 6.25, the map q_2 is smooth away from $\mathbf{X}\star\mathbf{n}$. Since the vanishing cycles sheaf $i^*\mathcal{I}_!\mathcal{I}^!\mathbb{Q}$ is trivial along the smooth locus, the restriction of the lower line of 6.28 to $q_2^{-1}(0)\setminus(\mathbf{X}\star\mathbf{n})$ is $0\to\mathbb{Q}\to i^*\mathcal{J}_*\mathcal{J}^*\xrightarrow{[1]}$. The map $\mathbb{Q}\to i^*\mathcal{J}_*\mathcal{J}^*\xrightarrow{[1]}$ must therefore be an isomorphism. Since the middle vertical arrow from $i^*\mathbb{Q}$ to $i^*\mathbb{Q}$ is simply the identity map, and the diagram commutes, the rightmost vertical arrow $\mathbb{Q}=i^*J_*J^*\to i^*\mathcal{J}_*\mathcal{J}^*$ must also be an isomorphism. This shows that we have an isomorphism of triangles over $q_2^{-1}(0)\setminus(\mathbf{X}\star\mathbf{n})$. □

Let $\mathbf{B} := \mu_{\mathfrak{D}}^{-1}(\zeta - \mathfrak{U})$. $\mathbf{B}$ is an open neighbordhood of $\mathbf{n} \in \mathfrak{D}$, hence the following proposition will complete the proof of Proposition 6.28.

Proposition 6.30. *The restriction of Diagram 6.28 to* $q_2^{-1}(0) \cap (\mathbf{X} \star \mathbf{B})$ *is an isomorphism.*

Since the proof of this proposition is somewhat lengthy, we begin by sketching the argument. First consider the simplest case, when $\mathbf{X} = [\mathbf{Y} \times \mathbb{U}_1]$ where $\mathbf{Y}$ is a free $\mathbb{U}_1$-space. Then we have $\mathbf{X} \star \mathbf{B} = \mathbf{Y}/\mathbb{U}_1 \times \mathbf{B} = \mathbf{X} /\!\!/ \mathbb{U}_1 \times \mathbf{B}$. Moreover, Diagram 6.28 is identified with Diagram 6.3, up to a box product with $\mathbb{Q}_{\mathbf{X}/\!\!/\mathbb{U}_1}$. The proposition follows immediately from the corresponding statement for Diagram 6.3, Proposition 6.21.

In general, $\mathbf{X}$ will not have such a simple product form. However, our assumptions guarantee that *locally*, $\mathbf{X}$ can be put in such a form, at least along the locus which contributes to the star product. More precisely, we will cover $q_2^{-1}(0) \cap (\mathbf{X} \star \mathbf{B})$ with open charts of the form $\mathbf{U} \times \mathbf{W}$ where $\mathbf{W} \subset \mathbf{B}$ and $\mathbf{U} \subset \mathbf{X} /\!\!/ \mathbb{U}_1$.

We use this cover to obtain a local identification of Diagram 6.28 with Diagram 6.3, up to a box product with $\mathbb{Q}_{\mathbf{U}}$. Since on each chart, the restriction of the diagram is an isomorphism by Proposition 6.21, it is an isomorphism globally, which proves our claim.

Proof of 6.30. Recall that $\mathbf{X} \star \mathbf{B}$ is defined as $(\mu_{\mathbf{X}} + \mu_{\mathfrak{D}})^{-1}(\zeta)/\mathbb{U}_1$. Since $\mu_{\mathfrak{D}}(\mathbf{B}) = \zeta - \mathfrak{U}$, we have

$$\mathbf{X} \star \mathbf{B} = \mu_{\mathbf{X}}^{-1}(\mathfrak{U}) \star \mathbf{B}.$$

In other words, the complement of $\mu_{\mathbf{X}}^{-1}(\mathfrak{U})$ does not contribute to the $\star$ product with $\mathbf{B}$, since it cannot possibly satisfy the moment map condition in the definition of $\mathbf{X} \star \mathbf{B} := \mathbf{X} \times \mathbf{B} /\!\!/_{\zeta} \mathbb{U}_1$.

Recall that by assumption, we have an isomorphism of $(\mathbb{U}_1, \mathbb{U}_1)$-spaces

$$\mu_{\mathbf{X}}^{-1}(\mathfrak{U}) \cong [\mu_{\mathbf{X}}^{-1}(\zeta) \times \mathfrak{U}] \tag{37}$$

where the moment map on the RHS is the projection $\mu_{\mathbf{X}}^{-1}(\zeta) \times \mathfrak{U} \to \mathfrak{U}$.

The isomorphism 37 determines an isomorphism

$$\begin{aligned}
\mu_{\mathbf{X}}^{-1}(\mathfrak{U}) \star \mathbf{B} &= (\mu_{\mathbf{X}}^{-1}(\zeta) \times \mathfrak{U}) \star \mathbf{B} && (38)\\
&= (\mu_{\mathbf{X}}^{-1}(\zeta) \times \mathfrak{U}) \star \mu_{\mathfrak{D}}^{-1}(\zeta - \mathfrak{U}) && (39)\\
&= (\mu_{\mathbf{X}} + \mu_{\mathfrak{D}})^{-1}(\zeta)/\mathbb{U}_1 && (40)\\
&\cong (\mu_{\mathbf{X}}^{-1}(\zeta) \times \mathbf{B})/\mathbb{U}_1. && (41)
\end{aligned}$$

To go from line 39 to line 40, note that for any point $z \in \mu_{\mathfrak{D}}^{-1}(\zeta - \mathfrak{U})$ there is a unique value $p \in \mathfrak{U}$ such that $p + \mu_{\mathfrak{D}}(z) = \zeta$. Thus the moment map condition appearing in the star quotient amounts to forgetting the factor $\mathfrak{U}$ and taking the ordinary quotient.

By our assumptions on $\mathbf{X}$, the action of $\mathbb{U}_1$ on $\mu_{\mathbf{X}}^{-1}(\zeta)$ is free, hence defines a principle $\mathbb{U}_1$-bundle $\rho \colon \mu_{\mathbf{X}}^{-1}(\zeta) \to \mu_{\mathbf{X}}^{-1}(\zeta)/\mathbb{U}_1$. The projection $\pi \colon (\mu_{\mathbf{X}}^{-1}(\zeta) \times \mathbf{B})/\mathbb{U}_1 \to \mu_{\mathbf{X}}^{-1}(\zeta)/\mathbb{U}_1$ is a fiber bundle map with fibers isomorphic to $\mathbf{B}$ and structure group $\mathbb{U}_1$, defined by $\mathbf{B} \times_{\mathbb{U}_1} \mu_{\mathbf{X}}^{-1}(\zeta)$.

In order to prove the proposition, it suffices to find an open cover $\{\mathbf{V}_i\}$ of the base $\mu_{\mathbf{X}}^{-1}(\zeta)/\mathbb{U}_1$ of this fiber bundle, and show that the restriction of Diagram 6.28 to each $q_2^{-1}(0) \cap \pi^{-1}(\mathbf{V}_i)$ is an isomorphism. Unravelling the definitions, we see that given $\mathbf{V} \subset \mu_{\mathbf{X}}^{-1}(\zeta)/\mathbb{U}_1$, the preimage $\pi^{-1}(\mathbf{V}) \subset \mu_{\mathbf{X}}^{-1}(\mathfrak{U}) \star \mathbf{B}$ is equal to $(\rho^{-1}(\mathbf{V}) \times \mathfrak{U}) \star \mathbf{B} = (\rho^{-1}(\mathbf{V}) \times \mathbf{B})/\mathbb{U}_1$.

If we choose each $\mathbf{V}_i$ to be contractible, we can trivialize $\rho^{-1}(\mathbf{V}_i)$ as a $\mathbb{U}_1$-bundle. This trivialization determines an isomorphism $\pi^{-1}(\mathbf{V}_i) = (\rho^{-1}(\mathbf{V}_i) \times \mathbf{B})/\mathbb{U}_1 \cong (\rho^{-1}(\mathbf{V}_i)/\mathbb{U}_1) \times \mathbf{B} = \mathbf{V}_i \times \mathbf{B}$.

In fact this factorization is 'functorial' in the following sense. For any $\mathbb{U}_1$-invariant subset $\mathbf{K} \subset \mathfrak{D}$, replacing $\mathbf{B}$ by $\mathbf{K} \cap \mathbf{B}$ above determines an isomorphism $(\mathbf{X} \star \mathbf{K}) \cap \pi^{-1}(\mathbf{V}_i) = \mathbf{V}_i \times (\mathbf{K} \cap \mathbf{B})$. In particular, we have

$$(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}) \cap \pi^{-1}(\mathbf{V}_i) \cong \mathbf{V}_i \times (\mathbf{S}_{\mathfrak{D}} \cap \mathbf{B})$$

and likewise when $\mathbf{K}$ equals $\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}$, $q^{-1}(\{\mathrm{Re}(z) \leq 0\})$ or $q^{-1}(\{\mathrm{Re}(z) > 0\})$.

Hence after restricting to $\mathbf{X} \star \mathbf{B} \cap \pi^{-1}(\mathbf{V}_i) \cap q_2^{-1}(0) \cong \mathbf{V}_i \times q^{-1}(0)$, Diagram 6.28 is identified with Diagram 6.3, up to a box product with the constant sheaf $\mathbb{Q}_{\mathbf{V}_i}$.

Hence the proposition follows from the corresponding one for Diagram 6.3, i.e. Proposition 6.21. □

Corollary 6.31.

$$\Phi_{q_2}\mathbb{Q} = \mathbb{Q}_{\mathbf{X}\star\mathbf{n}}[-2]$$

Proof. This follows from Lemmas 6.26, 6.27, and Proposition 6.28, as in the proof of Corollary 6.22. □

Proposition 6.32. *Taking hypercohomology of the triangle 36 defines a long exact sequence*

$$\to \mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{n}, \mathbb{Q}) \to \mathrm{H}^{\bullet}(q_2^{-1}(0), \mathbb{Q}) \to \mathrm{H}^{\bullet}(q_2^{-1}(\epsilon), \mathbb{Q}) \to \tag{42}$$

Proof. The hypercohomology of the vanishing cycles gives the first term by Corollary 6.31. The second term is immediate. To show that the third term is as claimed, we need to check that the hypercohomology of the nearby cycles sheaf indeed equals the cohomology of a nearby fiber. Since the map q_2 is non-proper, this is not automatic. However, the retraction of $(\mathbb{U}_1, \mathbb{U}_1)$ spaces $\mathfrak{D} \to q^{-1}(0)$ induces a retraction $\mathbf{X} \star \mathfrak{D} \to \mathbf{X} \star q^{-1}(0)$, compatible with an exhaustion of of both spaces by compact subsets (see Lemma 3.23 and the remark following it). This retraction restricts to a surjective map $r\colon q_2^{-1}(\epsilon) \to q_2^{-1}(0)$. Thus $\Psi_{q_2}\mathbb{Q} = r_*\mathbb{Q}_{q_2^{-1}(\epsilon)}$ and the claim follows. □

We ultimately want apply the results above when $\mathbf{X} = \mathfrak{D}(\Gamma/e)$, with $(\mathbb{U}_1, \mathbb{U}_1)$ structure as in Definition 4.18 and $\zeta = \eta_e$ as in Definition 4.16. Thus let us now verify Hypothesis 6.24 for this space. Condition 2 follows from our requirement that η be generic; we turn to checking the local constancy asked in Condition (1).

Lemma 6.33. *Let $\mu_{\mathrm{res}}^{\mathbb{U}_1}\colon \mathfrak{D}(\Gamma) \to \mathrm{H}_1(\Gamma, \mathbb{U}_1)_\eta$ be the residual moment map. Let $\alpha\colon \mathrm{H}_1(\Gamma, \mathbb{U}_1)_\eta \to \mathbb{U}_1$ be the restriction of a character of $C_1(\Gamma, \mathbb{U}_1)$. For all but a finite set of $\zeta \in \mathbb{U}_1$, there exists* *an open neighborhood* $\mathfrak{U}$ *of* ζ *and an isomorphism of* $(\mathbb{U}_1, \mathbb{U}_1)$*-spaces* $(\alpha \circ \mu_{\text{res}}^{\mathbb{U}_1})^{-1}(\mathfrak{U}) \cong [(\alpha \circ \mu_{\text{res}}^{\mathbb{U}_1})^{-1}(\zeta) \times \mathfrak{U}]$.

That is, Condition (1) of Hypothesis 6.24 holds for generic choice of ζ.

Proof. In general, for a map $E \to B$, a (nonlinear) connection is an assignment, for each path in the base B with endpoints x, y, of a diffeomorphism $E_x \cong E_y$, compatible with composition of paths. Given a stratification of B, by a stratified connection, we mean the data of a connection on each stratum. In the presence of a group action, we can discuss equivariant connections (those which commute with the group action).

Recall from Lemma 6.2 that $\mathfrak{D}$ maps to $\mathbb{C} \times \mathbb{U}_1$ and is a $\mathbb{U}_1$-bundle with a connection ∇^0 over the complement of 0×1. Let $\mathcal{A}$ be the stratification of $\mathbb{C} \times \mathbb{U}_1$ by 0×1 and its complement. Then $\mathfrak{D}$ carries a stratified connection ∇, where $\nabla(\gamma)$ is defined by parallel transport using ∇^0 over the open stratum. Since the closed stratum 0×1 is a point, it requires no extra data.

We now turn to $\mathfrak{D}(\Gamma)$. Let $n = |E(\Gamma)|$. Let $\mathcal{A}^n$ be the product stratification over $C_1(\Gamma, \mathbb{C} \times \mathbb{U}_1)$. Then $\mathfrak{D}^n$ carries the stratified connection ∇^n. Consider the residual $\mathbb{C} \times \mathbb{U}_1$ moment map $\pi_{\text{res}} \colon \mathfrak{D}(\Gamma) \to \mathrm{H}_1(\Gamma, \mathbb{C} \times \mathbb{U}_1)_\eta$. Let $\mathcal{A}(\eta)$ be the stratification in $\mathrm{H}_1(\Gamma, \mathbb{C} \times \mathbb{U}_1)_\eta$ given by intersecting with $\mathcal{A}^n$. Since ∇^n is $C^0(\Gamma, \mathbb{U}_1)$-equivariant, it descends to a $\mathcal{A}(\eta)$-connection ∇_Γ on $\mathfrak{D}(\Gamma)$.

Let $\bar{\alpha} \colon \mathrm{H}_1(\Gamma, \mathbb{C} \times \mathbb{U}_1) \to \mathrm{H}_1(\Gamma, \mathbb{U}_1) \to \mathbb{U}_1$ be the composition of α with the natural projection. Then $\pi_{\text{res}} \circ \bar{\alpha} = \mu_{\text{res}}^{\mathbb{U}_1} \circ \alpha$.

Let $\nu \in \mathbb{U}_1$. The intersection of $\bar{\alpha}^{-1}(\nu)$ with $\mathcal{A}(\eta)$ defines a stratification $\mathcal{A}(\eta)_\nu$ of $\bar{\alpha}^{-1}(\nu)$. For all but a finite set of $\zeta \in \mathbb{U}_1$, there exists an open neighborhood $\mathfrak{U}$ of ζ such that the combinatorial type of this stratification remains constant for $\nu \in \mathfrak{U}$. Fix such a ζ. We may choose a smooth family of stratification-preserving diffeomorphisms $f_\nu \colon \bar{\alpha}^{-1}(\zeta) \to \bar{\alpha}^{-1}(\nu)$ for $\nu \in \mathfrak{U}$, such that $f_\zeta = \mathrm{id}$.

Parallel transport for the $\mathcal{A}(\eta)$-connection ∇_Γ along the paths defined by f_ν defines an isomorphism of $(\mathbb{U}_1, \mathbb{U}_1)$-spaces $(\bar{\alpha} \circ \pi)^{-1}(\mathfrak{U}) \cong [(\bar{\alpha} \circ \pi)^{-1}(\zeta) \times \mathfrak{U}]$. ∎

Remark 6.34. There is an alternative approach to the above lemma. If $\mathbf{X}$ is a variety with an action of a reductive group G, equipped with a G-equivariant bundle $\mathcal{L}$, then one can define the quotient $\mathbf{X} /\!\!/_{\mathcal{L}} G$ in the sense of geometric invariant theory. Under certain conditions, $\mathrm{H}^2(\mathbf{X}, \mathbb{Q})$ contains a collection of open chambers such that the quotient $\mathbf{X} /\!\!/_{\mathcal{L}} G$ is isomorphic for all $\mathcal{L}$ in a given chamber. Since in our setting, $\mathbf{X}$ is merely a complex manifold and not a variety, we prefer to use smooth methods, but it should be possible to adapt those arguments to this setting.

6.5. **Retractions.** We will want to replace the $q_2^{-1}(0)$ and $q_2^{-1}(\epsilon)$ of Sequence 42 by spaces with more familiar names. We do this by constructing certain retraction maps.

Recall that according to Lemma 6.2, there is a map $\pi_{\mathfrak{D}} := \mu_{\mathfrak{D}}^{\mathbb{U}_1} \times q \colon \mathfrak{D} \to \mathbb{U}_1 \times \mathbb{C}$ giving the structure of a principal $\mathbb{U}_1$ bundle away from 1×0.

Lemma 6.35. *Let $\mathbb{D}^\epsilon$ be the open disk of radius $0 < \epsilon < 1$. There is a $\mathbb{U}_1$-equivariant deformation retraction of $\mathfrak{D}$ onto $q^{-1}(\mathbb{D}^\epsilon)$.*

Proof. Pick any $\mathbb{U}_1$-connection on this bundle (we do not require it to be flat). Then the linear retraction $\mathbb{U}_1 \times \mathbb{D} \to \mathbb{U}_1 \times \mathbb{D}^\epsilon$ induces a retraction of $\mathfrak{D}$ via parallel transport. □

Lemma 6.36. *There is a $\mathbb{U}_1$-equivariant deformation retraction of $\mathfrak{D}$ onto $q^{-1}(0)$.*

Proof. Once again, pick a $\mathbb{U}_1$ connection on the $\mathbb{U}_1$-bundle over $\mathbb{U}_1 \times \mathbb{D} \setminus (0 \times 1)$, and consider the linear retraction from $\mathbb{D}$ to 0. Parallel transport along this retraction extends to a continuous map onto $q^{-1}(0)$. □

We constructed the Dolbeault space $\mathfrak{D}(\Gamma)$ by taking the quotient $\mathfrak{D}^{E(\Gamma)} /\!\!/_\eta \overline{\mathbb{U}_1^{V(\Gamma)}}$ and restricting to the locus $q_{\text{res}}^{-1}(\mathrm{H}_1(\Gamma;\mathbb{C}))$. In fact, we will show below that the cohomology of the resulting space is unchanged if we replace $\mathrm{H}_1(\Gamma;\mathbb{C})$ by any star-convex subset of $\mathbb{C}^{E(\Gamma)}$. As we will see, this invariance also holds for the quotient $\mathfrak{D}^{E(\Gamma)} /\!\!/_\eta \mathbb{T}$ where $\mathbb{T} \subset \mathbb{U}_1^{V(\Gamma)}$ is any subtorus. The key idea is to show that everything in sight retracts onto the subvariety $(q^{E(\Gamma)})^{-1}(0) /\!\!/ \mathbb{T}$.

Let $r_t \colon \mathfrak{D} \to \mathfrak{D}$, $t \in [0,1]$ be the retraction onto $q^{-1}(0)$ constructed in Lemma 6.35, and let $r^{E(\Gamma)} \colon \mathfrak{D}^{E(\Gamma)} \to q^{-1}(0)^{E(\Gamma)}$ be the product retraction.

Lemma 6.37. *Let $\Lambda \subset \mathbb{C}^{E(\Gamma)}$ be star-convex. Then $r^{E(\Gamma)}$ restricts to a retraction of $(q^{E(\Gamma)})^{-1}(\Lambda)$ onto $(q^{E(\Gamma)})^{-1}(0)$.*

Proof. Recall $q \circ r_t$ covers a linear retraction $\mathbb{D} \to 0$, and $\mu_{\mathfrak{D}}^{\mathbb{U}_1} \circ r_t = \mu_{\mathfrak{D}}^{\mathbb{U}_1}$. It follows that $r_t^{E(\Gamma)}$ preserves Λ. □

Lemma 6.38. *Let $\Lambda \subset \mathbb{C}^{E(\Gamma)}$ be star-convex. Let $\mathbb{T} \subset \mathbb{U}_1^{V(\Gamma)}$ be a subtorus and $\zeta \in \mathrm{Lie}(\mathbb{T})^*$. Then $r^{E(\Gamma)}$ descends to a retraction $(q^{E(\Gamma)})^{-1}(\Lambda) /\!\!/_\zeta \mathbb{T}$ onto $(q^{E(\Gamma)})^{-1}(0) /\!\!/_\zeta \mathbb{T}$.*

Proof. Recall that $\mu_{\mathfrak{D}}^{\mathbb{U}_1} \circ r_t = \mu_{\mathfrak{D}}^{\mathbb{U}_1}$. It follows that $r_t^{E(\Gamma)}$ preserves the $\mathbb{T}$-moment map. It is also $\mathbb{U}_1^{V(\Gamma)}$-invariant, hence it descends to a retraction on the quotient. □

Corollary 6.39. *For any pair $\Lambda \subset \Lambda'$, where Λ, Λ' are star-convex, and for any torus $\mathbb{T} \subset \mathbb{U}_1^{V(\Gamma)}$, the inclusion $(q^{E(\Gamma)})^{-1}(\Lambda) /\!\!/ \mathbb{T} \to (q^{E(\Gamma)})^{-1}(\Lambda') /\!\!/ \mathbb{T}$ induces an isomorphism in cohomology.*

As an illustration, we state some special cases.

Corollary 6.40. *The inclusions $q_\Gamma^{-1}(0) \to \mathfrak{D}(\Gamma)$ and $\mathfrak{D}(\Gamma) \to \mathfrak{V}(\Gamma, \eta)$ induce isomorphisms in cohomology.*

Proof. We have $\mathfrak{D}(\Gamma) = (q^{E(\Gamma)})^{-1}(\Lambda)/\mathbb{T}$ where $\Lambda = \mathrm{H}_1(\Gamma;\mathbb{C})$ and $\mathbb{T} = \mathbb{U}_1^{V(\Gamma)}$. The same holds for $\mathfrak{V}(\Gamma, \eta)$ with $\Lambda = \mathbb{C}^{E(\Gamma)}$ and $\mathbb{T} = \mathbb{U}_1^{V(\Gamma)}$. Hence both isomorphism are special cases of Corollary 6.39. □

We now draw some cohomological consequences of these retractions. Let $\epsilon > 0$ lie in the unit disk. We have inclusions $i\colon \mathbf{X} \star q^{-1}(0) = q_2^{-1}(0) \to \mathbf{X} \star \mathfrak{D}$, $i_{\mathbf{n}}\colon \mathbf{X} \star \mathbf{n} \to \mathbf{X} \star \mathbf{S}_{\mathfrak{D}}$, and $i_\epsilon\colon q_2^{-1}(\epsilon) \to \mathbf{X} \star \mathfrak{D}$.

Lemma 6.41. *The following restriction maps are isomorphisms.*

(1) $\mathrm{H}^\bullet(\mathbf{X} \star \mathfrak{D}, \mathbb{Q}) \xrightarrow{i^*} \mathrm{H}^\bullet(q_2^{-1}(0), \mathbb{Q})$.
(2) $\mathrm{H}^\bullet(\mathbf{X} \star \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \xrightarrow{i_\epsilon^*} \mathrm{H}^\bullet(q_2^{-1}(\epsilon), \mathbb{Q})$.
(3) $\mathrm{H}^\bullet(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \xrightarrow{i_{\mathbf{n}}^*} \mathrm{H}^\bullet(\mathbf{X} \star \mathbf{n}, \mathbb{Q})$.

Proof. In each case, the claim follows from the existence of a retraction on the basic space.

(1) The retraction $\mathfrak{D} \to q^{-1}(0)$ constructed in Lemma 6.36 is a retraction of $(\mathbb{U}_1, \mathbb{U}_1)$-spaces, and thus induces a retraction $\mathbf{X} \star \mathfrak{D} \to \mathbf{X} \star q^{-1}(0) = q_2^{-1}(0)$.
(2) We have a retraction of $(\mathbb{U}_1, \mathbb{U}_1)$-spaces $\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \to q^{-1}(\epsilon)$ (Lemma 6.20), which then induces a retraction $\mathbf{X} \star \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \to \mathbf{X} \star q^{-1}(\epsilon)$.
(3) Recall that $\mathbf{S}_{\mathfrak{D}}$ is the preimage of $0 \times \mathbb{R}^{\leq 0}$ under $\pi_{\mathfrak{D}}$. The retraction of $\mathbb{R}^{\leq 0}$ to 0 induces a retraction of $(\mathbb{U}_1, \mathbb{U}_1)$-spaces $\mathbf{S}_{\mathfrak{D}} \to \mathbf{n}$, which in turn induces a retraction $\mathbf{X} \star \mathbf{S}_{\mathfrak{D}} \to \mathbf{X} \star \mathbf{n}$.

□

Consider the adjunction triangle 35 and its image under i^*. Taking hypercohomology yields a pair of long exact sequences, together with a map between them.

(43)
$$\begin{tikzcd}
\arrow[r] & \mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(\mathbf{X} \star \mathfrak{D}, \mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{H}^\bullet(\mathbf{X} \star \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \arrow[r] \arrow[d] & \cdots \\
\arrow[r] & \mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathfrak{D}; i_* i^* I_* I^! \mathbb{Q}) \arrow[r] & \mathrm{H}^\bullet(\mathbf{X} \star \mathfrak{D}; i_* i^* \mathbb{Q}) \arrow[r] & \mathrm{H}^\bullet(\mathbf{X} \star \mathfrak{D}; i_* i^* J_* J^* \mathbb{Q}) \arrow[r] & \cdots
\end{tikzcd}$$

The top sequence in this diagram is the long exact sequence of a pair, where the pair in question is the inclusion $\mathbf{X} \star (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}) \to \mathbf{X} \star \mathfrak{D}$.

Lemma 6.42. *The map of long exact sequences 43 is an isomorphism.*

Proof. Lemma 6.41, parts 1 and 3, show that two out of three of the intertwining maps are isomorphisms. Thus the remaining map must also be an isomorphism. □

Taking hypercohomology of the diagram in Proposition 6.28 over $q_2^{-1}(0)$ (equivalently, of its image under $i_* i^*$) gives a map of long exact sequences. Composing with the equivalence 43, and using Corollary 6.31 in the bottom left corner, we obtain an isomorphism of long exact sequences

(44)
$$\begin{tikzcd}
\arrow[r] & \mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \arrow[r] \arrow[d, "i_{\mathbf{n}}^*"] & \mathrm{H}^{\bullet}(\mathbf{X} \star \mathfrak{D}, \mathbb{Q}) \arrow[r] \arrow[d, "i^*"] & \mathrm{H}^{\bullet}(\mathbf{X} \star (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}), \mathbb{Q}) \arrow[r] \arrow[d, "i_{\epsilon}^*"] & \cdots \\
\arrow[r] & \mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{n}, \mathbb{Q}) \arrow[r] & \mathrm{H}^{\bullet}(q_2^{-1}(0), \mathbb{Q}) \arrow[r] & \mathrm{H}^{\bullet}(q_2^{-1}(\epsilon), \mathbb{Q}) \arrow[r] & \cdots
\end{tikzcd}$$

The bottom row is Sequence 42.

6.6. The deletion-contraction sequence.

We now set $\mathbf{X} = \mathfrak{D}(\Gamma/e)$ in Sequence 44, where e is a nonloop, nonbridge edge of Γ. We choose the lower left, upper middle, and upper right terms to represent the sequence. This gives

(45)
$$\longrightarrow \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e) \star \mathbf{n}, \mathbb{Q}) \longrightarrow \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star \mathfrak{D}, \mathbb{Q}) \longrightarrow \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \longrightarrow$$

Recall that the convolution $\star$ products above are $\star_{\mathbb{U}_1}$. We want to replace these by their submanifolds given by the corresponding the $\star_{\mathbb{U}_1 \times \mathbb{C}}$ products. Let us name the restrictions:

$$\mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1} \mathbf{n}, \mathbb{Q}) \xrightarrow{\kappa_{\mathbf{n}}^*} \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} \mathbf{n}, \mathbb{Q}) = \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma \setminus e), \mathbb{Q})$$

(46)
$$\mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1} \mathfrak{D}, \mathbb{Q}) \xrightarrow{\kappa^*} \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} \mathbf{n}, \mathbb{Q}) = \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma), \mathbb{Q})$$

$$\mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1} (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}), \mathbb{Q}) \xrightarrow{i_{\epsilon}^*} \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}), \mathbb{Q}) = \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e), \mathbb{Q})$$

Theorem/Definition 6.43. *Each of the restriction maps is an isomorphism. We may therefore define the lower row in the following diagram by requiring that the diagram commute.*

(47)
$$\begin{tikzcd}
\arrow[r] & \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e) \star \mathbf{n}, \mathbb{Q}) \arrow[r] \arrow[d, "\kappa_{\mathbf{n}}^*"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star \mathfrak{D}, \mathbb{Q}) \arrow[r] \arrow[d, "\kappa^*"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e) \star \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}, \mathbb{Q}) \arrow[r] \arrow[d, "i_{\epsilon}^*"] & {} \\
\arrow[r, dashed, "c_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma \setminus e), \mathbb{Q}) \arrow[r, dashed, "a_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma), \mathbb{Q}) \arrow[r, dashed, "b_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e), \mathbb{Q}) \arrow[r, dashed, "c_e^{\mathfrak{D}}"] & {}
\end{tikzcd}$$

We term this lower row the Dolbeault deletion-contraction sequence ($\mathfrak{D}$-DCS).

Proof. To see that the map $\kappa^*_{\mathbf{n}}$ is an isomorphism, note that we are in the setting of Corollary 6.39, with $\Lambda = \mathrm{H}_1(\Gamma/e, \mathbb{C})$, $\Lambda' = \mathrm{H}_1(\Gamma \setminus e, \mathbb{C})$ and $\mathbb{T} = \mathbb{U}_1^{V(\Gamma)}$.

Similarly, the isomorphism κ^* follows from Corollary 6.39 with $\Lambda' = \mathrm{H}_1(\Gamma/e, \mathbb{C}))$, $\Lambda = \mathrm{H}_1(\Gamma; \mathbb{C})$ and $\mathbb{T} = \mathbb{U}_1^{V(\Gamma)}$.

Finally, the isomorphism i^*_ϵ is Lemma 6.41, part 2. □

Remark 6.44. The top row and rightmost column in diagram (47) match the corresponding parts of Diagram 44, if the leftmost term is identified with $\mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q})$ via the pullback i^*.

Just as with the maps a_e^{Υ} and $a_e^{\mathfrak{B}}$, the maps $a_e^{\mathfrak{D}}$ for different edges e commute when their composition is defined. One can prove this directly from the definition of $a^{\mathfrak{D}}$, in much the same way we proved it for $a_e^{\mathfrak{B}}$. We prefer to defer this proof to Corollary 9.15, which deduces the commutation of $a_e^{\mathfrak{D}}$ from that of $a_e^{\mathfrak{B}}$. The reader can easily check that the logical flow of the paper is not affected.

The commutativity allows us to make the following special case of Definition 1.4.

Definition 6.45. The *Dolbeault deletion filtration* is the increasing filtration $D_r\mathrm{H}^n(\mathfrak{D}(\Gamma), \mathbb{Q})$ obtained from Definition 1.4, where the functor A takes Γ to $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q})$ and takes $\Gamma' \to \Gamma$ to the composition, in any order, of $a_e^{\mathfrak{D}}$ for $e \in \Gamma \setminus \Gamma'$.

Note the similarity with Definition 5.35.

6.7. **Another sequence.** We now define a variant on the deletion contraction sequence, given by the long exact sequence of a pair. The two sequences are intertwined by a collection of isomorphisms, so they contain essentially the same information. The virtue of the variant defined here is that it more closely ressembles the Betti deletion-contraction sequence, and will be easier to compare to it later.

We can restrict κ to the open subset $\mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}) \subset \mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} \mathfrak{D} = \mathfrak{D}(\Gamma)$ to obtain the following.

$$\begin{tikzcd} \mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}) \arrow[r, "\kappa_\circ"] \arrow[d] & \mathfrak{D}(\Gamma/e) \star (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}) \arrow[d, "q_{res} \times q"] \\ \mathrm{H}_1(\Gamma, \mathbb{C}) \arrow[r] & \mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C} \end{tikzcd} \tag{48}$$

Definition 6.46. The *Dolbeault pairs sequence* is the long exact sequence associated to the pair $\mathfrak{D}(\Gamma) \leftarrow \mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}})$.

By construction, we have a commutative diagram

$$\begin{tikzcd} \mathfrak{D}(\Gamma/e) \star \mathfrak{D} & \mathfrak{D}(\Gamma/e) \star (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}) \arrow[l] \\ \mathfrak{D}(\Gamma) \arrow[u, "\kappa"] & \mathfrak{D}(\Gamma/e) \star_{\mathbb{U}_1 \times \mathbb{C}} (\mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}}). \arrow[l] \arrow[u, "\kappa_\circ"] \end{tikzcd} \tag{49}$$

The associated map of long exact sequences of relative cohomology groups is

(50)

$$\begin{tikzcd}
\mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e)\star\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[r] \arrow[d, "\kappa_{\mathbf{n}}^*"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e)\star\mathfrak{D},\mathbb{Q}) \arrow[r] \arrow[d, "\kappa^*"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e)\star\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[r] \arrow[d, "\kappa_{\circ}^*"] & {} \\
\mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[r, dashed, "A_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "B_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}(\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}}),\mathbb{Q}) \arrow[r, dashed, "C_e^{\mathfrak{D}}"] & {}
\end{tikzcd}$$

Lemma 6.47. *The Dolbeault deletion-contraction sequence is obtained from the Dolbeault pairs sequence by requiring that the following diagram commute.*

(51)

$$\begin{tikzcd}
\mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[r, dashed, "A_e^{\mathfrak{D}}"] \arrow[d, equal] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "B_e^{\mathfrak{D}}"] \arrow[dd, equal] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}(\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}}),\mathbb{Q}) \arrow[r, dashed, "C_e^{\mathfrak{D}}"] & {} \\
\mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{n},\mathbb{Q}) \arrow[d, equal] & & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1}\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[u, "\kappa_{\circ}^*"'] & \\
\mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma\setminus e),\mathbb{Q}) \arrow[r, dashed, "a_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "b_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet}(\mathfrak{D}(\Gamma/e),\mathbb{Q}) \arrow[r, dashed, "c_e^{\mathfrak{D}}"] \arrow[u, "i_{\epsilon}^*"'] & {}
\end{tikzcd}$$

We have used the equality $\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{n} = \mathfrak{D}(\Gamma\setminus e)$.

7. An integrable system

The moduli of Higgs bundles famously carries the structure of a complex integrable system [H2], and the perverse Leray filtration of interest to [dCHM] is defined in terms of the map whose coordinates are the Hamiltonians of this system.

It is easy to see that our spaces $\mathfrak{D}(\Gamma)$ also carry the structures of complex integrable systems, as they are built from the elliptic fibration $q:\mathfrak{D}\to\mathbb{D}$. In this section we investigate the properties of these degenerating families of abelian varieties.

Remark 7.1. Let C be a nodal curve with rational components and dual graph Γ. Let B be the base of a locally versal family of deformations of C; it has dimension (# of nodes of C) = $|E(\Gamma)|$. Given an auxiliary choice of stability parameter η, there is a family of compactified Jacobians $\overline{\mathcal{J}}\to B$. ($\mathfrak{V}$ is for "versal".)

The family $q_{\mathrm{res}}:\mathfrak{V}(\Gamma,\eta)\to\mathbb{C}^{E(\Gamma)}$ is very similar to this family, although neither one is the base-change of the other. Meanwhile $q_{\mathrm{res}}:\mathfrak{D}(\Gamma)\to\mathrm{H}_1(\Gamma,\mathbb{C})$ is similar to the relative compactified Jacobian of a subfamily of deformations of C existing within an ambient symplectic 2-fold S.

Some relevant background on compactified Jacobians can be found in [OS, AK, AK2, Ale].

7.1. **Structure of the generic fiber.** We write $\mathbb{D}^{E(\Gamma)}_{\mathrm{reg}}$ for the complement of the coordinate hyperplanes. Our first step is to give a natural presentation of the fundamental group of a fiber over $\mathbb{D}^{E(\Gamma)}_{\mathrm{reg}}$. Let $b \in \mathbb{D}^{E(\Gamma)}_{\mathrm{reg}}$. We write $\mathfrak{V}(\Gamma)_b := q_{\mathrm{res}}^{-1}(b)$.

Lemma 7.2. *There is a natural short exact sequence of groups*

$$\mathrm{H}^1(\Gamma, \mathbb{Z}) \to \pi_1(\mathfrak{V}(\Gamma)_b) \to \mathrm{H}_1(\Gamma, \mathbb{Z}).$$

Proof. The basic space $\mathfrak{D}$ is defined as a $\mathbb{Z}$-quotient. Let $b \in \mathbb{D}^*$, and let $\mathbb{C}^* \to \mathbb{C}^*/b^{\mathbb{Z}} \cong q^{-1}(b)$ be the restriction of this quotient to the fiber. It induces an inclusion of fundamental groups, defining a short exact sequence

$$\pi_1(\mathbb{C}^*) \to \pi_1(q^{-1}(b)) \to \mathbb{Z},$$

where the image is identified via the inclusion to $\mathfrak{D}$ with $\pi_1(\mathfrak{D})$.

Now let $b \in \mathbb{D}^{E(\Gamma)}_{\mathrm{reg}}$. The point b determines a product of elliptic curves $\mathbf{E}_b := \prod_{e \in E(\Gamma)} q_e^{-1}(b_e)$ in $\mathfrak{D}^{E(\Gamma)}$, and $\mathfrak{V}(\Gamma)_b$ is the Kähler reduction of $\mathbf{E}_b$ by $\overline{\mathbb{U}_1^{V(\Gamma)}}$. More precisely, there is a moment map $\mu_\Gamma^{\mathbb{U}_1} \colon \mathbf{E}_b \to \mathbb{U}_1^{V(\Gamma)}$ for the action of $\overline{\mathbb{U}_1^{V(\Gamma)}}$ on $\mathbf{E}_b$, and $\mathfrak{V}(\Gamma)_b$ is the quotient of the fiber $\mathbf{E}_b(\eta)$ over η. Taking cartesian products of the basic sequence of fundamental groups, we obtain a sequence which we may write as

$$C^1(\Gamma, \mathbb{Z}) \to \pi_1(\mathbf{E}_b) \to C_1(\Gamma, \mathbb{Z}).$$

The inclusion $\mathbf{E}_b(\eta) \to \mathbf{E}_b$ gives the embedded short exact sequence

$$C^1(\Gamma, \mathbb{Z}) \to \pi_1(\mathbf{E}_b(\eta)) \to \mathrm{H}_1(\Gamma, \mathbb{Z}).$$

The quotient $\mathbf{E}_b(\eta)/\mathbb{U}_1^{V(\Gamma)}$ defines the quotient short exact sequence

$$\mathrm{H}^1(\Gamma, \mathbb{Z}) \to \pi_1(\mathfrak{V}(\Gamma)_b) \to \mathrm{H}_1(\Gamma, \mathbb{Z}).$$

$\square$

We now give a description of $\mathfrak{V}(\Gamma)_b$ as a group quotient.

Recall that b_e for $e \in E(\Gamma)$ be the coordinates of b in $\mathbb{C}^{E(\Gamma)}$. To alleviate notation, we will number the edges of Γ e_1 through e_n and write b_i for b_{e_i}. Given $\beta \in \mathrm{H}_1(\Gamma, \mathbb{Z})$, consider $b^\beta := (b_1^{\beta_1}, ..., b_n^{\beta_n}) \in \mathbb{C}^n$ where β_i are the coordinates of the image of β under the pullback $\mathrm{H}_1(\Gamma, \mathbb{Z}) \to C_1(\Gamma, \mathbb{Z})$. Since by assumption. all of the b_i are nonzero, b^β defines an element of $C^1(\Gamma, \mathbb{C}^*)$, and we write b^β for its image in $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$. This defines a map $\tau_b \colon \mathrm{H}_1(\Gamma, \mathbb{Z}) \to \mathrm{H}^1(\Gamma, \mathbb{C}^*)$. We write $b^{\mathrm{H}_1(\Gamma, \mathbb{Z})}$ for the image of τ_b.

Proposition 7.3. $b^{\mathrm{H}_1(\Gamma,\mathbb{Z})}$ *is a discrete lattice in* $\mathrm{H}^1(\Gamma, \mathbb{C}^*)$. *The fiber* $q_{\mathrm{res}}^{-1}(b)$ *is naturally isomorphic to the quotient* $\mathrm{H}^1(\Gamma, \mathbb{C}^*)/b^{\mathrm{H}_1(\Gamma,\mathbb{Z})}$.

Proof. Consider the cover $(\mathbb{C}^*)^{E(\Gamma)} \to \mathbf{E}_b$, obtained by taking the Cartesian product of the maps $\mathbb{C}^* \to \mathbb{C}^*/b_e^{\mathbb{Z}} \cong q^{-1}(b_e)$. The torus-valued moment map $\mu_\Gamma^{\mathbb{U}_1}$ lifts to a real-valued moment map

$\mu^{\mathbb{R}}_{\Gamma}\colon (\mathbb{C}^*)^{E(\Gamma)} \to \mathbb{R}^{V(\Gamma)}$. Pick any lift $\widetilde{\eta}$ of η; the quotient $(\mu^{\mathbb{R}}_{\Gamma})^{-1}(\widetilde{\eta})/\overline{\mathbb{U}_1^{V(\Gamma)}}$ is the Galois cover of $\mathfrak{V}(\Gamma)_b$ corresponding to the subgroup $\mathrm{H}^1(\Gamma,\mathbb{Z}) \subset \pi_1(\mathfrak{V}(\Gamma)_b)$. By the Kempf-Ness theorem, we can identify it with $(\mathbb{C}^*)^{E(\Gamma)}/(\mathbb{C}^*)^{V(\Gamma)} = \mathrm{H}^1(\Gamma,\mathbb{C}^*)$. We can compute the action of an element $\gamma \in \pi_1(\mathfrak{V}(\Gamma)_b)/\mathrm{H}^1(\Gamma,\mathbb{Z}) = \mathrm{H}_1(\Gamma,\mathbb{Z})$ on the cover by choosing a lift to $\pi_1(\mathbf{E}_b)$; we find it is given by multiplication by $\tau_b(\gamma)$. This proves the second claim.

Discreteness of the image of τ_b can be deduced from the fact that the quotient is a manifold. Here we give a direct proof.

The torus $C^1(\Gamma,\mathbb{C}^*)$ splits into a real and a compact factor : $C^1(\Gamma,\mathbb{C}^*) = C^1(\Gamma,\mathbb{U}_1)\times C^1(\Gamma,\mathbb{R}^{>0})$. Likewise, we have $\mathrm{H}^1(\Gamma,\mathbb{C}^*) = \mathrm{H}^1(\Gamma,\mathbb{U}_1) \times \mathrm{H}^1(\Gamma,\mathbb{R}^{>0})$. The exponential map defines isomorphisms $C^1(\Gamma,\mathbb{R}) \cong C^1(\Gamma,\mathbb{R}^{>0})$ and $\mathrm{H}^1(\Gamma,\mathbb{R}) \cong \mathrm{H}^1(\Gamma,\mathbb{R}^{>0})$. Postcomposing τ_b with the projection $\mathrm{H}^1(\Gamma,\mathbb{C}^*) \to \mathrm{H}^1(\Gamma,\mathbb{R}^{>0}) \cong \mathrm{H}^1(\Gamma,\mathbb{R})$ defines a map $\mathrm{H}_1(\Gamma,\mathbb{R}) \to \mathrm{H}^1(\Gamma,\mathbb{R})$. Tensoring the left-hand side with $\mathbb{R}$, we obtain a map of vector spaces

$$\overline{\tau_b}\colon \mathrm{H}^1(\Gamma,\mathbb{R}) \to \mathrm{H}^1(\Gamma,\mathbb{R}).$$

It is enough to show that this map is an isomorphism. Let $c_i = \log|b_i| < 0$, and let $[e] \in \mathrm{H}^1(\Gamma,\mathbb{Z})$ be the element of cohomology corresponding to the oriented edge e. Then $\overline{\tau_b}(\beta) = \sum_{i=1}^n c_i\beta(v_i)[e]$.

Define an inner product on $C_1(\Gamma,\mathbb{R})$ by $\langle \mathbf{x},\mathbf{x}'\rangle_c := \sum_{i=1}^n -c_i\mathbf{x}_e\mathbf{x}'_e$. Since it is manifestly positive definite, so is its pullback along the injection $\mathrm{H}_1(\Gamma,\mathbb{R}) \to C_1(\Gamma,\mathbb{R})$. $\overline{\tau_b}$ is the map $\mathrm{H}_1(\Gamma,\mathbb{R}) \to (\mathrm{H}_1(\Gamma,\mathbb{R}))^\vee = \mathrm{H}^1(\Gamma,\mathbb{R})$ given by $\beta \to \langle\beta,-\rangle_c$. It follows that it is an isomorphism, as was to be shown. $\square$

Corollary 7.4. *The restriction of* $q_{\mathrm{res}}\colon \mathfrak{V}(\Gamma) \to \mathbb{D}^{E(\Gamma)}$ *has a section.*

Proof. We can define such a section by taking the image of the unit section of the trivial fibration $(\mathbb{C}^*)^{E(\Gamma)}$ under the quotient map $(\mathbb{C}^*)^{E(\Gamma)} \to \mathrm{H}^1(\Gamma,\mathbb{C}^*) \to \mathfrak{V}(\Gamma)_b$ from Proposition 7.3. $\square$

Proposition 7.5. *For* $b \in \mathbb{D}^{E(\Gamma)}$ *in the complement of the coordinate hyperplanes,* $q_{\mathrm{res}}^{-1}(b)$ *is an abelian variety, i.e. a compact complex group admitting a projective embedding.*

Proof. We have shown that $\mathfrak{V}(\Gamma)_b$ is a compact abelian group. We will now show that $\mathfrak{V}(\Gamma)_b$ carries a Kähler form ω_b with integral pairings $\omega_b(\beta)$ for $\beta \in \mathrm{H}_2(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$. The Kodaira embedding theorem then tells us that $\mathfrak{V}(\Gamma)_b$ admits a projective embedding. Let $\widetilde{\omega}_b$ be the Kähler form on $\mathbf{E}_b$; recall that we have chosen it to be integral. We can represent any curve class $\beta \in \mathrm{H}_2(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$ as the image under the quotient map of a curve $\widetilde{\beta}$ in $\mathbf{E}(\eta)$. The Kähler form on $\mathfrak{V}(\Gamma)_b$ is obtained by reduction of that on $\mathbf{E}_b$, and thus $\omega_b(\beta) = \widetilde{\omega}_b(\widetilde{\beta}) \in \mathbb{Z}$.

$\square$

7.2. **Monodromy.** For any compact torus A, we have a natural isomorphism $\mathrm{H}^\bullet(A,\mathbb{C}) = \bigwedge^\bullet \mathrm{H}^1(A;\mathbb{C})$. Hence we have a graded local system $R^\bullet {q_{\mathrm{res}}}_* \mathbb{Q}_{\mathfrak{D}(\Gamma)}$ on $\mathbb{D}^{E(\Gamma)}_{\mathrm{reg}}$, with fiber at b given by $\bigwedge^\bullet \mathrm{H}^1(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$. The monodromy of this local system is determined by the monodromy in degree one. This is described as follows.

Proposition 7.6. *Fix an edge e of Γ, and consider the corresponding hyperplane in $C_1(\Gamma,\mathbb{C})$. The logarithm of the monodromy of $R^1 q_{\mathrm{res}\,*}\mathbb{Q}_{\mathfrak{D}(\Gamma)}$ around this hyperplane is given by the composition* $\mathrm{H}^1(\mathfrak{V}(\Gamma)_b,\mathbb{Z}) \to \mathrm{H}_1(\Gamma,\mathbb{Z}) \xrightarrow{\langle e,-\rangle[e]} \mathrm{H}^1(\Gamma,\mathbb{Z}) \to \mathrm{H}^1(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$.

Proof. Fix a basepoint b near the hyperplane $b_e = 0$. Fix bases $e_1 = e, e_2, .., e_g$ and $\gamma_1, ..., \gamma_g$ of $\mathrm{H}^1(\Gamma,\mathbb{Z})$ and $\mathrm{H}_1(\Gamma,\mathbb{Z})$ such that $\langle \gamma_i, e\rangle = 0$ for $i \neq 1$. Thus $\langle e,\gamma\rangle$ picks out the coefficient of γ_1 in $\gamma = \sum_i c_i\gamma_i$.

Recall the equality $\mathfrak{V}(\Gamma)_b = \mathrm{H}^1(\Gamma,\mathbb{C}^*)/b^{\mathrm{H}_1(\Gamma,\mathbb{Z})}$. Choose branches of the logarithms $\log(b_{e_i})$, and define $\tilde{\gamma}_i \in \mathrm{H}_1(\mathfrak{V}(\Gamma)_b)$ as the cycle

$$r \in [0,1] \to \{\exp\big(r\log(b_{e_i})\langle\gamma_i, e\rangle\big)\}_{e\in E(\Gamma)}. \tag{52}$$

Let $\tilde{e}_i \in \mathrm{H}_1(\mathrm{H}^1(\Gamma,\mathbb{C}^*),\mathbb{Z})$ be the tautological cycles; we abusively use the same notation for their projections to $\mathrm{H}_1(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$. Then $\tilde{e}_i$ and $\tilde{\gamma}_i$ form a basis of $\mathrm{H}_1(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$. By following the explicit cycle 52 as $b_0 \to \exp(2\pi i\theta)b_0$, one can verify the proposition.

Note, however, that the general form of the answer follows without any further calculations. Consider a small loop around the hyperplane $b_e = 0$, starting and ending at b. By construction, all the basis elements but γ_1 are globally defined along this loop; it follows that the log monodromy factors through $\mathrm{H}_1(\mathfrak{V}(\Gamma)_b,\mathbb{Z}) \to \mathrm{H}_1(\Gamma,\mathbb{Z}) \xrightarrow{\langle e,-\rangle} \mathbb{Z}$. Applying the same reasoning to the Poincaré dual basis in $\mathrm{H}_{2g-1}(\mathfrak{V}(\Gamma)_b,\mathbb{Z}) = \bigwedge^{2g-1}\mathrm{H}_1(\mathfrak{V}(\Gamma)_b,\mathbb{Z})$, we see that the image of log monodromy must lie in the span of e. ☐

7.3. **Structure of the special fiber.** By construction, $q_{\mathrm{res}}^{-1}(0)$ is the symplectic reduction of $q^{-1}(0)^{E(\Gamma)}$ by $\overline{C}^0(\Gamma,\mathbb{U}_1)$. It is easier to understand this reduction by first passing to the universal cover $\widetilde{q^{-1}(0)}^{E(\Gamma)}$ of $q^{-1}(0)^{E(\Gamma)}$. The universal cover of $q^{-1}(0)$ is an infinite chain of rational curves $\mathbb{P}^1_n$ which we index by the integers $n \in \mathbb{Z}$. Thus $\widetilde{q^{-1}(0)}^{E(\Gamma)}$ is an infinite grid of irreducible components $\prod_{e\in E(\Gamma)}\mathbb{P}^1_{n_e}$.

To understand the reduction of $\widetilde{q^{-1}(0)}^{E(\Gamma)}$, we will use Delzant's dictionary between polytopes and toric varieties [Delz], according to which a toric variety is classified by its image under the moment map. The moment map

$$\mu_{\mathfrak{D}}^{E(\Gamma)} : \widetilde{q^{-1}(0)}^{E(\Gamma)} \to C_1(\Gamma,\mathbb{R}) = \mathbb{R}^{E(\Gamma)}$$

maps the component indexed by $\mathbf{n} = \{n_e\}$ to the cube $\square_{\mathbf{n}} := \prod_{e\in E(\Gamma)}[n_e, n_e+1]$. These cubes are the chambers of the coordinate periodic hyperplane arrangement on $C_1(\Gamma,\mathbb{R})$.

Identify $\mathbb{U}_1$ with $\mathbb{R}/\mathbb{Z}$, and suppose that $\eta \in \overline{C}_0(\Gamma,\mathbb{Q}/\mathbb{Z})$. Let $\widetilde{\eta}$ be a lift to $\overline{C}_0(\Gamma,\mathbb{Q})$ with all components in the range $0 \leq \widetilde{\eta}_v \leq 1$. The affine subspace $d_\Gamma^{-1}(\widetilde{\eta}) \subset C_1(\Gamma,\mathbb{R})$ intersects the coordinate periodic arrangement in a $\mathrm{H}_1(\Gamma,\mathbb{Z})$-periodic arrangement A^{per}. It is given by the hyperplanes $\langle\gamma, e\rangle = n + \tilde{\eta}_e$ for $e \in E(\Gamma), n \in \mathbb{Z}$. By our genericity assumptions on η, A^{per}

is a simple unimodular arrangement, i.e. any k hyperplanes intersects in codimension k, and the integral normal vectors at such an intersection span the lattice $\mathrm{H}_1(\Gamma,\mathbb{Z})$.

The chambers of A^{per} are given by $\Delta_{\mathbf{n}} := \Box_{\mathbf{n}} \cap d_\Gamma^{-1}(\widetilde{\eta})$, for those $\mathbf{n}$ such that the right-hand side is nonempty. Each such chamber $\Delta_{\mathbf{n}}$ corresponds to a component $\mathfrak{X}_{\mathbf{n}} := \prod_{e\in E(\Gamma)} \mathbb{P}^1_{n_e} /\!\!/_{\widetilde{\eta}} \overline{C}^0(\Gamma,\mathbb{U}_1)$ of the reduction. The reduction of a toric variety by a torus action is toric, with moment map obtained by restriction from the moment map of the prequotient, and in particular the moment image of $\mathfrak{X}_{\mathbf{n}}$ is $\Delta_{\mathbf{n}}$. The components $\mathfrak{X}_{\mathbf{n}}$ and $\mathfrak{X}_{\mathbf{m}}$ intersect along the sub-toric variety determined by the mutual face $\Delta_{\mathbf{n}} \cap \Delta_{\mathbf{m}}$ of their polytopes.

We thus obtain a description of $\widetilde{q_{\mathrm{res}}^{-1}(0)}$ as a union of smooth toric varieties glued along toric subvarieties, whose moment map defines an infinite periodic hyperplane arrangement. The fiber $q_{\mathrm{res}}^{-1}(0)$ itself is obtained by quotienting this picture by $\mathrm{H}_1(\Gamma,\mathbb{Z})$.

7.4. **Projectivity.** Here we show that q_{res} is projective, at least near the central fiber. The argument is independent of Proposition 7.5.

Recall the definition of $\widetilde{\mathfrak{D}}$ from Section 6.1. We define a line bundle $\mathcal{L}$ on $\widetilde{\mathfrak{D}}$ with transition function x_n on the overlaps $\mathbb{C}^2_n \cap \mathbb{C}^2_{n+1}$. $\mathcal{L}$ is naturally equivariant with respect to the $\mathbb{C}^*$ and $\mathbb{Z}$ actions. It is not, however, jointly equivariant. Instead, if $s_1^*\mathcal{L}$ denotes the shift of $\mathcal{L}$ by $1 \in \mathbb{Z}$, we have an equality of $\mathbb{C}^*$-equivariant bundles $s_1^*\mathcal{L} = \chi\mathcal{L}$ where χ is the fundamental character of $\mathbb{C}^*$.

The following proposition is direct from the definition of $\mathcal{L}$:

Proposition 7.7. *$\mathcal{L}$ restricts to an ample bundle on any finite chain of rational curves in the fiber $\tilde{q}^{-1}(0) \subset \widetilde{\mathfrak{D}}$.*

Theorem 7.8. *There is a line bundle $\mathcal{L}_\Gamma$ on $\mathfrak{V}(\Gamma,\eta)$ and an open neigborhood of $0 \in \mathrm{H}_1(\Gamma,\mathbb{C})$ such that for any b in this neighborhood, the restriction of $\mathcal{L}_\Gamma$ to $q_{\mathrm{res}}^{-1}(b)$ defines a projective embedding.*

Proof. We will construct such a bundle starting from the bundle $\mathcal{L}$ in Proposition 7.7. By Proposition 1.4 of [Nak], if $f\colon X \to S$ is a proper map of complex manifolds, and a line bundle L on X is ample on a given fiber, then it is relatively ample over a neighborhood of the image. Thus after constructing $\mathcal{L}_\Gamma$, it will be enough to check its ampleness on the central fiber.

Let $\phi \in C_1(\Gamma,\mathbb{Z})$ satisfy $d_\Gamma(\phi) = N\widetilde{\eta}$ for some integer N. Consider the $C^1(\Gamma,\mathbb{C}^*)$-equivariant bundle

$$\mathcal{L}_\phi^N := \phi \otimes \left(\boxtimes_{e\in E(\Gamma)} \mathcal{L}_e^N \right)$$

on $\widetilde{\mathfrak{D}}^{E(\Gamma)}$. It also carries an action of $C_1(\Gamma,\mathbb{Z})$ which does not commute with the torus action. The image of $\overline{C}^0(\Gamma,\mathbb{C}^*) \to C^1(\Gamma,\mathbb{C}^*) = (\mathbb{C}^*)^n$, however, commutes with the action of $\mathrm{H}_1(\Gamma,\mathbb{Z})$. Thus $\mathcal{L}_\phi^N$ descends to a $\mathrm{H}_1(\Gamma,\mathbb{Z})$-equivariant bundle $\widetilde{\mathcal{L}}_\Gamma$ on $\widetilde{\mathfrak{D}}^n /\!\!/_\eta \overline{C}^0(\Gamma,\mathbb{U}_1)$.

A component by component application of the dictionary between polytopes and toric varieties shows that $\widetilde{q_{\mathrm{res}}^{-1}(0)}$ is the union of GIT quotients $\prod_{e\in E(\Gamma)} \mathbb{P}^1_{n_e} /\!\!/_{\mathcal{L}_\phi^N} \overline{C}^0(\Gamma,\mathbb{C}^*)$, glued along GIT quotients of subvarieties, such that $\widetilde{\mathcal{L}}_\Gamma$ restricts to the GIT bundle $\mathcal{O}(1)$ on any component. It follows that $\widetilde{\mathcal{L}}_\Gamma$ is ample on any finite union of components.

We can now conclude that the descent $\mathcal{L}_\Gamma$ of $\widetilde{\mathcal{L}}_\Gamma$ to $q_{\mathrm{res}}^{-1}(0)$ is also ample, by the same argument as in [Mum2], Theorem 3.10. □

8. Some compatibilities of perverse Leray filtrations

Given a map $f : X \to B$ of algebraic varieties, the middle perverse t-structure on B induces a filtration – the perverse Leray filtration – on the cohomology of X. We recall some facts about this filtration in Appendix A.5.

Convention 8.1. When we speak without further qualification of 'the' perverse Leray filtration on $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q})$, we mean the one associated to the map $q_{\mathrm{res}} \colon \mathfrak{D}(\Gamma) \to \mathrm{H}_1(\Gamma, \mathbb{C})$. Likewise, by 'the' perverse Leray filtration on $\mathrm{H}^\bullet(\mathfrak{V}(\Gamma), \mathbb{Q})$, we mean the one associated to $q_{\mathrm{res}} \colon \mathfrak{V}(\Gamma) \to \mathbb{C}^{E(\Gamma)}$.

Per this convention, each term of the Dolbeault deletion-contraction sequence (dashed sequence in Diagram 47) carries a perverse Leray filtration. We will translate the filtration on the left-hand term by one step - this is analogous to the Tate twist occuring in the $\mathfrak{B}$-DCS. We denote the resulting filtered vector space by $\mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q})\{-1\}$, so that $P_k\mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q})\{-1\} = P_{k-1}\mathrm{H}^{\bullet-2}(\mathbf{X} \star \mathbf{S}_{\mathfrak{D}}, \mathbb{Q})$. That is, we consider:

$$\xrightarrow{c_e^{\mathfrak{D}}} \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma \setminus e), \mathbb{Q})\{-1\} \xrightarrow{a_e^{\mathfrak{D}}} \mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q}) \xrightarrow{b_e^{\mathfrak{D}}} \mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e), \mathbb{Q}) \xrightarrow{c_e^{\mathfrak{D}}} \tag{53}$$

We wish to investigate the extent to which this sequence is compatible with perverse Leray filtrations. To our knowledge, compatibility (much less strictly compatibility) does not follow from general considerations. (Recall for comparison that it did follow from general considerations that the Betti deletion contraction sequence strictly preserves the weight filtrations.) We will eventually show this strict compatibility in Corollary 9.34, but only by *first* proving that P=W compatibly with an intertwining of the deletion-contraction sequences. However, to prove those results, we will need to know something about the perverse Leray filtrations. We establish the necessary results here.

If we identify the top middle term of Diagram 47 with $\mathrm{H}^\bullet(q_2^{-1}(0), \mathbb{Q})$, then the sequence composed of the top-left, top-middle and bottom-left terms is sequence 42. We show in Proposition 8.2 below that this sequence (not necessarily strictly) preserves the perverse Leray filtrations on each term. To transfer this result to the sequence of interest, we must show that the isomorphisms $\kappa_{\mathbf{n}}^*$ and κ^* preserve the perverse Leray filtrations in Diagrams 55 and 56. In Section 8.2 we take up this problem.

8.1. Compatibility of Sequence 42 with perverse filtrations.

Suppose the space $\mathbf{X}$ comes with a proper $\mathbb{U}_1$-invariant map $q_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{B}$. This allows us to define three filtrations:

- $\mathrm{H}^\bullet(\mathbf{X}, \mathbb{Q})$ carries the perverse Leray filtration associated to $q_{\mathbf{X}}$.
- The map $q_{\mathbf{X}} \times q \colon \mathbf{X} \star \mathfrak{D} \to \mathbf{B} \times \mathbb{C}$ is proper, and endows the cohomology of the latter space with a perverse Leray filtration. We can transport this filtration to $\mathrm{H}^\bullet(q_2^{-1}(0), \mathbb{Q})$ via the pullback by $i \colon q_2^{-1}(0) \to \mathbf{X} \star \mathfrak{D}$ (the middle vertical map in Diagram 44).
- One can restrict $q_{\mathbf{X}}$ to the closed submanifold $\mathbf{X} \star \mathbf{n} \subset \mathbf{X} \star \mathfrak{D}$ to obtain a proper map $q_{\mathbf{X}}^{\mathbf{n}}$ and endow the cohomology of that space with a perverse Leray filtration.

Proposition 8.2. *Fix a $\mathbb{U}_1$-invariant map $q_{\mathbf{X}} \colon \mathbf{X} \to \mathbf{B}$ and filter the terms of sequence 42 as above. Then the maps of this sequence are compatible with the filtrations on each term.*

Proof. We will need the following general fact. Suppose $\mathcal{K}$ is a complex of sheaves on a space $\mathbf{Y}$ equipped with a map $f \colon \mathbf{Y} \to \mathbb{C}$. Then the nearby-vanishing triangle defines a long exact sequence

$$\to \mathbb{H}^\bullet(f^{-1}(0); \Phi_f \mathcal{K}) \to \mathrm{H}^\bullet(f^{-1}(0), \mathbb{Q}) \to \mathbb{H}^\bullet(f^{-1}(0); \Psi_f \mathcal{K}) \xrightarrow{[1]}$$

Let $\iota \colon f^{-1}(0) \to \mathbf{Y}$ be the inclusion, and suppose it induces an isomorphism in cohomology. In general, the perverse filtrations on $\mathrm{H}_0 := \mathrm{H}^\bullet(f^{-1}(0), \mathbb{Q})$ arising from $\mathcal{K}$ and $\iota^*\mathcal{K}$ will differ. Suppose we endow H_0 with the filtration coming from $\mathcal{K}$. Then, since nearby and vanishing cycles functors preserve respect the perverse t-structure, the sequence

$$\to \mathrm{H}^\bullet(\Phi_f \mathcal{K}) \to \mathrm{H}_0 \to \mathrm{H}^\bullet(\Psi_f \mathcal{K}) \to \tag{54}$$

preserves perverse filtrations.

Returning to our setting, we have a map $(q_{\mathbf{X}} \times q_2) \colon \mathbf{X} \star \mathfrak{D} \to \mathbf{B} \times \mathbb{C}$. Recall that the long exact sequence of Proposition 6.32 was obtained by taking the hypercohomology of the triangle 36 in $D^b(q_2^{-1}(0))$. We may instead consider the pushforward of this triangle by $q_{\mathbf{X}}$ to obtain a triangle $q_{\mathbf{X}*}\Phi_{q_2}\mathbb{Q} \to q_{\mathbf{X}*}\mathbb{Q}_{q_2{}^{-1}(0)} \to q_{\mathbf{X}*}\Psi_{q_2}\mathbb{Q} \to$ in $D^b(\mathbf{B})$, inducing the same long exact sequence upon taking hypercohology. In fact, the image is the nearby-vanishing triangle $\Phi_r(q_{\mathbf{X}} \times q_2)_*\mathbb{Q} \to ((q_{\mathbf{X}} \times q_2)_*\mathbb{Q})_{r^{-1}(0)} \to \Psi_r(q_{\mathbf{X}} \times q_2)_*\mathbb{Q} \to$ over the zero fiber of the map $r \colon \mathbf{B} \times \mathbb{C} \to \mathbb{C}$. Here we are using the properness of $q_{\mathbf{X}} \times q_2$ to switch the order in which we take vanishing cycles and pushforwards.

The associated long exact sequence is a special case of sequence 54, and preserves the corresponding perverse filtrations on the cohomologies. In order to conclude the proof, we must identify the filtered spaces $\mathbb{H}^\bullet(\Phi_r(q_{\mathbf{X}} \times q_2)_*\mathbb{Q})$ and $\mathrm{H}^\bullet(\mathbf{X} \star \mathbf{n}, \mathbb{Q}[2])$, and the filtered spaces $\mathbb{H}^\bullet(\Psi_r(q_{\mathbf{X}} \times q_2)_*\mathbb{Q})$ and $\mathrm{H}^\bullet(\mathbf{X}, \mathbb{Q})$. The former identification is immediate from $\Phi_r(q_{\mathbf{X}} \times q_2)_*\mathbb{Q} = q_{\mathbf{X}*}\Phi_{q_2}\mathbb{Q} = q_{\mathbf{X}*}\mathbb{Q}_{\mathbf{X}\star\mathbf{n}}[2]$. The latter follows from the fact that along the open half-disk $\mathbb{D}^1_> := \mathbb{D}^1 \cap \Re(z) > 0$, $q_2 \colon \mathbf{X} \star \mathfrak{D} \to \mathbb{C}$ can be trivialized as $\mathbf{X} \times \mathbb{D}^1_>$, so that $q_{\mathbf{X}} \times q_2$ is the product of the natural maps on each factor. $\square$

Remark 8.3. Note we have not claimed in the proposition that the maps in question are *strictly* compatible with the perverse filtration. We do not know if this is true in general, though it seems possible the question is related to the decomposition theorem. In the case of interest, we will establish strict compatibility only after identifying the perverse filtration with the deletion filtration.

8.2. **Compatibility of the Dolbeault deletion map with the perverse Leray filtration.** We now show that the maps $\kappa_{\mathbf{n}}^*$ and κ^* from Equation 46 preserve the perverse filtration. The role of this result is to show that the deletion map $a_e^{\mathfrak{D}}$, introduced above, (not necessarily strictly) preserves the perverse Leray filtrations.

The target and domain of κ and $\kappa_{\mathbf{n}}$ are all quotients of subspaces of $\mathfrak{D}^{E(\Gamma)}$. The map $q^{E(\Gamma)} \colon \mathfrak{D}^{E(\Gamma)} \to \mathbb{C}^{E(\Gamma)}$ descends to a map from the target (resp domain) to $\mathbb{C}^{E(\Gamma)}$, endowing their cohomologies with perverse Leray filtrations. More precisely, the relevant filtrations are defined by vertical maps in the following diagrams:

For $\kappa_{\mathbf{n}}$,

$$\begin{array}{ccc} \mathfrak{D}(\Gamma \setminus e) & \xrightarrow{\kappa_{\mathbf{n}}} & \mathfrak{D}(\Gamma/e) \star \mathbf{n} \\ \downarrow & & \downarrow{\scriptstyle q_{res}} \\ \mathrm{H}_1(\Gamma \setminus e, \mathbb{C}) & \longrightarrow & \mathrm{H}_1(\Gamma/e, \mathbb{C}) \end{array} \tag{55}$$

Here the bottom arrow is the pushforward along the composition $\Gamma \setminus e \to \Gamma \to \Gamma/e$. The right-hand vertical map defines the perverse Leray filtration on $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e) \star \mathbf{n}, \mathbb{Q})$.

Similarly, we have

$$\begin{array}{ccc} \mathfrak{D}(\Gamma) & \xrightarrow{\kappa} & \mathfrak{D}(\Gamma/e) \star \mathfrak{D} \\ \downarrow & & \downarrow{\scriptstyle q_{res} \times q} \\ \mathrm{H}_1(\Gamma, \mathbb{C}) & \longrightarrow & \mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C} \end{array} \tag{56}$$

Here the bottom row is the pushforward along $\Gamma \to \Gamma/e$ on the first factor, and the projection $\gamma \to \langle \gamma, e \rangle$ on the second factor. The right-hand vertical map defines the perverse Leray filtration on $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e) \star \mathfrak{D}, \mathbb{Q})$.

8.2.1. *Review of relevant maps.* We will start by studying the discriminants of the simple maps $q \colon \mathfrak{D} \to \mathbb{C}$ and $q^{E(\Gamma)} \colon \mathfrak{D}^{E(\Gamma)} \to \mathbb{C}^{E(\Gamma)}$. We will then deduce the discriminant locus of the versal family

$$q_{\mathrm{res}} \colon \mathfrak{V}(\Gamma) \to \mathbb{C}^{E(\Gamma)}.$$

It is descended from the second map above using the $\mathbb{U}_1$-invariance of q. Next, we will study the discriminants of various subfamilies of the versal family. The first is

$$\begin{array}{ccc} \mathfrak{D}(\Gamma) & \longrightarrow & \mathfrak{V}(\Gamma) \\ \downarrow & & \downarrow \\ \mathrm{H}_1(\Gamma, \mathbb{C}) & \longrightarrow & \mathbb{C}^{E(\Gamma)} \end{array} \tag{57}$$

The second subfamily is produced as follows. We start with the previous diagram, but substituting Γ/e for Γ:

$$\begin{tikzcd} \mathfrak{D}(\Gamma/e) \arrow[r] \arrow[d] & \mathfrak{V}(\Gamma/e) \arrow[d] \\ \mathrm{H}_1(\Gamma/e,\mathbb{C}) \arrow[r] & \mathbb{C}^{E(\Gamma/e)} \end{tikzcd} \tag{58}$$

By taking the $\star$-product with $\mathfrak{D}$ and using Lemma 4.19 on the top-right corner, we obtain the diagram

$$\begin{tikzcd} \mathfrak{D}(\Gamma/e)\star\mathfrak{D} \arrow[r] \arrow[d] & \mathfrak{V}(\Gamma/e)\star\mathfrak{D} = \mathfrak{V}(\Gamma) \arrow[d] \\ \mathrm{H}_1(\Gamma/e,\mathbb{C})\times\mathbb{C} \arrow[r] & \mathbb{C}^{E(\Gamma/e)}\times\mathbb{C} = \mathbb{C}^{E(\Gamma)} \end{tikzcd} \tag{59}$$

It extends Diagram 56 to the right.

We pause to remark on the somewhat confusing bottom arrow. The image of $\mathrm{H}_1(\Gamma/e,\mathbb{C})$ is the subspace of $\mathbb{C}^{E(\Gamma)} = C_1(\Gamma;\mathbb{C})$ consisting of chains in $C_1(\Gamma\setminus e;\mathbb{C})$ with boundary a multiple of $t(e)-h(e)$. The image of the second factor $\mathbb{C}$ is $\mathbb{C}e$. Thus the image of the lower map is the joint span of $\mathrm{H}_1(\Gamma,\mathbb{C})$ and $\mathbb{C}e$. Similarly, we have the diagram

$$\begin{tikzcd} \mathfrak{D}(\Gamma/e)\star\mathbf{n} \arrow[r] \arrow[d] & \mathfrak{V}(\Gamma/e)\star\mathbf{n} = \mathfrak{V}(\Gamma\setminus e) \arrow[d] \\ \mathrm{H}_1(\Gamma/e,\mathbb{C}) \arrow[r] & \mathbb{C}^{E(\Gamma/e)} \end{tikzcd} \tag{60}$$

where we have used Lemma 4.20 in the top right corner. It extends Diagram 55 to the right.

8.2.2. *Determination of higher discriminants.* Microlocal methods [KS] can be used to control the interaction of the perverse filtration with base change; we review the relevant facts in Appendix A.5. One must determine various degeneracy loci of maps, which can be characterized in terms of derivatives.

Consider first the basic map $q\colon \mathfrak{D}\to\mathbb{D}^1$. It is a submersion away from a single point, namely $(q\times\mu_{\mathfrak{D}})^{-1}(0\times 1)$. In particular, $\Delta^0(q)=\mathbb{D}^1$ and $\Delta^1(q)=\{0\}$. The following observation will be useful later:

Lemma 8.4. *At any point $z\in\mathfrak{D}$, we have* $\ker dq + \ker d\mu_{\mathfrak{D}} = T_z\mathfrak{D}$.

Proof. Over $(q\times\mu_{\mathfrak{D}})^{-1}(0\times 1)$, this holds since $\ker dq$ is the entire tangent space. At any other point $q\times\mu_{\mathfrak{D}}$ is a submersion, and $\dim(\ker dq+\ker d\mu_{\mathfrak{D}}) = \dim\ker dq + \dim\ker d\mu_{\mathfrak{D}} - \dim\ker d(q\times\mu_{\mathfrak{D}}) = 2+3-1=4$. ∎

Consider now $\mathfrak{D}^{E(\Gamma)}$. We can calculate the images of the differential by taking direct sums. For convenience of notation, we introduce for $z \in \mathfrak{D}^{E(\Gamma)}$ the subset $R(z) \subset E(\Gamma)$ of edges e with the property that $(q_e \times \mu_{\mathfrak{D},e})(z) = 0 \times 1$. Evidently:

$$dq(T_z\mathfrak{D}^{E(\Gamma)}) = \mathbb{C}^{E(\Gamma\setminus R(z))} \subset T^*_{q(z)}\mathbb{D}^{E(\Gamma)} \cong \mathbb{C}^{E(\Gamma)} \tag{61}$$

Definition 8.5. A subset $R \subset E(\Gamma)$ is said to be independent if its elements are linearly independent in $\mathrm{H}^1(\Gamma, \mathbb{C})$. Equivalently, $h_1(\Gamma \setminus R) = h_1(\Gamma, \mathbb{C}) - |R|$.

Lemma 8.6. *For η generic (see Definition 4.12), $R(z)$ is independent for all $z \in \mu_\Gamma^{-1}(\eta)$.*

Proof. By definition, η is generic if for all $z \in \mu_\Gamma^{-1}(\eta)$, the graph $\Gamma \setminus R(z)$ is connected. Thus $\chi(\Gamma \setminus R(z)) = 1 - h_1(\Gamma \setminus R(z))$. On the other hand, we have

$$\chi(\Gamma \setminus R(z)) = \chi(\Gamma) + |R(z)| = 1 - h_1(\Gamma) + |R(z)|.$$

The proposition follows. $\square$

Remark 8.7. In fact, all such independent subsets appear as some $R(z)$. We will not need this fact for our results, though we use it below in stating characterizations of discriminants with $=$ rather than $\subset$ (for us, only the bound $\subset$ is important).

Proposition 8.8. *Consider the smooth submanifold $\mu_\Gamma^{-1}(\eta) \xrightarrow{j} \mathfrak{D}^{E(\Gamma)}$. Let $\widetilde{q}\colon Y = \mu_\Gamma^{-1}(\eta) \to \mathbb{C}^{E(\Gamma)}$ be the restriction of $q^{E(\Gamma)}$. Then $\widetilde{q}_\dagger(T^*_Y Y)$ is the union of conormals over all independent subsets $R \subset E(\Gamma)$ of the conormal bundles to the coordinate spaces $\mathbb{D}^{E(\Gamma\setminus R)} \subset \mathbb{D}^{E(\Gamma)}$. Equivalently*

$$\Delta^i(\widetilde{q}) = \bigcup_{\substack{R\subset E(\Gamma)\\ |R|=i\\ R \text{ independent}}} \bigcap_{e\in R} \{t_e = 0\}.$$

Proof. Let us determine the image of $d\widetilde{q} = dq^{E(\Gamma)} \circ dj$.

As the image of j is a smooth fiber of μ_Γ, we have $\mathrm{Image}(dj) = \ker(d\mu_\Gamma)$. We claim that $\ker d\mu_\Gamma + \ker dq^{E(\Gamma)} = T_z\mathfrak{D}^{E(\Gamma)}$. Indeed, since μ_Γ factors through $\mu^{E(\Gamma)}$, we have $\ker d\mu_\Gamma \supset \ker d\mu^{E(\Gamma)}$. Hence it is enough to show $\ker d\mu_{\mathfrak{D}}^{E(\Gamma)} + \ker dq^{E(\Gamma)} = T_z\mathfrak{D}^{E(\Gamma)}$, which follows from Lemma 8.4. Thus

$$\mathrm{Image}(d\widetilde{q})(T_z\mu_\Gamma^{-1}(\eta)) = dq^{E(\Gamma)}(\mathrm{Image}(dj)) = dq^{E(\Gamma)}(\ker d\mu_\Gamma) =$$

$$= dq^{E(\Gamma)}(\ker d\mu_\Gamma + \ker dq^{E(\Gamma)}) = dq^{E(\Gamma)}(T_z\mathfrak{D}^{E(\Gamma)}) = \mathbb{C}^{E(\Gamma\setminus R(z))}$$

Combined with observation that the $R(z)$ which may appear for $z \in \mu_\Gamma^{-1}(\eta)$ are independent (Lemma 8.6), the result follows. $\square$

Corollary 8.9. *Let $q_{\mathrm{res}}\colon \mathfrak{V}(\Gamma) \to \mathbb{D}^{E(\Gamma)}$ be the above map. Then $\Delta^i(q_{\mathrm{res}})$ is the union of $\mathbb{D}^{E(\Gamma\setminus R)}$ over all independent sets R satisfying $|R| \geq i$. In other words:*

$$\Delta^i(q_{\mathrm{res}}) = \bigcup_{\substack{R\subset E(\Gamma)\\ |R|=i \\ R \text{ independent}}} \bigcap_{e\in R} \{t_e = 0\}.$$

Proof. $\overline{\mathbb{U}_1^{V(\Gamma)}}$ acts freely on $\mu_\Gamma^{-1}(\eta)$ preserving the map $\widetilde{q}$, and the induced map on the quotient is by definition q_{res}. Thus $\widetilde{q}$ and q_{res} have the same discriminants, so the statement follows from Proposition 8.8. □

We now turn to $q_{\mathrm{res}}\colon \mathfrak{D}(\Gamma) \to \mathrm{H}_1(\Gamma, \mathbb{C})$.

Lemma 8.10. *The inclusion $\mathrm{H}_1(\Gamma, \mathbb{C}) \to \mathbb{C}^{E(\Gamma)}$ is transverse to $\mathbb{D}^{E(\Gamma\setminus R)}$ for all independent sets $R \subset E(\Gamma)$.*

Proof. We have $\dim \mathrm{H}_1(\Gamma, \mathbb{C}) + \dim \mathbb{D}^{E(\Gamma\setminus R)} - \dim \mathbb{C}^{E(\Gamma)} = h_1(\Gamma, \mathbb{C}) - |R|$. Thus it is enough to show that $\dim(H_1(\Gamma, \mathbb{C}) \cap \mathbb{D}^{E(\Gamma\setminus R)}) \leq h_1(\Gamma, \mathbb{C}) - |R|$. But this is immediate from the condition that R be independent. □

Corollary 8.11. *Any linear subspace containing $\mathrm{H}_1(\Gamma, \mathbb{C})$ is transverse to $\mathbb{D}^{E(\Gamma\setminus R)}$ for all independent sets $R \subset E(\Gamma)$.*

Proposition 8.12. *The discriminant locus $\Delta^i(q_{\mathrm{res}})$ of $q_{\mathrm{res}}\colon \mathfrak{D}(\Gamma) \to \mathrm{H}_1(\Gamma, \mathbb{C})$ is the union of $\mathbb{D}^{E(\Gamma\setminus R)} \cap \mathrm{H}_1(\Gamma, \mathbb{C})$ over all independent sets $R \subset E(\Gamma)$ of size i.*

Proof. By Lemma A.16 applied to Corollary 8.9, it suffices to check that $\mathrm{H}_1(\Gamma, \mathbb{C})$ is transverse to all independent subsets, which is true by Lemma 8.10. □

Lemma 8.13. *The inclusion $\mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C} \to \mathbb{C}^{E(\Gamma)}$ is transverse to $\mathbb{D}^{E(\Gamma\setminus R)}$ for all independent sets $R \subset E(\Gamma)$.*

Proof. The image of $\mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C}$ contains $\mathrm{H}_1(\Gamma, \mathbb{C})$. Thus the result follows from Corollary 8.11.

□

Proposition 8.14. *The discriminant locus $\Delta^i(q_{\mathrm{res}})$ of $q_{\mathrm{res}}\colon \mathfrak{D}(\Gamma/e) \star \mathfrak{D} \to \mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C}$ is the union of $\mathbb{D}^{E(\Gamma\setminus R)} \cap \mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C}$ over all independent sets $R \subset E(\Gamma)$ of size i.*

Proof. By Lemma A.16 applied to Corollary 8.9, it suffices to check that $\mathrm{H}_1(\Gamma/e, \mathbb{C}) \times \mathbb{C}$ is transverse to all independent subsets, which is true by Lemma 8.13. □

Finally, we consider $q_{\mathrm{res}}\colon \mathfrak{D}(\Gamma/e) \star \mathbf{n} \to \mathrm{H}_1(\Gamma/e, \mathbb{C})$.

Lemma 8.15. *$\Delta^i(q_{\mathrm{res}})$ has components $\mathbb{D}^{E(\Gamma\setminus R)}$ for $R \subset \Gamma \setminus e$ independent.*

Proof. As above, it is enough to show that $\mathrm{H}_1(\Gamma/e)$ is transverse to $\mathbb{D}^{E(\Gamma\setminus e\setminus R)}$ for R as stated. We have $\dim \mathrm{H}_1(\Gamma/e,\mathbb{C})+\dim \mathbb{D}^{E(\Gamma\setminus e\setminus R)}-\dim \mathbb{C}^{E(\Gamma\setminus e)} = h_1(\Gamma,\mathbb{C})-|R|$. But $\mathrm{H}_1(\Gamma/e)\cap\mathbb{C}^{E(\Gamma\setminus e\setminus R)} = \mathrm{H}_1(\Gamma/e\setminus R)$, which has dimension $h_1(\Gamma/e)-|R|$. □

8.2.3. *Preservation of perverse filtration.*

Theorem 8.16. *The isomorphism* $\mathrm{H}^\bullet(\mathfrak{V}(\Gamma),\mathbb{Q}) \to \mathrm{H}^\bullet(\mathfrak{D}(\Gamma),\mathbb{Q})$ *of Corollary 6.40 preserves the perverse filtrations induced by the projections to* $\mathbb{C}^{E(\Gamma)}$ *and* $\mathrm{H}^1(\Gamma,\mathbb{C})$*, respectively.*

Proof. Follows from Corollary A.15, Proposition 8.9, and Lemma 8.10. □

Proposition 8.17. *The restriction map* $\mathrm{H}^\bullet(\mathfrak{V}(\Gamma),\mathbb{Q}) \to \mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e)\star\mathfrak{D},\mathbb{Q})$ *preserves the perverse Leray filtrations, with respect to the projections to* $\mathbb{C}^{E(\Gamma)}$ *and* $\mathrm{H}_1(\Gamma/e,\mathbb{C})\times\mathbb{C}_e$*, respectively.*

Proof. The proof is similar to that of the previous theorem. The right-hand space is the preimage of $\mathrm{H}_1(\Gamma/e,\mathbb{C})\times\mathbb{C}_e$ in the left-hand space. The latter has complex dimension $h_1(\Gamma)+1$. Its intersection with $\mathbb{C}^{E(\Gamma\setminus R)}$ has dimension $h_1(\Gamma\setminus R)+1 = h_1(\Gamma)-|R|+1$, assuming R is independent.

It follows that it is transverse to all $\Delta^i(q_{\mathrm{res}})$, from which the conclusion follows. □

Corollary 8.18. *The restriction map* $\kappa^*\colon \mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e)\star\mathfrak{D},\mathbb{Q}) \to \mathrm{H}^\bullet(\mathfrak{D}(\Gamma),\mathbb{Q})$ *defined from Equation 46 preserves the perverse Leray filtrations coming from the projections to* $\mathrm{H}_1(\Gamma/e,\mathbb{C})\times\mathbb{C}_e$ *and* $\mathrm{H}^1(\Gamma,\mathbb{C})$*, respectively.*

Proof. Follows by composing to the two previous equivalences. □

Proposition 8.19. *The restriction map* $\kappa_{\mathbf{n}}^*\colon \mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e)\star\mathbf{n},\mathbb{Q}) \to \mathrm{H}^\bullet(\mathfrak{D}(\Gamma\setminus e),\mathbb{Q})$ *defined in Equation 46 preserves perverse filtrations coming from projection to* $\mathrm{H}_1(\Gamma/e;\mathbb{C})$ *and* $\mathrm{H}_1(\Gamma\setminus e,\mathbb{C})$ *respectively.*

Proof. Recall the description of the higher discriminants in Lemma 8.15. We must show that $\mathrm{H}_1(\Gamma\setminus e,\mathbb{C})$ intersects all such subspaces transversally. Indeed, $\dim \mathrm{H}_1(\Gamma\setminus e,\mathbb{C})+\dim \mathbb{C}^{E(\Gamma\setminus e\setminus R)} - \dim \mathbb{C}^{E(\Gamma\setminus e)} = h_1(\Gamma\setminus e)-|R|$. But $\dim \mathrm{H}_1(\Gamma\setminus e,\mathbb{C})\cap\mathbb{C}^{E(\Gamma\setminus e\setminus R)} = h_1(\Gamma\setminus e)-|R|$. □

8.2.4. *A compatibility we are not concerned with.* There is also the commutative diagram

$$\begin{tikzcd} \mathfrak{D}(\Gamma/e) \arrow[r, "i_\epsilon"] \arrow[d] & \mathfrak{D}(\Gamma/e)\star(\mathfrak{D}\setminus \mathbf{S}_{\mathfrak{D}}) \arrow[d, "q_{\mathrm{res}}\times q"] \\ \mathrm{H}_1(\Gamma/e) \arrow[r, "id\times\epsilon"] & \mathrm{H}_1(\Gamma/e,\mathbb{C})\times\mathbb{C} \end{tikzcd} \tag{62}$$

The perverse Leray filtration on $\mathfrak{D}(\Gamma/e)\star(\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}})$ arising from the right-hand map plays no role in our story. We are interested in the perverse filtration is defined on the left-hand space by the left-hand vertical map.

8.3. **Perverse and deletion filtrations.** Note by combining Proposition 8.2, Corollary 8.18, and Proposition 8.19, we learn that the maps $a_e^{\mathfrak{D}}$, $b_e^{\mathfrak{D}}$ of the $\mathfrak{D}$-DCS (not necessarily strictly) preserves the perverse filtration.

Proposition 8.20. *The Dolbeault deletion filtration is bounded by the perverse Leray filtration.*

Proof. Because the map $a_e^{\mathfrak{D}}$ of Equation (not necessarily strictly) preserves the perverse filtration, we learn that $D_k \mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q}) \subset P_k \mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q})$, by the same argument as we used in Proposition 5.36 for the Betti counterpart of this statement. (Note the shift of weight filtration by the Tate twist on the Betti side matches the shift of perverse Leray filtrations we impose on the first term on the Dolbeault side.) □

8.4. **The perverse Leray filtration and the Υ filtration.**

Theorem 8.21. *There is an isomorphism* $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q}) \cong \mathrm{H}^\bullet(\Upsilon, \mathbb{Q})$ *identifying the perverse filtration with the* Υ *filtration.*

We do not make this isomorphism explicit here; we will do so later via the Betti space.

Proof. By Theorem 8.16, we may work with the 'versal space' and study the perverse filtration on the central fiber induced by the map $q_{\mathrm{res}} : \mathfrak{V}(\Gamma, \eta) \to \mathbb{C}^{E(\Gamma)}$.

We have shown in Theorem 7.8 that q_{res} is projective in a neighborhood of the central fiber; we henceforth restrict to this neighborhood. Recalling that $\mathfrak{V}(\Gamma, \eta)$ is nonsingular, we may therefore apply the decomposition theorem of [BBD] to conclude $q_{\mathrm{res}\,*}\mathbb{Q}$ is a direct sum of semisimple perverse sheaves.

Let q_{res}° be the restriction of q_{res} to the complement of the coordinate hyperplanes. One summand of $q_{\mathrm{res}\,*}\mathbb{Q}$ is therefore $\bigoplus_j IC(R^j q^\circ_{\mathrm{res}\,*}\mathbb{Q})$. Of these, the summands with $j \leq k$ contribute to the k'th step of the perverse Leray filtration.

The family q_{res} is similar to the relative compactified Jacobian for the versal family of a nodal curve with dual graph Γ. Comparing Proposition 7.6 to [MSV, Eq. 3.7], we see that the local systems $R^j q^\circ_{\mathrm{res}\,*}\mathbb{Q}$ considered here are the same as what are called $\bigwedge^i R^1\pi_*\mathbb{Q}|_{B_{reg}}$ in [MSV].

In general, there is a formula [CKS] for the stalks of the intermediate extension of a local system across a normal crossing divisor. [MSV] explicitly computed in the case of the local system at hand; the result [MSV, Lem. 3.6] was that $IC(R^j q^\circ_{\mathrm{res}\,*}\mathbb{Q})_0$ is computed by the complex we have here called $\Upsilon^\bullet$.

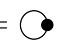
We have seen there is a summand $\mathrm{H}^\bullet(\Upsilon(\Gamma), \mathbb{Q}) \subset \mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q})$, such that the perverse filtration on the later restricts to the Υ filtration on the former. It remains to show this inclusion is an equality. By the argument of [MS, Prop. 15], it suffices to check the equality of weight polynomials. The calculation for $\Upsilon(\Gamma)$ is carried out in [MSV, Cor. 3.8], and for the central fiber of $\mathfrak{D}(\Gamma)$ in Theorem 6.15 above. The results agree: each is $t^{2h_1(\Gamma)}$ times the number of spanning trees of Γ. □

Remark 8.22. We will later show in Theorem 9.6 that $\mathfrak{B}(\Gamma)$ retracts to $\mathfrak{D}(\Gamma)$, hence in particular has the same cohomology. We also know that $\mathrm{H}^\bullet(\mathfrak{B}(\Gamma), \mathbb{Q}) \cong \mathrm{H}^\bullet(\Upsilon(\Gamma), \mathbb{Q})$ from Corollary 5.32. It follows that $\mathrm{H}^\bullet(\Upsilon(\Gamma), \mathbb{Q})$ and $\mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q})$ have the same total dimension, hence that the inclusion $\mathrm{H}^\bullet(\Upsilon(\Gamma), \mathbb{Q}) \subset \mathrm{H}^\bullet(\mathfrak{D}(\Gamma), \mathbb{Q})$ must be an isomorphism. This is an independent argument from the weight polynomial one given above.

We record the following corollary.

Corollary 8.23. *The perverse Leray filtration on* $\mathrm{H}^\bullet(\mathfrak{D}, \mathbb{Q})$ *with respect to the map* $q : \mathfrak{D} \to \mathbb{D}^1$ *is the filtration by cohomological degree. In other words,*

$$P_0\mathrm{H}^\bullet(\mathfrak{D}, \mathbb{Q}) = \mathrm{H}^0(\mathfrak{D}, \mathbb{Q}),$$
$$P_1\mathrm{H}^\bullet(\mathfrak{D}, \mathbb{Q}) = \mathrm{H}^{\leq 1}(\mathfrak{D}, \mathbb{Q}),$$
$$P_2\mathrm{H}^\bullet(\mathfrak{D}, \mathbb{Q}) = \mathrm{H}^{\leq 2}(\mathfrak{D}, \mathbb{Q}).$$

Proof. One can of course verify this by direct geometric arguments. Here we simply note that since the cohomology of $\mathfrak{D}$ has rank at most one in any given degree, it is enough to determine the associated graded of the perverse Leray filtration. This in turn is computed by the case $\Gamma =$  of Theorem 8.21. $\square$

Remark 8.24. More generally, comparing the formula for weight polynomials in [MSV, Cor. 3.8] with the formula in Remark 6.17, we see that in fact every summand of $(q_{\mathrm{res}} : \mathfrak{V}(\Gamma, \eta) \to \mathbb{C}^{E(\Gamma)})_*\mathbb{Q}$ has full support; in particular, ${}^pR^jq_{res*}\mathbb{Q} = IC(R^j\widetilde{q}_{\mathrm{res}\,*}\mathbb{Q})$.

By contrast, $(q : \mathfrak{D}(\Gamma, \eta) \to H_1(\Gamma, \mathbb{C}))_*\mathbb{Q}$ can have summands supported in positive codimension.

Remark 8.25. By combining the proof of Theorem 8.21 with the results of [MSV], we establish the isomorphism $D(X) \cong \mathfrak{D}(\Gamma_X)$ asserted in the introduction: both sides are computed by $\Upsilon^\bullet(\Gamma)$.

9. Comparisons

9.1. **"Hodge" correspondence.** We will construct a homotopy equivalence $\mathfrak{D} \to \mathfrak{B}$, and induce one to $\mathfrak{D}(\Gamma) \to \mathfrak{B}(\Gamma)$.

Lemma 9.1. *Let* $\kappa \colon \mathbb{U}_1 \times \mathbb{C} \cong \mathbb{C}^* \times \mathbb{R}$ *be the group isomorphism* $(e^{2\pi i\theta}, z) \to (e^{2\pi i\theta + Im(z)}, Re(z))$. *There is a (non-unique!)* $\mathbb{U}_1$*-equivariant* $\mathcal{C}^\infty$ *embedding* $\mathfrak{F} \colon \mathfrak{D} \to \mathfrak{B}$ *such that the following diagram commutes.*

$$\begin{tikzcd} \mathfrak{D} \arrow[r, "\mathfrak{F}"] \arrow[d, "\mu_{\mathfrak{D}}^{\mathbb{U}_1} \times q"'] & \mathfrak{B} \arrow[d, "\mu_{\mathfrak{B}}^{\mathbb{C}^*} \times \mu_{\mathfrak{B}}^{\mathbb{R}}"] \\ \mathbb{U}_1 \times \mathbb{D} \arrow[r, "\kappa"] & \mathbb{C}^* \times \mathbb{R} \end{tikzcd}$$

Proof. We write $0_{\mathfrak{D}} = 1 \times 0 \in \mathbb{U}_1 \times \mathbb{D}$ and $0_{\mathfrak{B}} = 1 \times 0 \in \mathbb{C}^* \times \mathbb{R}$. Evidently $\kappa(0_{\mathfrak{D}}) = 0_{\mathfrak{B}}$.

The maps $\mu_{\mathfrak{B}}^{\mathbb{C}^*} \times \mu_{\mathfrak{B}}^{\mathbb{R}}$ and $\mu_{\mathfrak{D}}^{\mathbb{U}_1} \times q$ define principal $\mathbb{U}_1$-bundles $\mathcal{P}_{\mathfrak{B}}, \mathcal{P}_{\mathfrak{D}}$ away from the point $0_{\mathfrak{D}}$ and $0_{\mathfrak{B}}$ (Lemma 5.4 and Proposition 6.3). The restrictions of these bundles to $\mathbb{U}_1 \times \mathbb{D} \setminus 0_{\mathfrak{D}}$ are classified by their Chern characters $c_1(\mathcal{P}_i) \in \mathrm{H}^2(\mathbb{D} \times \mathbb{U}_1 \setminus 0_{\mathfrak{D}}, \mathbb{Z})$. The group $\mathrm{H}^2(\mathbb{U}_1 \times \mathbb{D} \setminus 0_{\mathfrak{D}}, \mathbb{Z})$ is spanned by a small sphere around $0_{\mathfrak{D}}$. It follows that a $\mathbb{U}_1$-equivariant isomorphism over a small disk around $0_{\mathfrak{D}} \to 0_{\mathfrak{B}}$ can be extended to such an isomorphism over all of $\mathbb{U}_1 \times \mathbb{D} \setminus 0_{\mathfrak{D}}$. We must show that some such isomorphism extends over $0_{\mathfrak{D}}$.

Both spaces $\mathfrak{B}$ and $\mathfrak{D}$ have a single $\mathbb{U}_1$ fixed point, with image $0_{\mathfrak{B}}$ and $0_{\mathfrak{D}}$ respectively. By the differentiable slice theorem, the fibration near a small neighborhood of $0_{\mathfrak{B}}$ (or $0_{\mathfrak{D}}$) is equivariantly diffeomorphic to that given by a linear circle action on the unit ball in $\mathbb{R}^4$. We have seen that our actions have no nontrivial stabilizers away from the fixed point; it follows from this that the circle acts by the identity character, its inverse, or a sum of these. In any case, in the coordinates of our descriptions of both of these spaces, the $\mathbb{U}_1$ action around the fixed point was explicitly given in complex coordinates as $(x, y) \mapsto (\tau x, \tau^{-1} y)$.

The above argument suffices to establish that *some* map κ lifts to an equivariant embedding. To see that this is true for the specific map given (or indeed, any homotopy equivalence carrying $0_{\mathfrak{D}} \to 0_{\mathfrak{B}}$), it suffices to note that if $\mathbb{R}^4 \to \mathbb{R}^3$ is the Hopf fibration, then any diffeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ fixing the origin lifts to an equivariant diffeomorphism of Hopf fibrations. This again follows from local linearity at the fixed point. $\square$

Let $\mathfrak{B}^<$ be the image of $\mathfrak{F}$; alternatively, $\mathfrak{B}^<$ is the preimage of $\kappa(\mathbb{U}_1 \times \mathbb{D})$.

Lemma 9.2. *$\mathfrak{B}^<$ is a homotopy retract of $\mathfrak{B}$.*

Proof. Let $r \colon \mathbb{C} \times I \to \mathbb{C}$ be a linear retraction of $\mathbb{C}$ onto $\mathbb{D}$. As in the proof of Lemma 6.35, this induces a diffeomorphism and homotopy retract $\mathfrak{B} \to \mathfrak{B}^<$. $\square$

Lemma 9.3. $\mathrm{S}_{\mathfrak{D}} = \mathfrak{F}^{-1}(\mathrm{S}_{\mathfrak{B}})$

Proof. Each is the moment preimage of the a half line, and we have chosen κ to identify these half-lines. $\square$

Combining 9.9, 9.6 and 6.38, we see that $\mathfrak{B}(\Gamma, \eta)$ also retracts onto $q_\Gamma^{-1}(0)$, viewed as a subset of $\mathfrak{B}^<(\Gamma, \eta)$ via 9.6.

We turn to the case of $\mathfrak{D}(\Gamma)$ and $\mathfrak{B}(\Gamma)$. We restore now the moment map parameter η in our notation, since it plays a priori different roles for $\mathfrak{B}$ and $\mathfrak{D}$.

We have defined $\mathfrak{B}(\Gamma)$ as a complex algebraic (GIT) quotient; by the Kempf-Ness theorem we can instead understand it as a symplectic reduction, as the following proposition shows.

Proposition 9.4. *There is a diffeomorphism*

$$\mathfrak{B}^{\mathbb{U}_1, \mathbb{R} \times \mathbb{C}^*}(\Gamma, \eta) \to \mathfrak{B}^{\mathbb{C}^*, \mathbb{C}^*}(\Gamma, \eta).$$

Proof. In the construction of $\mathfrak{B}(\Gamma,\eta)$, we took a $(\mathbb{C}^*)^{V(\Gamma)}$ quotient of $\mu_\Gamma^{-1}(\eta)$. In fact we had the structure of a $\mathbb{U}_1^{V(\Gamma)} \subset (\mathbb{C}^*)^{V(\Gamma)}$ action on the complex manifold $\mu_\Gamma^{-1}(\eta) \subset \mathfrak{B}^{E(\Gamma)}$ with $\mathbb{R}^{V(\Gamma)}$-valued moment map, induced from the $\mathbb{U}_1 \subset \mathbb{C}^*$ action on $\mathfrak{B}$ with moment map $|x|^2 - |y|^2$. By the Kempf-Ness theorem [KN], we can replace the $(\mathbb{C}^*)^{V(\Gamma)}$ quotient of $\mu_\Gamma^{-1}(\eta)$ by the symplectic reduction by $\mathbb{U}_1^{V(\Gamma)}$. This is a particularly simple application of the Kempf-Ness theorem, since the $(\mathbb{C}^*)^{V(\Gamma)}$ action is free on $\mu_\Gamma^{-1}(\eta)$ and all orbits are closed. The resulting diffeomorphism takes a point $z \in \mathfrak{B}^{\mathbb{U}_1,\mathbb{R}\times\mathbb{C}^*}(\Gamma,\eta)$ to the $(\mathbb{C}^*)^{V(\Gamma)}$-orbit of its image. □

The virtue of the symplectic reduction picture of the Betti space is that it is more readily comparable to the Dolbeault space. More precisely we would like to compare a retracted version:

Definition 9.5.

$$\mathfrak{B}^<(\Gamma,\eta) := (\mathfrak{B}^<)^{\mathbb{U}_1,\mathbb{R}\times\mathbb{C}^*}(\Gamma,\eta)$$

Theorem/Definition 9.6. *There is a diffeomorphism $\mathfrak{F}_\Gamma \colon \mathfrak{D}(\Gamma,\eta) \to \mathfrak{B}^<(\Gamma,\eta)$ making the following diagram commutative.*

$$\begin{tikzcd}
\mathfrak{D}(\Gamma,\eta) \arrow[r, "\mathfrak{F}_\Gamma"] \arrow[d, "\mu^{\mathbb{U}_1}_{\mathfrak{D},\mathrm{res}}\times q_{\mathrm{res}}"'] & \mathfrak{B}^<(\Gamma,\eta) \arrow[d, "\mu^{\mathbb{C}^*}_{\mathfrak{B},\mathrm{res}}\times\mu^{\mathbb{R}}_{\mathfrak{B},\mathrm{res}}"] \\
\mathrm{H}_1(\Gamma,\mathbb{U}_1\times\mathbb{C})_\eta \arrow[r, "\kappa_\Gamma"] & \mathrm{H}_1(\Gamma,\mathbb{C}^*\times\mathbb{R})_\eta
\end{tikzcd}$$

Here κ_Γ is the isomorphism induced by κ.

Proof. Let $\mathfrak{F}\colon \mathfrak{D} \to \mathfrak{B}^<$ be the diffeomorphism in 9.1. By construction, it is $\mathbb{U}_1$-equivariant and covers a group isomorphism $\kappa\colon \mathbb{U}_1\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}^*$. Hence it induces a diffeomorphism

$$\mathfrak{F}_\Gamma\colon \mathfrak{D}^{(\mathbb{U}_1,\mathbb{U}_1\times\mathbb{C})}(\Gamma,\eta) \cong (\mathfrak{B}^<)^{\mathbb{U}_1,\mathbb{R}\times\mathbb{C}^*}(\Gamma,\eta).$$

covering the group isomorphism $\kappa\colon \mathrm{H}_1(\Gamma,\mathbb{U}_1\times\mathbb{C}) \to \mathrm{H}_1(\Gamma,\mathbb{R}\times\mathbb{C}^*)$. □

Via this theorem, we will often consider $\mathfrak{D}(\Gamma,\eta)$ as a subset of $\mathfrak{B}(\Gamma,\eta)$, with a different complex structure.

We see from Proposition 4.7 that $\mathfrak{D}(\Gamma,\eta)$ is an $(\mathrm{H}^1(\Gamma;\mathbb{U}_1),\mathrm{H}_1(\Gamma;\mathbb{U}_1\times\mathbb{C}))$-space and $\mathfrak{B}^<(\Gamma,\eta)$ is an $(\mathrm{H}^1(\Gamma;\mathbb{U}_1),\mathrm{H}_1(\Gamma;\mathbb{R}\times\mathbb{C}^*))$-space. The isomorphism $\kappa\colon \mathbb{U}_1\times\mathbb{C}\to\mathbb{R}\times\mathbb{C}^*$ allows us to view both as $(\mathrm{H}^1(\Gamma;\mathbb{U}_1),\mathrm{H}_1(\Gamma;\mathbb{U}_1\times\mathbb{C}))$-spaces. Then one can reformulate Theorem 9.6 as follows.

Lemma 9.7. $\mathfrak{F}_\Gamma\colon \mathfrak{D}(\Gamma,\eta) \cong \mathfrak{B}^<(\Gamma,\eta)$ *is an isomorphism of* $(\mathrm{H}^1(\Gamma;\mathbb{U}_1),\mathrm{H}_1(\Gamma;\mathbb{U}_1\times\mathbb{C}))$*-spaces.*

Remark 9.8. Note that $\mathfrak{F}$ was not unique, and thus nor is $\mathfrak{F}_\Gamma$. However, the group of smooth maps $\mathrm{H}_1(\Gamma,\mathbb{U}_1\times\mathbb{C}) \to \mathrm{H}^1(\Gamma,\mathbb{U}_1)$ (not respecting any group structure) acts transitively on the set of choices. We will not need this fact in what follows.

Proposition 9.9. *The retraction $\mathfrak{B}\to\mathfrak{B}^<$ in Lemma 9.2 descends to a homotopy retraction from $\mathfrak{B}(\Gamma,\eta)$ to $\mathfrak{B}^<(\Gamma,\eta)$.*

Proof. We have the following diagram.

$$\begin{tikzcd} \mu_\Gamma^{-1}(\eta)/\mathbb{U}_1^{V(\Gamma)} & \mu_\Gamma^{-1}(\eta) \arrow[l] \arrow[r, hook] \arrow[d] & \mathfrak{B}^{E(\Gamma)} \arrow[d] \\ & d_\Gamma^{-1}(\eta) \arrow[r, hook] & C_1(\Gamma, \mathbb{C}\times\mathbb{U}_1) = (\mathbb{C}\times\mathbb{U}_1)^{E(\Gamma)} \end{tikzcd} \tag{63}$$

Pick a diffeomorphism $\psi\colon \mathbb{C}^{E(\Gamma)} \to \mathbb{D}^{E(\Gamma)}$ which preserves $\mathbb{R}$-lines through the origin. Linear interpolation between z and $\psi(z)$ defines a deformation retraction $r\colon \mathbb{C}^{E(\Gamma)} \times [0,1] \to \mathbb{D}^{E(\Gamma)}$. Let $R\colon (\mathbb{U}_1 \times \mathbb{C})^{E(\Gamma)} \times [0,1] \to (\mathbb{U}_1 \times \mathbb{C})^{E(\Gamma)}$ be the induced deformation retraction, which is constant in the $(\mathbb{U}_1)^{E(\Gamma)}$ factor. Since R preserves lines in $\mathbb{C}^{E(\Gamma)}$, R preserves $d_\Gamma^{-1}(\eta)$ and its stratification by coordinate hyperplanes. As in Lemma 9.2, this induces a $\mathbb{U}_1^{E(\Gamma)}$-equivariant retraction of $\mu_\Gamma^{-1}(\eta)$ onto $\mu_\Gamma^{-1}(\eta) \cap (\mathfrak{B}^<)^{E(\Gamma)}$. Passing to $\mathbb{U}_1$-quotients, we obtain the desired retraction. □

Corollary 9.10. *The diffeomorphism $R(-,-,1) = \mathrm{id}\times\psi$ is covered by a diffeomorphism $\mathfrak{B}(\Gamma) \to \mathfrak{B}^<(\Gamma,\eta)$.*

9.2. **Intertwining of deletion maps.** In this section, we show that the Hodge map intertwines the Betti and Dolbeault deletion maps, and thus identifies the Betti and Dolbeault deletion-filtrations.[4]

We wish to relate the Betti deletion contraction sequence with the Dolbeault pairs sequence. To that end, consider the following map of pairs, as a special case of Diagram 74.

$$\begin{tikzcd} \mathfrak{B}(\Gamma/e) = \mathbf{Y} \star_{\mathbb{G}_m} (\mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}}) \arrow[r, "\mathcal{J}_{\mathfrak{B}}"] & \mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B} = \mathfrak{B}(\Gamma) \\ \mathbf{X} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \arrow[u] \arrow[r, "\mathcal{J}_{\mathfrak{D}}"] & \mathbf{X} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{D} = \mathfrak{D}(\Gamma) \arrow[u, "\mathfrak{F}_\Gamma"] \end{tikzcd} \tag{64}$$

It is not immediately clear that this map of pairs exists; this is established by the following lemma.

Lemma 9.11. *The restriction of $\mathfrak{F}_\Gamma$ in the above diagram to the bottom subspace has image in the top subspace.*

Proof. By construction, the Hodge map $\mathfrak{F}_\Gamma$ covers the map $\kappa_\Gamma : \mathrm{H}_1(\Gamma, \mathbb{U}_1 \times \mathbb{C}) \to \mathrm{H}_1(\Gamma, \mathbb{C}^* \times \mathbb{R})$. The coefficient of any given edge e determines a $\mathbb{U}_1 \times \mathbb{C}$-valued function $t_e^{\mathfrak{D}}$ on $\mathrm{H}_1(\Gamma, \mathbb{U}_1 \times \mathbb{C})$, and the bottom subspace is the preimage of the locus $\{t_e \neq 1 \times 0\}$. Likewise, the top subspace is the preimage of the locus $\{t_e^{\mathfrak{B}} \notin 1 \times \mathbb{R}^{<0}\}$ where $t_e^{\mathfrak{B}}$ is the coordinate of e in $\mathrm{H}_1(\Gamma, \mathbb{C}^* \times \mathbb{R})$. Since these loci are intertwined by κ_Γ, $\mathfrak{F}_\Gamma$ intertwines their preimages. □

Thus $\mathfrak{F}_\Gamma$ defines a map of long exact sequences from the Betti deletion contraction sequence to the Dolbeault pairs sequence. Composing with Lemma 6.47 gives a map from the Betti to the

[4] In fact, the Hodge map intertwines not just the deletion maps, but the entire deletion-contraction sequences. This will be proven as Theorem 9.20 in Section 9.4. We postpone it to a later section because (1) the proof is more involved, requiring a homotopy of maps of topological spaces rather than an equality of such maps, and (2) the more general intertwining is not necessary to establish the 'P=W' statement.

Dolbeault deletion contraction sequences, which we record in the following monstrous diagram. All of the vertical arrows are isomorphisms.

(65)

$$\begin{tikzcd}
{} \arrow[r, dashed, "c_e^{\mathfrak{B}}"] & H^{\bullet-2}(\mathfrak{B}(\Gamma\setminus e),\mathbb{Q}) \arrow[r, dashed, "a_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}^*_{\Gamma,\mathrm{rel}}"] & H^{\bullet}(\mathfrak{B}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "b_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}^*_{\Gamma}"] & H^{\bullet}(\mathfrak{B}(\Gamma/e),\mathbb{Q}) \arrow[r, dashed, "c_e^{\mathfrak{B}}"] \arrow[d, "(\mathfrak{F}_{\Gamma,\mathrm{res}})^*"] & {} \\
{} \arrow[r, dashed, "C_e^{\mathfrak{D}}"] & H^{\bullet-2}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[r, dashed, "A_e^{\mathfrak{D}}"] \arrow[d, equal] & H^{\bullet}(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "B_e^{\mathfrak{D}}"] \arrow[dd, equal] & H^{\bullet}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}(\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}}),\mathbb{Q}) \arrow[r, dashed, "C_e^{\mathfrak{D}}"] \arrow[d, "(\kappa_\circ^*)^{-1}"] & {} \\
 & H^{\bullet-2}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{n},\mathbb{Q}) \arrow[d, equal] & & H^{\bullet}(\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1}\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}},\mathbb{Q}) \arrow[d, "i_\epsilon^*"] & \\
{} \arrow[r, dashed, "c_e^{\mathfrak{D}}"] & H^{\bullet-2}(\mathfrak{D}(\Gamma\setminus e),\mathbb{Q}) \arrow[r, dashed, "a_e^{\mathfrak{D}}"] & H^{\bullet}(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "b_e^{\mathfrak{D}}"] & H^{\bullet}(\mathfrak{D}(\Gamma/e),\mathbb{Q}) \arrow[r, dashed, "c_e^{\mathfrak{D}}"] & {}
\end{tikzcd}$$

Here the top-left map $\mathfrak{F}^*_{\Gamma,\mathrm{rel}}$ is defined as in Section A.1. We do not need this entire diagram right away - we will focus only on the left-hand side in this section. We must determine the composition of the left-hand vertical maps. For this, we will use Lemma A.1. Since $\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{S}_{\mathfrak{D}}$ retracts onto $\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{n}$, it is enough to understand the restriction of $\mathfrak{F}_\Gamma$ to the latter.

Lemma 9.12. *The restriction of $\mathfrak{F}_\Gamma$ to $\mathfrak{D}(\Gamma/e)\star_{\mathbb{U}_1\times\mathbb{C}}\mathbf{n}=\mathfrak{D}(\Gamma\setminus e)$ has image in $\mathfrak{B}(\Gamma/e)\star_{\mathbb{G}_m}\mathbf{n}=\mathfrak{B}(\Gamma/e)$, and is thereby identified with $\mathfrak{F}_{\Gamma\setminus e}$.*

Corollary 9.13. *We have a map of long exact sequences*

(66)
$$\begin{tikzcd}
\ldots \arrow[r] & H^{\bullet-2}(\mathfrak{B}(\Gamma\setminus e),\mathbb{Q}) \arrow[r, "a_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}^*_{\Gamma\setminus e}"] & H^{\bullet}(\mathfrak{B}(\Gamma),\mathbb{Q}) \arrow[r, "b_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}^*_{\Gamma}"] & H^{\bullet}(\mathfrak{B}(\Gamma/e),\mathbb{Q}) \arrow[r, "c_e^{\mathfrak{B}}"] \arrow[d, "i_\epsilon^*(\kappa_\circ^*)^{-1}\mathfrak{F}^*_{\Gamma,\mathrm{res}}"] & \ldots \\
\ldots \arrow[r] & H^{\bullet-2}(\mathfrak{D}(\Gamma\setminus e),\mathbb{Q}) \arrow[r, "a_e^{\mathfrak{D}}"] & H^{\bullet}(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r] & H^{\bullet}(\mathfrak{D}(\Gamma/e),\mathbb{Q}) \arrow[r] & \ldots
\end{tikzcd}$$

The vertical arrows are isomorphisms.

Remark 9.14. We will show that the right-hand map equals $\mathfrak{F}^*_{\Gamma/e}$ in Section 9.4.

Corollary 9.15. *The maps $a_e^{\mathfrak{D}}$ for different edges commute when their composition is defined.*

Proof. The claimed commutativity holds for the maps $a_e^{\mathfrak{B}}$ by Corollary 5.34. Thus the claim follows from Lemma 9.13, and in particular the commutativity of the left-hand square. ☐

Corollary 9.16. *The Hodge map $\mathfrak{F}^*_\Gamma$ identifies the Betti and Dolbeault deletion filtrations.*

Proof. Since both deletion filtrations are defined purely in terms of the deletion maps, this follows from Lemma 9.13, and in particular the commutativity of the left-hand square. ☐

9.3. **P=W.**

Proposition 9.17. $D_k\mathrm{H}^\bullet(\mathfrak{D}(\Gamma),\mathbb{Q}) = P_k\mathrm{H}^\bullet(\mathfrak{D}(\Gamma),\mathbb{Q})$. *In other words, the Dolbeault deletion filtration equals the perverse Leray filtration.*

Proof. We know that the Dolbeault deletion filtration is bounded by the perverse Leray filtration by Proposition 8.20. Thus it is enough to show that the filtrations are abstractly isomorphic, i.e. that there exists is an isomorphism of vector spaces taking the deletion filtration to the perverse Leray filtration. We produce this isomorphism via the Hodge map to the Betti space.

Namely, by Corollary 9.16, the Dolbeault deletion filtration is isomorphic to the Betti deletion filtration. In turn, the Betti deletion filtration is isomorphic to the Υ filtration by Corollary 5.37. Finally, Theorem 8.21 shows that the Υ and perverse Leray filtrations are isomorphic. □

Remark 9.18. To show Proposition 9.17 without appeal to the $\mathfrak{D} \subset \mathfrak{B}$ comparison, we would have to fix an isomorphism between the cohomology of the $\mathfrak{D}$ space and the Υ complex, and show that said isomorphism intertwines the deletion maps. This is presumably straightforward since both the Υ complex and the deletion map are built from nearby-vanishing cycle operations, but we have not done it here.

Theorem 9.19 (P=W)**.** *The map $\mathfrak{F}_\Gamma^*$ identifies $W_{2k}\mathrm{H}^\bullet(\mathfrak{B}(\Gamma),\mathbb{Q})$ with $P_k\mathrm{H}^\bullet(\mathfrak{D}(\Gamma),\mathbb{Q})$.*

Proof. We have identified both filtrations with the deletion filtrations on the respective spaces in Proposition 5.39 and Proposition 9.17. The result thus follows from Corollary 9.16. □

9.4. **Intertwining of deletion-contraction sequences.** Finally, we show that the homotopy equivalences $\mathfrak{F}_\Gamma$ intertwine the Betti and Dolbeault deletion contraction sequences, in such a way as to match the weight and perverse Leray filtrations. We note this fact *cannot* be deduced by the five-lemma, even assuming that we knew that the perverse Leray filtration is strictly compatible with the Dolbeault deletion sequence. In any case, we don't yet know this compatibility, though it will follow as a corollary of the following result.

Theorem 9.20. *The Hodge maps define isomorphisms of deletion-contraction sequences, depicted below.*

$$\begin{tikzcd}
{} \arrow[r, dashed, "c_e^{\mathfrak{B}}"] & \mathrm{H}^{\bullet-2}(\mathfrak{B}(\Gamma\setminus e),\mathbb{Q})(-1) \arrow[r, dashed, "a_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}_{\Gamma/e}^*"] & \mathrm{H}^\bullet(\mathfrak{B}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "b_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}_\Gamma^*"] & \mathrm{H}^\bullet(\mathfrak{B}(\Gamma/e),\mathbb{Q}) \arrow[r, dashed, "c_e^{\mathfrak{B}}"] \arrow[d, "\mathfrak{F}_{\Gamma\setminus e}^*"] & {} \\
{} \arrow[r, dashed, "c_e^{\mathfrak{D}}"] & \mathrm{H}^{\bullet-2}(\mathfrak{D}(\Gamma\setminus e),\mathbb{Q})\{-1\} \arrow[r, dashed, "a_e^{\mathfrak{D}}"] & \mathrm{H}^\bullet(\mathfrak{D}(\Gamma),\mathbb{Q}) \arrow[r, dashed, "b_e^{\mathfrak{D}}"] & \mathrm{H}^\bullet(\mathfrak{D}(\Gamma/e),\mathbb{Q}) \arrow[r, dashed, "c_e^{\mathfrak{D}}"] & {}
\end{tikzcd} \tag{67}$$

Recall that in Corollary 9.13 we showed the commutativity of a very similar looking diagram, save only with a different map in place of $\mathfrak{F}_{\Gamma/e}^*$. In Corollary 9.16, we remarked that this diagram was tautologically compatible with the deletion filtrations, and by now we have learned that the deletion filtrations agree with the weight and perverse Leray filtrations on the Betti and Dolbeault sides, respectively. Thus to complete the proof of Theorem 9.20, it remains only to show that the right-hand vertical map in Corollary 9.13 equals $\mathfrak{F}^*_{\Gamma/e}$, i.e., to prove:

Proposition 9.21. *The image of the following diagram under the functor* $\mathrm{H}^\bullet(-,\mathbb{Q})$ *(with arrow* κ^* *replaced by the inverse arrow* $(\kappa^*)^{-1}$*) is a commutative diagram.*

$$\begin{tikzcd} \mathbf{Y} \arrow[rr, equal] & & \mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} \\ \mathbf{X} \arrow[u, "\mathfrak{F}_{\Gamma/e}"] \arrow[r, "i_\epsilon"] & \mathbf{X} \star_{\mathbb{U}_1} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} & \mathbf{X} \star_{\mathbb{U}_1 \times \mathbb{C}} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \arrow[l, "\kappa_\circ"] \arrow[u, "\mathfrak{F}_{\Gamma,\mathrm{res}}"'] \end{tikzcd} \tag{68}$$

In other words, the right-hand vertical map in Corollary 9.13 is the map $\mathfrak{F}^*_{\Gamma/e}$.

We note the diagram *does not* commute at the level of spaces. We will instead need to construct certain homotopies between the various compositions. As the proof is rather long, we split it two parts. In Subsection 9.4.1, we get close as we can to the result using only commutative diagrams of spaces. The result is an equivalence between Proposition 9.21 and the more tractable Proposition 9.26. We then prove Proposition 9.26 by constucting a certain homotopy in Subsection 9.4.2.

9.4.1. *Some diagrams of spaces.* We begin by extending the maps κ and $\mathfrak{F}_\Gamma$ slightly, as follows.

Definition 9.22. If we forget the complex part of the moment map, a $(\mathbb{U}_1, \mathbb{U}_1 \times \mathbb{C})$ structure becomes a $(\mathbb{U}_1, \mathbb{U}_1)$ structure. The moment fiber for the latter is contained in the moment fiber for the former. Let $\kappa^{\mathbf{Y}}\colon \mathbf{Y} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} \to \mathbf{Y} \star_{\mathbb{U}_1} (\mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}})$ be the resulting inclusion.

Definition 9.23. The product embedding $\mathfrak{F}_{\Gamma/e} \times \mathfrak{F}\colon \mathbf{X} \times \mathfrak{D} \to \mathbf{Y} \times \mathfrak{B}$ descends to an embedding $\mathfrak{F}_\times\colon \mathbf{X} \star_{\mathbb{U}_1} \mathfrak{D} \to \mathbf{Y} \star_{\mathbb{U}_1} \mathfrak{B}$, and likewise $\mathfrak{F}_{\times,\mathrm{res}}\colon \mathbf{X} \star_{\mathbb{U}_1} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \to \mathbf{Y} \star_{\mathbb{U}_1} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}}$.

By construction, the restriction of $\mathfrak{F}_\times$ to $\mathbf{X} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{D} = \mathfrak{D}(\Gamma)$ gives the usual Hodge map $\mathfrak{F}_\Gamma$: $\mathfrak{D}(\Gamma) \to \mathfrak{B}(\Gamma)$. Thus we have the commutative diagram

$$\begin{tikzcd} \mathbf{Y} \star_{\mathbb{U}_1} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} & \mathbf{Y} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} \arrow[l, "\kappa^{\mathbf{Y}}"'] \\ \mathbf{X} \star_{\mathbb{U}_1} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \arrow[u, "\mathfrak{F}_{\times,\mathrm{res}}"] & \mathbf{X} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{D} \setminus \mathbf{S}_{\mathfrak{D}} \arrow[l, "\kappa"'] \arrow[u, "\mathfrak{F}_{\Gamma,\mathrm{res}}"'] \end{tikzcd} \tag{69}$$

Lemma 9.24. *The top map of this diagram induces an isomorphism in cohomology.*

Proof. All but the top map have been shown to induce isomorphisms in cohomology; since the square commutes, the top map must also induce an isomorphism. □

Definition 9.25. Let $\phi\colon \mathbf{Y} \to \mathbf{Y} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}}$. be the isomorphism resulting from the composition

$$\mathbf{Y} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} \to \mathbf{Y} \star_{\mathbb{G}_m} \mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} = \mathbf{Y} \tag{70}$$

where the second equality combines the isomorphism $\mathfrak{B} \setminus \mathbf{S}_{\mathfrak{B}} \cong [\mathbb{G}_m \times \mathbb{G}_m]$ with Lemma 3.29.

By commutativity of Diagram 69, Proposition 9.21 is equivalent to the following.

Proposition 9.26. *The image under the contravariant functor* $\mathrm{H}^\bullet(-,\mathbb{Q})$ *of the following square is commutative.*

$$\begin{tikzcd} \mathbf{Y} \arrow[r, "\phi"] & \mathbf{Y} \star_{\mathbb{U}_1\times\mathbb{C}} \mathfrak{B}\setminus \mathbf{S}_{\mathfrak{B}} \arrow[r, "\kappa^{\mathbf{Y}}"] & \mathbf{Y}\star_{\mathbb{U}_1}\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}} \\ \mathbf{X} \arrow[u, "\mathfrak{F}_{\Gamma/e}"] \arrow[rr, "i_\epsilon"] & & \mathbf{X}\star_{\mathbb{U}_1}\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}} \arrow[u, "\mathfrak{F}_{\times,\mathrm{res}}"'] \end{tikzcd} \tag{71}$$

9.4.2. *A certain homotopy.* The rest of this section will be occupied by the proof of Proposition 9.26. Since Diagram 71 is not commutative at the level of spaces, we will have to produce a homotopy between its two halves.

Let F_+ be the composition $\kappa^{\mathbf{Y}}\ \phi\ \mathfrak{F}_{\Gamma/e}\colon \mathbf{X}\to\mathbf{Y}\star_{\mathbb{U}_1}(\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}})$ along the left and top sides of Diagram 71. Let F_- be the composition $\mathfrak{F}_{\times,\mathrm{res}}\ i_\epsilon\colon \mathbf{X}\cong\mathbf{X}\star_{\mathbb{U}_1}q^{-1}(\epsilon)\to\mathbf{X}\star_{\mathbb{U}_1}(\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}})\to\mathbf{Y}\star_{\mathbb{U}_1}(\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}})$ along the bottom and right sides. Then Proposition 9.26 would follow if we knew F_+ and F_- were homotopic.

To prove this, we first establish some preliminary structural results. The codomain of $F_\pm$ is the space $\mathbf{Y}\star_{\mathbb{U}_1}(\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}})$. Recall that the latter is obtained from $\mathbf{Y}\times\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}}$ by taking the subspace $\mu_{\mathbb{U}_1}^{-1}(\zeta)$ and quotienting by $\mathbb{U}_1$.

Below, we will write $\mu_{\mathbb{U}_1}^{-1}(\zeta)_{\mathfrak{D}}\subset\mathbf{X}\times\mathfrak{D}$ for the moment fiber in the product of Dolbeault spaces. The Hodge map $\mathfrak{F}_{\Gamma/e}\times\mathfrak{F}:\mathbf{X}\times\mathfrak{D}\to\mathbf{Y}\times\mathfrak{B}$ identifies it with a subset of $\mu_{\mathbb{U}_1}^{-1}(\zeta)$.

Definition 9.27. Define $\tilde{f}_+\colon\mathbf{Y}\to\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}}=\mathbb{G}_m\times\mathbb{G}_m$ by $\tilde{f}_+(y)=(\zeta/\mu_{\mathbb{G}_m}(y),1)$.

The following is just a spelling out of the definition of ϕ.

Lemma 9.28. *The image of* $id\times\tilde{f}_+$ *lies in* $\mu_{\mathbb{U}_1\times\mathbb{C}}^{-1}(\zeta)$*, and* ϕ *is the composition of* $id\times\tilde{f}_+$ *with the quotient by* $\mathbb{U}_1$.

Consider $q^{-1}(\epsilon)\subset\mathfrak{D}$ where $\epsilon\in\mathbb{D}^1$ is a small positive real number. We have an isomorphism $q^{-1}(\epsilon)\cong\mathbb{U}_1\times\mathbb{U}_1$ as a $(\mathbb{U}_1,\mathbb{U}_1)$-manifold.

Definition 9.29. Define $\tilde{f}_-\colon\mathbf{X}\to\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}}$ by $\tilde{f}_-(x)=(\zeta/\mu_{\mathbb{U}_1}(x),1)\in q^{-1}(\epsilon)$.

The following is just a spelling out of the definition of i_ϵ.

Lemma 9.30. *The image of* $id\times\tilde{f}_-$ *lies in* $\mu_{\mathbb{U}_1}^{-1}(\zeta)_{\mathfrak{D}}$*, and* i_ϵ *is the composition of* $id\times\tilde{f}_-$ *with the quotient by* $\mathbb{U}_1$.

We thus have the following (non-commutative) diagram, where the horizontal maps are inclusions and the vertical maps are quotients. F_+ and F_- are the compositions along the boundaries of the sloping square.

(72)
$$\begin{tikzcd}
 & \mathbf{X} \arrow[rrr, "\mathfrak{F}_{\Gamma/e}"] \arrow[dl, "id\times\tilde f_-"'] \arrow[ddl, dashed, "i_\epsilon"] & & & \mathbf{Y} \arrow[dl, "id\times\tilde f_+"'] \arrow[ddl, dashed, "\phi"] \\
\mu_{\mathbb{U}_1}^{-1}(\zeta)_{\mathfrak{D}} \arrow[rr, "\mathfrak{F}_{\Gamma/e}\times\mathfrak{F}"] \arrow[d] & & \mu_{\mathbb{U}_1}^{-1}(\zeta) \arrow[d] & \mu_{\mathbb{U}_1\times\mathbb{C}}^{-1}(\zeta) \arrow[l] \arrow[d] & \\
\mathbf{X}\star_{\mathbb{U}_1}\mathfrak{D}\setminus\mathbf{S}_{\mathfrak{D}} \arrow[rr, "\mathfrak{F}_{\times,\mathrm{res}}"] & & \mathbf{Y}\star_{\mathbb{U}_1}\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}} & \mathbf{Y}\star_{\mathbb{U}_1\times\mathbb{C}}\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}} \arrow[l, "\kappa^{\mathbf{Y}}"] &
\end{tikzcd}$$

Diagram 72 is commutative if one removes the top horizontal arrow; we wish to show it commutes up to homotopy with this arrow included. It is enough to show that the roof is commutative up to homotopy.

Definition 9.31. Let $f_-\colon \mathbf{Y}\to\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}}$ be the map $f_-(y)=(\zeta/\mu_{\mathbb{U}_1}(y),1)\in q^{-1}(\epsilon)\to\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}}$, where the last map is the restriction of the Hodge map $\mathfrak{F}\colon\mathfrak{D}\to\mathfrak{B}$.

By construction, we have the following.

Lemma 9.32. *The diagram*

(73)
$$\begin{tikzcd}
 & \mathbf{X} \arrow[r, "\mathfrak{F}_{\Gamma/e}"] \arrow[dl, "id\times\tilde f_-"'] & \mathbf{Y} \arrow[dl, "id\times f_-"] \\
\mu_{\mathbb{U}_1}^{-1}(\zeta)_{\mathfrak{D}} \arrow[r, "\mathfrak{F}_{\Gamma/e}\times\mathfrak{F}"] & \mu_{\mathbb{U}_1}^{-1}(\zeta) &
\end{tikzcd}$$

commutes.

Proposition 9.33. *F_+ and F_- are homotopic.*

Proof. By the previous Lemma, it is enough to show that the maps $\tilde f_+, f_-$ are isotopic via a family of maps $\mathbf{Y}\to\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}}$ whose graph lies in $\mu_{\mathbb{U}_1}^{-1}(\zeta)$.

The space $\mathfrak{B}\setminus\mathbf{S}_{\mathfrak{B}}$ is a trivial $\mathbb{U}_1$-bundle over $(\mathbb{D}^1\times\mathbb{U}_1)\setminus(\mathbb{R}^{\le 0}\times 1)$. In fact, both $\tilde f_+$ and f_- have image contained in the image of the unit section and their projections to the $\mathbb{U}_1$-factor both coincide with $\mu_{\mathbb{U}_1}$. Retracting $\mathbb{D}^1\times\mathbb{U}_1\setminus(\mathbb{R}^{\le 0}\times 1)$ to $\epsilon\times\mathbb{U}_1$, keeping the second coordinate constant, defines an isotopy from $\tilde f_+$ to f_-. $\square$

Proof of Propositions 9.21 and 9.26. As previously noted, the two propositions are equivalent.

Since F_- and F_+ are the compositions along the two sides of the diagram in Proposition 9.26, the claimed commutativity follows from the homotopy in Proposition 9.33. $\square$

Proof of Theorem 9.20. Combine Corollary 9.13 and Proposition 9.21. ☐

Corollary 9.34. *The Dolbeault deletion contraction sequence strictly preserves the perverse Leray filtration.*

Proof. Follows immediately from Theorem 9.20, Theorem 9.19 and the corresponding fact for the Betti deletion contraction sequence. ☐

Appendix A. Recollections on cohomology and filtrations

A.1. **The long exact sequence of a pair.** Let $A \xrightarrow{f} X$ be an embedding of topological spaces. The pullback $\mathrm{H}^\bullet(X, \mathbb{Q}) \xrightarrow{f^*} \mathrm{H}^\bullet(A, \mathbb{Q})$ extends to a long exact sequence

$$\ldots \to \mathrm{H}^\bullet(X, A, \mathbb{Q}) \xrightarrow{\mathrm{co}(f)^*} \mathrm{H}^\bullet(X, \mathbb{Q}) \xrightarrow{f^*} \mathrm{H}^\bullet(A, \mathbb{Q}) \to \ldots$$

where $\mathrm{co}(f)$ is the map of pairs $X, \emptyset \to X, A$. Given a second embedding $B \to Y$ and a map of pairs given by a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "F_{\mathrm{res}}"] & X \arrow[d, "F"] \\ B \arrow[r] & Y \end{tikzcd} \tag{74}$$

we obtain a map of long exact sequences of pairs

$$\begin{tikzcd} \arrow[r] & \mathrm{H}^\bullet(Y, B, \mathbb{Q}) \arrow[r] \arrow[d, "F^*_{\mathrm{rel}}"] & \mathrm{H}^\bullet(Y, \mathbb{Q}) \arrow[r] \arrow[d, "F^*"] & \mathrm{H}^\bullet(B, \mathbb{Q}) \arrow[r] \arrow[d, "F^*_{\mathrm{res}}"] & {} \\ \arrow[r] & \mathrm{H}^\bullet(X, A, \mathbb{Q}) \arrow[r] & \mathrm{H}^\bullet(X, \mathbb{Q}) \arrow[r] & \mathrm{H}^\bullet(A, \mathbb{Q}) \arrow[r] & {} \end{tikzcd} \tag{75}$$

Lemma A.1. *If A, B are codimension d submanifolds of X, Y, we can identify F^*_{rel} with $F^*_{X\setminus A}\colon \mathrm{H}^{\bullet-d}(Y \setminus B, \mathbb{Q}) \to \mathrm{H}^{\bullet-d}(X \setminus A, \mathbb{Q})$.*

A.2. **The residue exact triangle.** Recall that if X is a topological space and $i\colon V \to X$ is a closed subset, and $j\colon X \setminus V \to X$ its open complement, then there is the Verdier dual exact triangle of a pair:

$$j_! j^! \mathbb{C} \to \mathbb{C}_X \to i_* \mathbb{C}_{X \setminus V} \xrightarrow{[1]}$$

In case X is a smooth complex manifold and D is a smooth divisor, this becomes

$$\mathbb{C}_D[-2] \to \mathbb{C}_X \to i_* \mathbb{C}_{X \setminus D} \xrightarrow{[1]}$$

We can take analytic de Rham resolutions of the constant sheaves to obtain

$$j_!\Omega^\bullet_D[-2] \to \Omega^\bullet_X \to i_*\Omega_{X\setminus D} \xrightarrow{[1]}$$

In this setting one can replace $i_*\Omega_{X\setminus D}$ by $\Omega_X\langle D\rangle$, the complex of differential forms with log poles along D. Having made this replacement, the connecting map may be identified with the residue (see e.g. [Del0, II.3]). Finally we may write $\Omega^\bullet_X \cong \mathrm{Cone}(i_*\Omega^\bullet_X\langle D\rangle \xrightarrow{Res} \Omega^\bullet_D[-1])[-1]$.

More generally, if X is a topological space with an increasing filtration by closed subsets $X_n \subset X_{n-1} \subset \cdots \subset X_0 = X$, then we may iterate this procedure to obtain an expression for $\mathbb{C}_X$ as a twisted complex on the sum of shifts of the star pushforwards of the $\mathbb{C}_{X_i\setminus X_{i+1}}$.

When X is a smooth complex manifold, $D = \bigcup D_k$ is a smooth normal crossings divisor, and X_i above is the codimension i intersections of the D_k, this complex can be explicitly described in terms of differential forms with log poles, as in the case of a single divisor above. This construction is presumably standard in e.g. the theory of mixed Hodge modules, but we did not find a convenient reference, so give some details here.

Again we write $\Omega^\bullet_X\langle D\rangle$ for the complex of holomorphic differential forms with log poles along D. Recall this means that in coordinates where D is cut out by $\prod_i z_i = 0$, the sheaf $\Omega^1_X\langle D\rangle$ is locally free and generated over Ω^1_X by $d\log z_i$ and $\Omega^\bullet_X\langle D\rangle$ is the exterior algebra on $\Omega^1_X\langle D\rangle$, here equipped with the de Rham differential.

Let us fix some notation for indexing divisors. Given a subset $J \subset \{1, ..., n\}$, let D_J be the intersection of components D_j for $j \in J$, and D^J be their union. Write J^c for the complement of J. Note that $\mathbf{K}_J := D_J \cap D^{J^c}$ is a normal crossings divisor in D_J; let $\mathbf{U}_J := D_J \setminus \mathbf{K}_J$ be its complement.

Suppose $j \notin J$. Then we can take the residue of a form in $\Omega^\bullet_{D_J}\langle \mathbf{K}_J\rangle$ along the component $D_j \cap D^J$ of $\mathbf{K}_J$. Let $J' = J \cup j$. This defines the residue map

$$\mathrm{res}_{J\to J'} \colon \Omega^l_{D_J}\langle \mathbf{K}_J\rangle \to \Omega^{l-1}_{D_{J'}}\langle \mathbf{K}_{J'}\rangle.$$

Definition A.2. Let $\Omega^{2k,l}_{X,D} := \bigoplus_{|J|=k} \Omega^l_{D_J}\langle \mathbf{K}_J\rangle$ and $\Omega_{X,D} = \bigoplus \Omega^{2k,l}_{X,D}$. The latter carries the de Rham differential d_{dR} of bidegree $(0,1)$, and an endomorphism d_{res} given by the sum of all residue maps, of bidegree $(2,-1)$.

Lemma A.3. *We have* $d_{dR}^2 = d_{\mathrm{res}}^2 = (d_{dR} + d_{\mathrm{res}})^2 = 0$.

Proof. $d_{dR}^2 = 0$ is of course standard. To show $d_{\mathrm{res}}^2 = 0$, we must check that for any two distinct edges e, e', the corresponding residue maps in d_{res} anti-commute. The two different compositions correspond to integration over a 2-torus with the two opposite orientations, which implies the result.

To verify $(d_{dR}+d_{\mathrm{res}})^2 = 0$, it remains to check that $d_{dR}d_{\mathrm{res}} = -d_{\mathrm{res}}d_{dR}$. Let us focus on the term $\mathrm{res}_{J\to J'}$ of d_{res} taking the residue along z_1. We can locally write any form as $f(z)\frac{dz_1}{z_1}\omega$ or $f(z)\omega$, where ω is a d_{dR}-closed form nonsingular along z_1. In the first case, we have $\mathrm{res}_{J\to J'}\, d_{dR} f(z)\frac{dz_1}{z_1}\omega =$

$-df(z)\omega$ whereas $d_{dR}\operatorname{res}_{J\to J'} f(z)\frac{dz_1}{z_1}\omega = df(z)\omega$. Here we have used the same notation for a form nonsingular along z_1 and its restriction to $z_1 = 0$. In the second case, both sides vanish. This concludes the proof. □

We henceforth regard $\Omega^\bullet_{X,D}$ as a singly-graded complex (the sum of the previous gradings) equipped with the differential $d_{dR} + d_{\mathrm{res}}$. This complex retains a filtration by the size of J (the first degree of the bidegree).

As $\mathbf{K}_J$ is a snc divisor in D_J, we analogously have $\Omega^\bullet_{D_J,\mathbf{K}_J}$.

Proposition A.4. *For any J, there is an exact triangle*

$$\Omega^\bullet_{D_J,\mathbf{K}_J}[-2|J|] \to \Omega^\bullet_{X,D} \to \Omega^\bullet_{X\setminus D_J, D\setminus D_J} \xrightarrow{[1]} \tag{76}$$

Proof. It is immediate from the definitions that $\Omega^\bullet_{D_J,\mathbf{K}_J}[2|J|]$ is a subcomplex (indeed, before imposing the differential, a summand) of $\Omega^\bullet_{X,D}$; we use this inclusion to induce the first map. (The shift comes from the choice of bigrading.) The second map is restriction of forms; evidently the image of $\Omega^\bullet_{D_J,\mathbf{K}_J}[-2|J|]$ lies in the kernel. The sequence of complexes is not exact in the middle, because elements of $\Omega^\bullet_{X\setminus D_J, D\setminus D_J}$ need not have log poles along D_J. Nevertheless, the map $\Omega^\bullet_{X,D}/\Omega^\bullet_{D_J,\mathbf{K}_J}[-2|J|] \to \Omega^\bullet_{X\setminus D_J, D\setminus D_J}$ is a quasi-isomorphism. Indeed, this can be seen after passing to the associated graded with respect to the filtration on these complexes. □

We have a natural inclusion $\Omega^\bullet_X \to \Omega^\bullet_{X,D}$, defined by the inclusion $\Omega^\bullet_X \to \Omega^\bullet_X\langle D\rangle$, the latter being a summand of $\Omega^\bullet_{X,D}$ (and d_{res} restricted to the image of $\Omega^\bullet_X$ is trivial.)

Proposition A.5. *The inclusion of complexes $\Omega^\bullet_X \to \Omega^\bullet_{X,D}$ is a quasi-isomorphism.*

Proof. We proceed by induction on the number of components of D. The statement is tautologous when D has no components. Choose a component D_j of D.

Consider the morphism of exact triangles:

$$\begin{tikzcd}
\Omega^\bullet_{D_j}[-2] \arrow[r] \arrow[d] & \Omega^\bullet_X \arrow[r] \arrow[d] & \Omega^\bullet_{X\setminus D_j} \arrow[r] \arrow[d] & {} \\
\Omega^\bullet_{D_j,\mathbf{K}_j}[-2] \arrow[r] & \Omega^\bullet_{X,D} \arrow[r] & \Omega^\bullet_{X\setminus D_j, D\setminus D_j} \arrow[r] & {}
\end{tikzcd}$$

Here the exact triangles are from Proposition A.4, and the vertical arrows are of the sort just described. By induction the first and last map are quasi-isomorphisms, as in each case is one fewer divisor. Thus the center map is a quasi-isomorphism as well. □

Corollary A.6. *The exact triangle $\mathbb{C}_{D_j}[-2] \to \mathbb{C}_X \to j_*\mathbb{C}_{X\setminus D_j}$ is quasi-isomorphic to the triangle $\Omega^\bullet_{D_j,\mathbf{K}_j}[-2] \to \Omega^\bullet_{X,D} \to \Omega^\bullet_{X\setminus D_j, D\setminus D_j}$.*

A.3. **Filtrations.** If V is a vector space with an increasing filtration F, we write the steps of the filtration as

$$\cdots \subset F_{-1}V \subset F_0 V \subset F_1 V \subset \cdots$$

We recall

Definition A.7. Let V, W be filtered vector spaces. A map $g : V \to W$ is said to be:

- compatible with the filtrations if $g(F_k V) \subset F_k W$
- strictly compatible with the filtrations if $F_k W \cap g(V) = g(F_k V)$

We will also synonymously say the map (strictly) preserves or (strictly) respects the filtration.

The significance of the strictness condition is:

Lemma A.8. *Let ... $\to V_{-1} \xrightarrow{a} V_0 \xrightarrow{b} V_1 \to$ ... be a long exact sequence of filtered vector spaces, whose maps strictly preserve the filtrations. Then the sequence defined by the associated graded spaces ... $\to \mathrm{gr}_k V_{-1} \xrightarrow{\mathrm{gr}(a)} \mathrm{gr}_k V_0 \xrightarrow{\mathrm{gr}(b)} \mathrm{gr}_k V_1 \to$ ... is also exact.*

Proof. We first show check that the kernel of $\mathrm{gr}(b)$ contains the image of $\mathrm{gr}(a)$. Given $u \in F_i V$, write $\mathrm{gr}_i(u)$ for the associated element of $\mathrm{gr}_i V$. Consider $w \in F_k V_{-1}$. By exactness of the original sequence, $b(a(w)) = 0$. It follows that $\mathrm{gr}(b)(\mathrm{gr}(a)(\mathrm{gr}_k(w))) = 0$.

We now show that the image of $\mathrm{gr}(a)$ contains the kernel of $\mathrm{gr}(b)$. Suppose $v \in F_k V_0$ satisfies $\mathrm{gr}(b)(\mathrm{gr}_k(v)) = 0$. By definition, this means $b(v) \in F_{k-1}V_1$. By strictness of b, there exists $v' \in F_{k-1}V_0$ such that $b(v') = b(v)$. By exactness of the sequence, there exists $w \in V_{-1}$ such that $a(w) = v - v'$. By strictness, there exists $w' \in F_k V_{-1}$ with the same property. It follows that $\mathrm{gr}(a)(\mathrm{gr}_k(w')) = \mathrm{gr}_k(v - v') = \mathrm{gr}_k(v)$. ☐

Caution A.9. For a fixed short exact sequence $0 \to A \to B \to C \to 0$ of vector spaces, and fixed filtrations on A, C, there are many filtrations on B such that the maps strictly preserve filtrations. Indeed let A and C be one-dimensional, with $\mathrm{gr}_1 A = A, \mathrm{gr}_0 C = C$. Then the filtration on B is determined by the subspace $F_0 B \cong C$. The only condition on $F_0 B$ is that is must intersect the image of A trivially.

A.4. **Weight filtration.** According to [Del1, Del2, Del3], cohomology of an algebraic varieties carry various filtrations, strictly preserved by pullback with respect to any morphism of algebraic varieties. Of relevance to us here is the weight filtration, defined on the rational cohomology, and denoted $W_\bullet \mathrm{H}^\bullet(X, \mathbb{Q})$. It is an increasing filtration, with $W_\bullet \mathrm{H}^n(X, \mathbb{Q})$ supported in degrees $[0, 2n]$ in general, and in degrees $[0, n]$ and $[n, 2n]$ if X is projective and smooth respectively.

The weight filtration of a smooth variety X is defined by choosing a normal crossings compactification. Then the complex cohomology of X is computed by a complex of differential forms with log singularities along the boundary, and $W_{\bullet+k}\mathrm{H}^\bullet(X, \mathbb{Q})$ is generated by the subsheaf of forms singular along at most k different boundary components near any given point.

In this sense, the size of $Gr^W_{\bullet+k}\mathrm{H}^\bullet(X, \mathbb{Q})$ for $k > 0$ is a measure of the non-compactness of X.

A.5. **Perverse Leray filtration.** Let $\mathbf{B}$ be a topological space. For a complex of sheaves K on $\mathbf{B}$, one can define a filtration on the (hyper)cohomology of $K^\bullet$ by cutting off the complex:

$$P_k \mathrm{H}^\bullet(\mathbf{B}; K) := Image(\mathrm{H}^\bullet(\mathbf{B}; \tau^{\leq k} K) \to \mathrm{H}^\bullet(\mathbf{B}; K))$$

In case one has another t-structure available — in our case the middle perverse t-structure for constructible sheaves on algebraic varieties — one gets a similar filtration by using the truncations of the t-structure. We term the resulting filtration the *perverse filtration*.

In the setting where one has a map $\pi\colon \mathbf{X} \to \mathbf{B}$ and $K = \pi_* F$, the perverse filtration on K is the filtration which arises on $\mathrm{H}^\bullet(\mathbf{X}; F)$ from the perverse t-structure Leray spectral sequence. Thus it is called the *perverse Leray filtration* on $\mathrm{H}^\bullet(\mathbf{X}; F)$. See [dCHM] and the references therein for discussion of this filtration.

Convention A.10. Let $f : X \to Y$ be a map of algebraic or complex analytic spaces. When we discuss the perverse Leray filtration $\mathrm{H}^\bullet(X, \mathbb{Q})$ associated by f, i.e. the perverse filtration on $\mathrm{H}^\bullet(f_*\mathbb{Q})$, we always shift the filtration so that $P_{-1} = 0$ and $1 \in P_0 \mathrm{H}^\bullet(X, \mathbb{Q})$.

In some circumstances, we may wish to further shift the filtration. We will write $\mathrm{H}^\bullet(X, \mathbb{Q})\{n\}$ to indicate we have shifted the filtration by n steps, i.e. $P_i \mathrm{H}^\bullet(X, \mathbb{Q})\{n\} = P_{i+n} \mathrm{H}^\bullet(X, \mathbb{Q})$.

Caution A.11. Let $f\colon \mathbf{X} \to \mathbb{C}$ be a function such that $\mathrm{H}^\bullet(\mathbf{X}; F) \xrightarrow{\sim} \mathrm{H}^\bullet(f^{-1}(0), F|_{f^{-1}(0)})$. Then these two cohomologies nonetheless will generally acquire different perverse Leray filtrations from the maps $\pi\colon \mathbf{X} \to \mathbf{B}$ and $\pi|_{f^{-1}(0)}\colon f^{-1}(0) \to \mathbf{B}$.

Pullback and pushforward operations generally respect only 'half' of the perverse t-structure. As a consequence, the perverse Leray filtration is not generally preserved by base-change. (In particular, the perverse Leray filtration of the base-change to a point is always trivial, in the sense of agreeing with the filtration by cohomological degree.) However, base-change which is "transverse to the singularities of the sheaf" does respect perversity. This can be precisely formulated in the language of microsupport of Kashiwara-Schapira, as we now recall.

Proposition A.12. [KS, Cor. 10.3.16] *Let $N \subset M$ be an inclusion of smooth complex submanifolds and $\mathcal{F}$ a perverse sheaf on M. If N is noncharacteristic for $\mathcal{F}$, i.e., $ss(\mathcal{F}) \cap T^*_N M \subset T^*_M M$, then $\mathcal{F}|_N$ is perverse.*

To use this fact, we recall the standard estimate on the the microsupport of a push-forward. Let $f : Y \to X$ be a map of manifolds, and consider the diagram

$$T^*Y \xleftarrow{df^*} Y \times_X T^*X \xrightarrow{\tilde{f}} T^*X.$$

Let $f_\dagger = \tilde{f} \circ (df^*)^{-1}$ be the composition; it maps subsets of T^*Y to subsets of T^*X. We will be interested in the image $f_\dagger(T^*_Y Y)$ for certain maps. This is a conical subspace of T^*X.

Given $y \in Y$, thought of as an element of the zero section of T^*Y, $f_\dagger(y) \subset T^*_{f(y)} X$ is the orthogonal complement to $df(T_y Y) \subset T_{f(y)} Y$.

By construction, the intersection of $f_\dagger(T^*_Y Y)$ with the cotangent fiber to $x \in X$ is the union over all $y \in f^{-1}(x)$ of $f_\dagger(y)$. Thus $f_\dagger(T^*_Y Y)$ measures the failure of df to be surjective at any point in the fiber $f^{-1}(x)$, and is a coarse measure of the variation in the fiber of f.

The relation to microsupports is:

Proposition A.13. [KS, Prop. 5.4.4] *For f proper, $ss(f_*\mathcal{F}) \subset f_\dagger(ss(\mathcal{F}))$, and in particular, $ss(f_*\mathbb{C}_Y) \subset f_\dagger(T^*_Y Y)$.*

The microsupport of a constructible sheaf is always a conical Lagrangian, as is $f_\dagger(T^*_Y Y)$. A conical Lagrangian is the union of closures to conormals of some finite collection of locally closed submanifolds, and it is convenient to describe it in terms of these submanifolds. Indeed, for conical Lagrangian $\Lambda = \bigcup \overline{T}^*_{N_\alpha} X \subset T^*X$, we have Y noncharacteristic to Λ if Y is transverse to all N_α.

Decompose $f_\dagger(T^*_Y Y) = \bigcup \overline{T}^*_{N_\alpha} X \subset T^*X$. We write $\Delta^i(f)$ for the union of the N_α which have codimension i.

Remark A.14. $\Delta^1(f)$ is the usual discriminant locus of f. In [MS2], the other $\Delta^i(f)$ are termed "higher discriminants" and given various alternative characterizations.

Combining the above recalled facts:

Corollary A.15. *Assume $f\colon Y \to X$ is proper, and $V \subset X$ is a smooth subspace. Assume V is transverse to all $\Delta^i(f)$. Then the restriction map $\mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(Y|_V)$ carries the perverse Leray filtration on $\mathrm{H}^\bullet(Y)$ induced by f to the perverse Leray filtration on $\mathrm{H}^\bullet(Y|_V)$ induced by $f|_V$.*

Proof. The first paragraph follows from the above recalled [KS, Prop. 5.4.4] and [KS, Cor. 10.3.16]. The last statement is elementary. □

We note also:

Lemma A.16. *Let $W \subset X$ be a smooth subspace, transverse to f (i.e. $T_xW + df(T_yY) = T_xX$ for all $x \in W, y \in f^{-1}(w)$). Then $\Delta^i(f|_W) = \Delta^i(f) \cap W$.*

REFERENCES

[Ale] Valery Alexeev, *Compactified Jacobians and Torelli map*, Publ. RIMS Kyoto Univ. **40** (2004), 1241–1265.

[AK] Allen Altman and Steven Kleiman, *Compactifying the Picard Scheme*, Adv. in Math. **35** (1980), 50–112.

[AK2] Allen Altman and Steven Kleiman, *Compactifying the Picard scheme. II*, Amer. J. Math., 101.1 (1979) 10–41.

[BBD] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, *Faisceaux pervers*, Astèrisque **100** (1982), 5–171.

[BK] Roman Bezrukavnikov and Mikhail Kapranov, *Microlocal sheaves and quiver varieties*, arXiv:1506.07050.

[BD] Roger Bielawski and Andrew Dancer, *The geometry and topology of toric hyperkähler manifolds*, Communications in Analysis and Geometry 8.4 (2000), 727–760.

[BB] Olivier Biquard and Philip Boalch, *Wild non-abelian Hodge theory on curves*, Compositio Mathematica 140.1 (2004), 179–204.

[Bol] Bela Bollabás *Modern graph theory*, Springer-Verlag, Berlin, New York (1998)

[dC] Mark de Cataldo, *Hodge theoretic splitting mechanisms for projective maps*, J. Singularities **7** (2013), 134–156.

[dCHM] Mark de Cataldo, Tamas Hausel, and Luca Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case A_1*, Annals of Mathematics 175.3 (2012), 1329–1407.

[dCHM2] Mark de Cataldo, Tamas Hausel, and Luca Migliorini, *Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces*, Journal of Singularities 7 (2013), 23–38.

[dCM] Mark de Cataldo and Davesh Maulik, *The perverse filtration for the Hitchin fibration is locally constant*, arXiv:1808.02235.

[dCMS] Mark de Cataldo, Davesh Maulik, and Junliang Shen. *Hitchin fibrations, abelian surfaces, and the P=W conjecture*, arXiv:1909.11885.

[CKS] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *L^2 and intersection cohomology for a polarized variation of Hodge structure*, Invent. Math. **87** (1987), 217–252.

[CL] Pierre-Henri Chaudouard and Gérard Laumon, *Le lemme fondamental pondéré I : constructions géométriques*, Compos. Math. **146**.6 (2010), 1416–1506.

[Cor] Kevin Corlette, *Flat Q-bundles with canonical metrics*, J. Diff. Geom., 28 (1988), 361–382.

[CDP1] Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Guang Pan, *Wall-crossing and cohomology of the moduli space of Hitchin pairs*, Commun. Num. Theor. Phys. 5 (2011), 1–56.

[CDP2] Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Guang Pan, *BPS states and the P = W conjecture*, in: Moduli Spaces, vol. 411 of London Math. Soc. Lecture Note Ser. (Cambridge University Press, 2014), 132–150.

[CDDP] Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Ron Donagi, and Tony Pantev, *Parabolic refined invariants and Macdonald polynomials* Commun. Math. Phys. 335.3 (2015), 1323–1379.

[CDDNP] Wu-yen Chuang, Duiliu-Emanuel Diaconescu, Ron Donagi, Satoshi Nawata, Tony Pantev, *Twisted spectral correspondence and torus knots*, arxiv:1804.08364.

[CBS] William Crawley-Boevey and Peter Shaw, *Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem*, Advances in Mathematics 201.1 (2006), 180-208.

[Del0] Pierre Deligne, Équations différentielles à points singuliers réguliers. LNM 163. Springer, 2006.

[Del1] Pierre Deligne, *Theorie de Hodge, I*, Actes du Congrès international des mathématiciens, Nice (1970).

[Del2] Pierre Deligne, *Theorie de Hodge, II*, Publications Mathématiques de l'IHES **40**.1 (1971), 5-57.

[Del3] Pierre Deligne, *Theorie de Hodge, III*, Publications Mathématiques de l'IHES **44**.1 (1974), 5-77.

[Del4] Pierre Deligne, *Decompositions dans la categorie dérivée*, in Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. 55, Part 1, (Amer. Math. Soc., 1994) 115–128.

[DR] Pierre Deligne and Michael Rapoport, *Les Schémas de Modules de Courbes Elliptiques*, in Modular Functions of One Variable II, Springer LNM 349 (1973), 143–316.

[Delz] Thomas Delzant, *Hamiltoniens périodiques et images convexes de l'application moment.* Bulletin de la Société mathématique de France 116.3 (1988), 315-339.

[Deo] Vinay Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Inventiones mathematicae 79.3 (1985): 499–511.

[Dia] Duiliu-Emanuel Diaconescu, *Local curves, wild character varieties, and degenerations*, arXiv:1705.05707.

[DDP] Duiliu-Emanuel Diaconescu, Ron Donagi, and Tony Pantev, *BPS states, torus links and wild character varieties*, Comm. Math. Phys. 357 (2018).

[D'S] Cyril D'Souza, *Compactification of generalised Jacobians*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88**.5 (1979) 419–457.

[Don] Simon Donaldson, *Twisted harmonic maps and self-duallty equations*, Proc. London Math. Soc (3), 55 (1987), 127–131.

[ER] Edna Jasso-Hernandez and Yongwu Rong, *A categorification for the Tutte polynomial*, Algebraic & Geometric Topology 6.5 (2006), 2031-2049.

[Est] Eduardo Esteves, *Compactifying the relative Jacobian over families of reduced curves*, Trans. Amer. Math. Soc. **353** (2001), 3045–3095.

[GMN] Davide Gaiotto, Gregory Moore and Andrew Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, arXiv:0807.4723.

[GMN2] Davide Gaiotto, Gregory Moore and Andrew Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, arXiv:0907.3987.

[GMcW] Benjamin Gammage, Michael McBreen, and Ben Webster. *Homological Mirror Symmetry for Hypertoric Varieties II*, arXiv:1903.07928.

[GS] Benjamin Gammage and Vivek Shende, *Mirror symmetry for very affine hypersurfaces*, arXiv:1707.02959.

[GPS1] Sheel Ganatra, John Pardon, and Vivek Shende, *Covariantly functorial wrapped Floer theory on Liouville sectors*, arXiv:1706.03152.

[GPS2] Sheel Ganatra, John Pardon, and Vivek Shende, *Structural results in wrapped Floer theory*, arXiv:1809.03427.

[GPS3] Sheel Ganatra, John Pardon, and Vivek Shende, *Microlocal Morse theory of wrapped Fukaya categories*, arxiv:1809.08807.

[Gie] David Gieseker, *On the moduli of vector bundles on an algebraic surface*, Annals of Math. **106**.1 (1977) 45–60.

[GORS] Eugene Gorsky, Alexei Oblomkov, Jacob Rasmussen, Vivek Shende, *Torus knots and the Rational DAHA*, Duke Math. J. 163.14 (2014), 2709–2794.

[Go] Ryushi Goto, *On hyper-Kähler manifolds of type* A_∞ , Geometric & Functional Analysis GAFA 4.4 (1994), 424-454.

[Gr] Mark Gross, *Topological mirror symmetry*, Inventiones mathematicae, 144 (2001) 75–137.

[GrWi] Mark Gross and Pelham Wilson, *Large complex structure limits of K3 surfaces*, Journal of Differential Geometry 55.3 (2000): 475–546.

[Har] Andrew Harder, *Torus fibers and the weight filtration*, arXiv:1908.05110.

[HKP] Andrew Harder, Ludmil Katzarkov, Victor Przyjalkowski, *P=W phenomena*, arxiv:1905.08706.

[HLSY] Andrew Harder, Zhiyuan Li, Junliang Shen, and Qizheng Yin, *P=W for Lagrangian fibrations and degenerations of hyperkähler manifolds*, arxiv:1908.07510.

[HS] Tamás Hausel and Bernd Sturmfels, *Toric hyperkähler varieties*, Doc. Math 7 (2002), 495–534.

[HP] Tamás Hausel and Nicholas Proudfoot, untitled note, circa 2006.

[HRV] Tamás Hausel and Fernando Rodrigues-Villegas, *Mixed Hodge polynomials of character varieties*, Inventiones mathematicae 174.3 (2008), 555-624.

[HR] Laure Helme-Guizon and Yongwu Rong, *A categorification for the chromatic polynomial*, Algebraic & Geometric Topology 5.4 (2005), 1365–1388.

[HR2] Laure Helme-Guizon and Yongwu Rong, *Graph cohomologies from arbitrary algebras*, arXiv:math/0506023.

[H1] Nigel Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc (3), 55 (1987), 59–126.

[H2] Nigel Hitchin, *Stable bundles and integrable systems*, Duke Mathematical Journal 54.1 (1987), 91–114.

[HKLR] Nigel Hitchin, Anders Karlhede, Ulf Lindström, and Martin Roçek, *Hyperkähler metrics and supersymmetry*, Communications in Mathematical Physics, 108.4 (1987), 535–589.

[K] Maxim Kontsevich, *Symplectic geometry of homological algebra*, available at the author's webpage (2009).

[Kf] George Kempf, *The Grothendieck-Cousin complex of an induced representation*, Advances in Mathematics, 29.3 (1978) : 310–396.

[KN] George Kempf and Linda Ness, *The length of vectors in representation spaces.* Algebraic geometry. Springer, Berlin, Heidelberg, 1979. 233-243.

[KS] Masaki Kashiwara and Pierre Schapira, Sheaves on Manifolds, Springer (1990).

[Kho] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Mathematical Journal 101.3 (2000), 359–426.

[MV] Margarida Melo and Filippo Viviani, *Fine compactified Jacobians*, Math. Nach. **285** (2012), no. 8-9, 997–1031.

[MRV] Margarida Melo, Antonio Rapagnetta, and Filippo Viviani, *Fine compactified Jacobians of reduced curves*, Trans AMS 369.8 (2017), 5341–5402.

[Mau] Davesh Maulik, *Stable pairs and the HOMFLY polynomial*, Inventiones mathematicae 204.3 (2016): 787–831.

[Mel] Anton Mellit, *Cell decompositions of character varieties*, arXiv:1905.10685.

[MS] Luca Migliorini and Vivek Shende, *A support theorem for Hilbert schemes of planar curves*, J. EMS 15.6 (2013): 2353–2367.

[MS2] Luca Migliorini and Vivek Shende, *Higher discriminants and the topology of algebraic maps*, Algebraic Geometry 5.1 (2018): 114–130.

[MSV] Luca Migliorini, Vivek Shende, and Filippo Viviani, *A support theorem for Hilbert schemes, II*, arxiv:1508.07602.

[McW] Michael McBreen and Ben Webster, *Homological Mirror Symmetry for Hypertoric Varieties I*, arXiv:1804.10646.

[Mum1] David Mumford, *An analytic construction of degenerating curves over complete local rings.* Compositio Mathematica 24.2 (1972): 129-174

[Mum2] David Mumford, *An analytic construction of degenerating abelian varieties over complete rings.* Compositio Mathematica 24.3 (1972): 239-272.

[N] David Nadler, *Wrapped microlocal sheaves on pairs of pants*, arXiv:1604.00114.

[Nak] Noboru Nakayama, *The lower semi-continuity of the plurigenera of complex varieties.* Algebraic geometry, Sendai, 1985. Mathematical Society of Japan, 1987.

[Ngô] Bâu Cháu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. IHES **111** (2010), 1–169.

[OS] Tadao Oda and C. S. Seshadri, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90.

[ObS] Alexei Oblomkov and Vivek Shende, *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link*, Duke Mathematical Journal 161.7 (2012): 1277–1303.

[ORS] Alexei Oblomkov, Jacob Rasmussen, and Vivek Shende, The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link, Geometry & Topology 22.2 (2018): 645–691.

[OV] Hirosi Ooguri and Cumrun Vafa, *Summing up Dirichlet instantons*. Phys. Rev. Lett. 77.16 (1996): 3296–3298.

[Ses] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **96** (1982).

[SZ] Junliang Shen and Zili Zhang, *Perverse filtrations, Hilbert schemes, and the P=W conjecture for parabolic Higgs bundles*, arXiv:1810.05330.

[S1] Carlos Simpson, *Higgs bundles and local systems*, Publ. Math. de l'IHES, 75 (1992), 5–95.

[S2] Carlos Simpson, *Moduli of representations of the fundamental group of a smooth projective variety, I*, Publ. math. de l'IHES 79.1 (1994), 47–129.

[S3] Carlos Simpson, *Moduli of representations of the fundamental group of a smooth projective variety, II*, Publ. math. de l'IHES 80.1 (1994), 5–79.

[S4] Carlos Simpson, *Harmonic bundles on noncompact curves*, Journal of the American Mathematical Society, 3.3 (1990): 713–770.

[S5] Carlos Simpson, *The dual boundary complex of the SL_2 character variety of a punctured sphere*, Ann. Fac. Sci. Toulouse, Math. (6) 25.2-3, Part A, (2016), 317–361.

[Sto] Marko Stošić, *Categorification of the dichromatic polynomial for graphs*, Journal of Knot Theory and Its Ramifications 17.01 (2008), 31–45.

[Sto2] Marko Stošić, *New categorifications of the chromatic and the dichromatic polynomials for graphs*, Fund. Math. 190 (2006), 231–243.

[STZ] Vivek Shende, David Treumann, Eric Zaslow, *Legendrian knots and constructible sheaves*, Inventiones mathematicae 207.3 (2017): 1031–1133.

[STWZ] Vivek Shende, David Treumann, Harold Williams, Eric Zaslow, *Cluster varieties from Legendrian knots*, arXiv:1512.08942.

[SW] Nathan Seiberg and Edward Witten, *Electric-magnetic duality, monopole condensation, and confinement in N= 2 supersymmetric Yang-Mills theory*, Nuclear Physics B 426.1 (1994), 19–52.

[Szi] Szilárd Szabó, *Perversity equals weight for Painlevé spaces*, arXiv:1802.03798.

[Th] Michael Thaddeus, two letters to Hausel and Proudfoot, circa 2008.

[Thi] M. B. Thistlewaite, *A spanning tree expansion of the Jones polynomial*, Topology 26 (1987), 297–309.

[Y] Daisuke Yamakawa, *Geometry of multiplicative preprojective algebra*, International Mathematics Research Papers 2008 (2008).

[Z] Zili Zhang, *The P=W identity for cluster varieties*, arxiv:1903.07014.