REGULARIZING EFFECT OF HOMOGENEOUS EVOLUTION EQUATIONS WITH PERTURBATION

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CONTENTS

ABSTRACT. Since the pioneering works by Aronson & Bénilan [C. R. Acad. Sci. Paris Sér., 1979], and Bénilan & Crandall [Johns Hopkins Univ. Press, 1981] it is well-known that first-order evolution problems governed by a nonlinear but homogeneous operator admit the smoothing effect that every corresponding mild solution is Lipschitz continuous at every positive time. Moreover, if the underlying Banach space has the Radon-Nikodým property, then these mild solution is a.e. differentiable, and the time-derivative satisfies global and point-wise bounds.

In this paper, we show that these results remain true if the homogeneous operator is perturbed by a Lipschitz continuous mapping. More precisely, we establish global L^1 *Aronson-Bénilan type estimates* and *point-wise Aronson-Bénilan type estimates*. We apply our theory to derive global L^q-L^∞-estimates on the time-derivative of the perturbed diffusion problem governed by the Dirichlet-to-Neumann operator associated with the p-Laplace-Beltrami operator and lower-order terms on a compact Riemannian manifold with a Lipschitz boundary.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we establish global regularity estimates on the time-derivative $\frac{du}{dt}$ of *mild* solutions u (see Definition 3.2) to the Cauchy problem associated

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with the perturbed operator $A+F$;

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) + F(u(t)) \ni f(t) & \text{for } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

for sufficiently regular $f : [0,T] \to X$ and initial data u_0. To ensure the well-posedness of Cauchy problem (1.1), we assume that A is an m-accretive, possibly, multi-valued operator $A : D(A) \to 2^X$ on a Banach space $(X, \|\cdot\|_X)$ (see Definition 3.1) with *effective domain* $D(A) := \{u \in X \,|\, Au \neq \emptyset\}$ and $F : X \to X$ a Lipschitz continuous mapping with constant $\omega \geq 0$ satisfying $F(0) = 0$.

The crucial condition to obtain global regularity estimates of $\frac{du}{dt}$ for mild solutions u of (1.1) is that A is *homogeneous* of order $\alpha \neq 1$; that is, $(0,0) \in A$ and

$$A(\lambda u) = \lambda^{\alpha} Au \qquad \text{for all } \lambda \geq 0 \text{ and } u \in D(A). \tag{1.2}$$

We emphasize that the governing operator $A+F$ in Cauchy problem (1.1) is not anymore homogeneous. Thus, our first main result can be understood as a perturbation theorem.

Theorem 1.1 (L^1 Aronson-Bénilan type estimates)**.** *For given $\alpha \in \mathbb{R} \setminus \{1\}$, let A be an m-accretive operator in X which is homogeneous of order α and suppose, the mapping $F : X \to X$ is Lipschitz continuous on X with constant $\omega \geq 0$, $F(0) = 0$, and let $f \in BV(0,T;X)$. Then for every $u_0 \in D(A)$, the mild solution u of* (1.1) *satisfies*

$$\limsup_{h\to 0+} \frac{\|u(t+h) - u(t)\|_X}{h} \leq \frac{1}{t}\Big[a_\omega(t) + \omega \int_0^t a_\omega(s) e^{\omega(t-s)} \mathrm{d}s\Big] \tag{1.3}$$

for a.e. $t \in (0,T)$, where

$$\begin{aligned} a_\omega(t) := V_0(f,t) + \frac{1}{|1-\alpha|}\Big[& \left(1 + e^{\omega t}\right) \|u_0\|_X \\ & + \int_0^t \|f(s)\|_X \,\mathrm{d}s + \omega \int_0^t \int_0^s e^{-\omega r} \|f(r)\|_X \mathrm{d}r\, \mathrm{d}s\Big]. \end{aligned} \tag{1.4}$$

and $V_0(f,\cdot)$ is given by (2.7) *below. In particular, if for $u_0 \in D(A)$, the right-hand side derivative $\frac{\mathrm{d}u}{\mathrm{d}t_+}$ exists, then*

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)\right\|_X \leq \frac{1}{t}\Big[a(t) + \omega \int_0^t a(s) e^{\omega(t-s)} \mathrm{d}s\Big] \quad \text{for a.e. } t \in (0,T). \tag{1.5}$$

At the first view, it seems that in Theorem 1.1, the hypothesis $u_0 \in D(A)$ merely provides a global point-wise estimate on the time-derivative $\frac{\mathrm{d}u}{\mathrm{d}t}(t)$, but not a regularization effect. This hypothesis together with the condition $f \in BV(0,T;X)$ imply that the mild solution u is Lipschitz continuous (see Proposition 3.6), which is required to apply Gronwall's lemma (see Lemma 2.7. But, starting from this, a standard density argument combined with an appropriate compactness result yield that estimate (1.3) holds for all mild solutions u of Cauchy problem (1.1).

For example, under the additional hypothesis that the Banach space X is *reflexive*, one has that the closed unit ball of X is weakly sequentially compact. Now, for every given $u_0 \in \overline{D(A)}^X$, there is a sequence $(u_n^{(0)})_{n\geq 1}$ in $D(A)$

such that $u_n^{(0)} \to u_0$ in X and by the ω-quasi contractivity of the semigroup $\{T_t\}_{t=0}^T$ generated by $-(A+F)$ on $\overline{D(A)}^X \times L^1(0,T;X)$ (see Definition 3.3), one has that $T_t(u_n^{(0)}, f) \to T_t(u_0, f)$ in X as $n \to \infty$. Thus, if for every $n \geq 1$, $u_n(t) := T_t(u_n^{(0)}, f)$, $t \geq 0$, satisfies (1.5), then the sequence $(\frac{\mathrm{d}u_n}{\mathrm{d}t})_{n\geq 1}$ is bounded $L^\infty(\delta, T; X)$ for every $\delta \in (0,T)$. From this, one can conclude the following smoothing effect of such semigroups acting on reflexive Banach spaces (see also Corollary 3.11 in Section 3).

Corollary 1.2. *Let A be an m-accretive operator on a reflexive Banach space X, $F : X \to X$ a Lipschitz continuous mapping with Lipschitz-constant $\omega \geq 0$ satisfying $F(0) = 0$, and $\{T_t\}_{t=0}^T$ the semigroup generated by $-(A+F)$ on $\overline{D(A)}^X \times L^1(0,T;X)$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $f \in BV(0,T;X)$, the unique mild solution u of Cauchy problem (1.1) is strong and satisfies (1.5) for a.e. $t \in (0,T)$.*

We outline the proof of this corollary in Section 3.

Our second main result of this paper is concerned with a point-wise estimate on the time-derivative $\frac{du}{dt}$ of positive[1] strong solutions u of the homogeneous Cauchy problem

$$
\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) + F(u(t)) \ni 0 & \text{for } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{1.6}
$$

under the additional hypothesis that the underlaying Banach space X is equipped with a partial ordering "$\leq$" such that the triple $(X, \|\cdot\|_X, \leq)$ defines an Banach lattice, and if for this ordering "$\leq$", every mild solution u of (1.6) is *order-preserving*; that is, for every u_0, $\hat{u}_0 \in \overline{D(A)}^X$ with corresponding mild solutions u and $\hat{u}$ of (1.6), one has that $u_0 \leq \hat{u}_0$ implies $u(t) \leq \hat{u}(t)$ for all $t \in (0,T]$.

Theorem 1.3 (Point-wise Aronson-Bénilan type estimates)**.** *Let A be an m-accretive operator on X, $(X, \|\cdot\|_X, \leq)$ a Banach lattice, and let $F : X \to X$ be a Lipschitz continuous mapping on X with constant $\omega \geq 0$ satisfying $F(0) = 0$. Suppose, for $\alpha \in \mathbb{R} \setminus \{1\}$, A is homogeneous of order α and every mild solution u of (1.6) is* order-preserving. *For every positive $u_0 \in \overline{D(A)}^X$, the mild solution u of (1.6) satisfies*

$$
\frac{u(t+h)-u(t)}{h} \geq \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h} \frac{u(t)}{t} + g_h(t) \qquad \textit{if } \alpha > 1
$$

and

$$
\frac{u(t+h)-u(t)}{h} \leq \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h} \frac{u(t)}{t} + g_h(t) \qquad \textit{if } \alpha < 1,
$$

for every t, $h > 0$, where $g_h : (0,\infty) \to X$ is a continuous function. Further, for positive $u_0 \in \overline{D(A)}^X$, if the right hand-side derivative $\frac{\mathrm{d}u}{\mathrm{d}t_+}$ belongs to $L^1_{loc}([0,\infty); X)$, then

$$
(\alpha - 1)\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) \geq -\frac{u(t)}{t} + (\alpha-1)g_0(t), \tag{1.7}
$$

for a.e. $t > 0$, where $g_0 : (0,\infty) \to X$ is a measurable function.

[1] That is, $u \geq 0$ for the given partial ordering "$\leq$" on X.

Theorem 1.3 follows from the slightly more general statement provided in Theorem 2.9 and by Corollary 2.11 in Section 2.

It is worth mentioning some words about the origin of the names assigned to the estimates (1.3) (respectively, (1.5)) and (1.7). Even though the result was already mentioned earlier in [4, p. 5] by Aronson, the point-wise estimate (1.7) was first proved by Aronson & Bénilan [5] for (strong) solutions u of the porous medium equation $u_t = \Delta u^m$ in $[0, +\infty) \times \mathbb{R}^d$ for $d \geq 1$ and $m > [d-2]^+/d$. In the same paper [5, Théorème 2.], they also proved that (strong) solutions of this porous media equation satisfy the L^1-estimate (1.5). Shortly afterwards, Bénilan and Crandall [9] made available the two global inequalities (1.3) and (1.7) for mild solutions u of the unperturbed Cauchy problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni 0 & \text{in } (0,\infty), \\ u(0) = u_0, \end{cases} \tag{1.8}$$

governed by nonlinear m-accretive operators A, which are homogeneous of order $\alpha > 0$, $\alpha \neq 0$. This class of operators include the local p-Laplace operator Δ_p, the local doubly nonlinear operator $\Delta_p u^m$, $1 < p < \infty$, $m > 0$, as well as the nonlocal fractional p-Laplace operator $(-\Delta_p)^s$, respectively equipped with various boundary conditions (see, for instance, [18] for more details to the analytic properties of these quasi-linear 2nd-order differential operators).

In the papers [19] and [20] Crandall and Pierre showed that every mild solution of the more general version of the porous medium equation $u_t = \Delta\varphi(u)$, where φ is an increasing function on $\mathbb{R}$, also satisfy the point-wise Aronson-Bénilan estimate (1.7). These two results by Crandall and Pierre were slightly improved in a short paper by Casseigne [16]. Estimate (1.7) has been established in various settings; on manifolds (see, e.g. [31, 14]), and with drift-term (see, e.g, [30]), or with a linear perturbation (see, e.g., [15]). One important reason among others, for the strong further development of the point-wise estimate (1.7) is that it can be used, for example, to derive Harnack-type inequalities (see, e.g., [6], but also [22, 23]) and to study the regularity of the free-boundaries (see, for instance, [34] or [39]). We refer the interested reader to the book [40] by Vázquez (and more recently [12]) for a detailed exposition concerning the development of the point-wise Aronson-Bénilan estimate (1.7) satisfied by solutions to the porous media equation.

Recently, the author and Mazón showed in [27] that the two Aronson-Bénilan type estimates (1.5) and (1.7) are satisfied by the mild solutions of the unperturbed Cauchy problem (1.8) for homogeneous operators of order zero (i.e., $\alpha = 0$). This class of operators includes, for example, the (negative) total variational flow operator $Au = -\mathrm{div}(\frac{Du}{|Du|})$, or the 1-fractional Laplacian $A = (-\Delta_1)^s$ for $s \in (0,1)$ respectively equipped with some boundary conditions. By tackling the the L^1 Aronson-Bénilan inequality (1.5) for mild solutions of the perturbed (homogeneous) Cauchy problem (1.6), their proof, unfortunately, contains a slightly wrong argument in the application of Gronwall's lemma. Thus, the proof of Theorem 2.6 presented here corrects this flaw.

If the operator A in (1.8) is *linear* (and hence $\alpha = 1$), then estimate

$$\|Au(t)\|_X \leq C\frac{\|u(0)\|_x}{t}, \qquad (t \in (0,1],\ u(0) \in D(A)), \tag{1.9}$$

yields that the operator $-A$ generates an analytic semigroup $\{T_t\}_{t\geq 0}$ (cf., [2, 33]). Thus, it is interesting to see that a regularity inequality (1.5), which is similar to (1.9), also holds for nonlinear operators of the type $A+F$, where A is homogeneous of order $\alpha \neq 1$. Further, if the norm $\|\cdot\|_X$ is induced by an inner product $(\cdot,\cdot)_X$ of a Hilbert space X and $A = \partial\varphi$ is the sub-differential operator $\partial\varphi$ in X of a semi-convex, proper, lower semicontinuous function $\varphi : X \to (-\infty, +\infty]$, then regularity inequality (1.9) is, in particular, satisfied by solutions u of (1.8) (cf., [13, 17]). It is worth mentioning that inequality (1.9) plays a crucial role in abstract 2^{nd}-order problems of elliptic type involving accretive operators A (see, for example, [35, (2.22) on page 525] or, more recently, [26, (1.8) on page 719]).

In many applications, the Banach space X is given by the classical Lebesgue space $(L^q := L^q(\Sigma,\mu), \|\cdot\|_q)$, $(1 \leq q \leq \infty)$, for a given σ-finite measure space (Σ,μ). If, in addition, the mild solutions u of Cauchy problem (1.6) satisfy a global *L^q-L^r regularity estimate* ($1 \leq q, r \leq \infty$, cf., [18])

$$\|u(t)\|_r \leq C\, e^{\omega t} \frac{\|u(0)\|_q^{\gamma}}{t^{\delta}} \qquad \text{for all } t > 0, \tag{1.10}$$

holding for some $C > 0$, γ, $\delta > 0$, then by combining (1.5) with (1.10) leads to

$$\limsup_{h\to 0+} \frac{\|u(t+h)-u(t)\|_r}{h} \leq C\, 2^{\delta+2}\, e^{\omega t}\, \frac{\|u_0\|_q^{\gamma}}{t^{\delta+1}}. \tag{1.11}$$

We outline this result in full details in Corollary 2.8. Regularity estimates similar to (1.10) have been studied recently by many authors (see, for example, [21, 38, 24] and the references therein for the linear theory, and we refer to [18] and the references therein for the nonlinear one). The idea to combine an L^q-L^r regularity estimate (1.10) for $q = 1$ and $r = \infty$ with the estimate (1.5) was already used by Alikakos and Rostamian [1] to obtain *gradient decay estimates* for solutions of the parabolic p-Laplace equation on the Euclidean space $\mathbb{R}^d$. Thus, Corollary 2.8 improves this result to a more general abstract framework with a Lipschitz perturbation. For further applications, we refer the interested reader to the book [18].

The structure of this paper is as follows. In the subsequent section, we collect some intermediate results to prove our main theorems (Theorem 1.1 and Theorem 1.3).

In Section 3, we consider the class of *quasi accretive operators* A (see Definition 3.1) and outline how the property that A is homogeneous of order $\alpha \neq 1$ is passed on the *nonlinear semigroup* $\{T_t\}_{t\geq 0}$ generated by $-A$ (see the paragraph after Definition 3.2). In particular, we discuss when solutions u of (1.1) are differentiable with values in X at a.e. $t > 0$, and give the proofs of Theorem 1.1, Corollary 1.2, and Theorem 1.3.

Section 4 focuses on the class of semigroups generated by a homogenous *quasi completely accretive* operators A of order $\alpha \neq 1$. The notion of *completely accretive* operators A (see Definition 4.5) was introduced by Bénilan and Crandall [8] and further extended by Jakubowski and Wittbold [29] to study nonlinear Volterra equations governed by this class of operators. More recently, Coulhon and the author [18] introduced the class of *quasi completely accretive* operators to study additional regularity properties of mild solutions to Cauchy problem (1.6) (respectively, (1.8)) when the infinitesimal generator satisfies a functional inequality of Sobolev, Gagliardo-Nirenberg, or Nash type. We prove in Section 5.4 a compactness result (see Lemma 4.13) and due to this, we obtain in Theorem 4.14 that every mild solution u of the homogeneous Cauchy problem (1.8) governed by a homogenous quasi completely accretive operators A of order $\alpha \neq 1$ defined on also-called *normal* Banach space, is differentiable for a.e. $t > 0$ and its right-hand side time-derivative satisfies point-wise Aronson-Bénilan type estimates and global L^1 Aronson-Bénilan type estimates.

We conclude this paper in Section 5 with an application; we derive in Theorem 5.2 global L^q-L^∞-regularity estimates of the time-derivative $\frac{\mathrm{d}u}{\mathrm{d}t}$ for solutions u to the perturbed evolution problem (1.1) when A is the Dirichlet-to-Neumann operator associated with the negative p-Laplacian $-\Delta_p$ plus lower order terms on a compact, smooth, Riemannian manifold (M, g) with a Lipschitz continuous boundary.

2. Preliminaries

In this section, we gather some intermediate results to prove the main theorems of this paper.

Suppose X is a linear vector space and $\|\cdot\|_X$ a semi-norm on X. Then, the main object of this paper is the following class of operators (cf., [9] and [27]).

Definition 2.1. An operator A on X is called *homogeneous of order* $\alpha \in \mathbb{R}$ if $0 \in A0$, and for every $u \in D(A)$ and $\lambda \geq 0$, one has that $\lambda u \in D(A)$ and A satisfies (1.2).

For the rest of this section suppose that A denotes a homogeneous operator on X of order $\alpha \neq 1$. We begin by considering the inhomogeneous Cauchy problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

and want to discuss the impact of the homogeneity of A on the solutions u to (2.1). For this, suppose $f \in C([0,T];X)$, $u_0 \in X$, and $u \in C^1([0,T];X)$ be a classical solution of (2.1). Further, for given $\lambda > 0$, set

$$v_\lambda(t) = \lambda^{\frac{1}{\alpha-1}} u(\lambda t), \qquad (t \in [0, \tfrac{T}{\lambda}]).$$

Then, v satisfies

$$\begin{aligned} \frac{\mathrm{d}v_\lambda}{\mathrm{d}t}(t) &= \lambda^{\frac{1}{\alpha-1}+1} \frac{\mathrm{d}u}{\mathrm{d}t}(\lambda t) \in \lambda^{\frac{\alpha}{\alpha-1}} \Big[f(\lambda t) - A(u(\lambda t)) \Big] \\ &= -A(v_\lambda(t)) + \lambda^{\frac{\alpha}{\alpha-1}} f(\lambda t) \end{aligned}$$

for every $t \in (0, T/\lambda)$ with initial value $v_\lambda(0) = \lambda^{\frac{1}{\alpha-1}} u(0) = \lambda^{\frac{1}{\alpha-1}} u_0$.

Now, if we assume that the Cauchy problem (2.1) is well-posed for given $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$ in the sense that there is a semigroup $\{T_t\}_{t=0}^T$ of mappings $T_t : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ given by

$$T_t(u_0, f) := u(t) \qquad \text{for every } u_0 \in \overline{D(A)}^X \text{ and } f \in L^1(0,T;X), \tag{2.2}$$

where u is the unique (mild) solution. Then, the previous reasoning can be formulated in terms of this semigroup $\{T_t\}_{t=0}^{T}$ as follows

$$T_t(0,0) = 0 \qquad \text{for all } t \in [0,T] \tag{2.3}$$

(i.e., $u(t) \equiv 0$ is the unique solution of (2.1) if $u_0 = 0$ and $f(t) \equiv 0$), and

$$\lambda^{\frac{1}{\alpha-1}} T_{\lambda t}(u_0, f) = T_t(\lambda^{\frac{1}{\alpha-1}} u_0, \lambda^{\frac{\alpha}{\alpha-1}} f(\lambda\cdot)) \quad \text{for every } t \in [0, T/\lambda], \lambda > 0. \tag{2.4}$$

Property (2.4) together with the standard growth estimate

$$\begin{aligned} &e^{-\omega t}\|T_t(u_0,f) - T_t(\hat{u}_0,\hat{f})\|_X \\ &\quad \le Le^{-\omega s}\|T_s(u_0,f) - T_s(\hat{u}_0,\hat{f})\|_X + L\int_s^t e^{-\omega r}\,\|f(r) - \hat{f}(r)\|_X\,\mathrm{d}r \\ &\qquad\qquad \text{for every } 0 \le s \le t \le T,\, u_0 \in \overline{D(A)}^X,\, f, \hat{f} \in L^1(0,T;X), \end{aligned} \tag{2.5}$$

holding for some $\omega \in \mathbb{R}$ and $L \ge 1$, are the main ingredients to obtain global regularity estimates of the form (1.5). This leads to our first intermediate result. This lemma also generalizes the case of homogeneous operators of order zero (cf., [27, Theorem 2.3]), and the case $\omega = 0$ treated in [9, Theorem 4].

Lemma 2.2. *Let $\{T_t\}_{t=0}^{T}$ be a family of mappings $T_t : C \times L^1(0,T;X) \to C$ defined on a subset $C \subseteq X$, and suppose there are $\omega \in \mathbb{R}$, $L \ge 1$, and $\alpha \neq 1$ such that $\{T_t\}_{t=0}^{T}$ satisfies (2.3)-(2.5). Then, the following statements hold.*

(1) For every $u_0 \in C$, $f \in L^1(0,T;X)$, $t \in (0,T]$ and $h > 0$ such that $t+h \in (0,T]$, one has that

$$\begin{aligned} &\|T_{t+h}(u_0,f) - T_t(u_0,f)\|_X \\ &\quad \le \left|\left(1+\tfrac{h}{t}\right) - \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\right| \; L\int_0^t e^{\omega(t-s)}\|f(s+\tfrac{h}{t}s)\|_X\,\mathrm{d}s \\ &\qquad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \; L\int_0^t e^{\omega(t-s)}\|f(s+\tfrac{h}{t}s) - f(s)\|_X\,\mathrm{d}s \\ &\qquad + \; Le^{\omega t}\left|\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right| \left(2\,\|u_0\|_X + \int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s\right). \end{aligned} \tag{2.6}$$

(2) If one denotes

$$V_\omega(f,t) := \limsup_{h\to 0+} \int_0^t e^{-\omega s}\frac{\|f(s+hs) - f(s)\|_X}{h}\,\mathrm{d}s, \tag{2.7}$$

and $\{T_t\}_{t=0}^{T}$ satisfies (2.5), then for every $t > 0$ and $u_0 \in C$, one has that

$$\begin{aligned} &\limsup_{h\to 0+}\left\|\frac{T_{t+h}(u_0,f) - T_t(u_0,f)}{h}\right\|_X \\ &\quad \le \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|} + \frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s + V_\omega(f,t)\right], \end{aligned} \tag{2.8}$$

and if $f \in W^{1,1}(0,T;X)$, then

$$\limsup_{h\to 0+}\left\|\frac{T_{t+h}(u_0,f)-T_t(u_0,f)}{h}\right\|_X \tag{2.9}$$
$$\le \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s + \int_0^t e^{-\omega s}\|f'(s)\|_X\,s\,\mathrm{d}s\right].$$

(3) *If for given $u_0 \in C$ and $f \in W^{1,1}(0,T;X)$, $\frac{\mathrm{d}}{\mathrm{d}t_+}T_t(u_0,f)$ exists (in X) at a.e. $t \in (0,T)$, then*

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t_+}T_t(u_0,f)\right\|_X \le \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|}+\frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s + \int_0^t e^{-\omega s}\|f'(s)\|_X\,s\,\mathrm{d}s\right]. \tag{2.10}$$

Our proof of Lemma 2.2 uses the same techniques as in [9].

Proof. Let $u_0 \in C$, $f \in L^1(0,T;X)$, $t>0$, and $h>0$ satisfying $t+h\le T$. If we choose $\lambda = 1+\frac{h}{t}$ in (2.4), then

$$\begin{aligned} T_{t+h}(u_0,f)-T_t(u_0,f) &= T_{\lambda t}(u_0,f)-T_t(u_0,f)\\ &= \lambda^{\frac{1}{1-\alpha}}T_t\left[\lambda^{\frac{1}{\alpha-1}}u_0,\lambda^{\frac{\alpha}{\alpha-1}}f(\lambda\cdot)\right]-T_t(u_0,f)\end{aligned} \tag{2.11}$$

and so,

$$\begin{aligned} T_{t+h}(u_0,f)-T_t(u_0,f) &= \lambda^{\frac{1}{1-\alpha}}\left[T_t\left[\lambda^{\frac{1}{\alpha-1}}u_0,\lambda^{\frac{\alpha}{\alpha-1}}f(\lambda\cdot)\right]-T_t(u_0,f(\lambda\cdot)\right]\\ &\quad+\lambda^{\frac{1}{1-\alpha}}\left[T_t\left[u_0,f(\lambda\cdot)\right]-T_t(u_0,f)\right]\\ &\quad+\left[\lambda^{\frac{1}{1-\alpha}}-1\right]T_t(u_0,f).\end{aligned} \tag{2.12}$$

Applying to this (2.5) and by using (2.3), one sees that

$$\begin{aligned} &\|T_{t+h}(u_0,f)-T_t(u_0,f)\|_X\\ &\le \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\left\|T_t\left[\lambda^{\frac{1}{\alpha-1}}u_0,\lambda^{\frac{\alpha}{\alpha-1}}f(\lambda\cdot)\right]-T_t(u_0,f(\lambda\cdot)\right\|_X\\ &\quad+\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\|T_t\left[u_0,f(\lambda\cdot)\right]-T_t(u_0,f)\|_X\\ &\quad+\left|\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}-1\right|\,\|T_t(u_0,f)\|_X\\ &\le \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\,L\,e^{\omega t}\left\|\left(1+\tfrac{h}{t}\right)^{\frac{1}{\alpha-1}}u_0-u_0\right\|_X\\ &\quad+\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\,L\int_0^t e^{\omega(t-s)}\left\|(1+\tfrac{h}{t})^{\frac{\alpha}{\alpha-1}}f(s+\tfrac{h}{t}s)-f(s+\tfrac{h}{t}s)\right\|_X\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \; L \int_0^t e^{\omega(t-s)} \| f(s+\tfrac{h}{t}s) - f(s)\|_X \,\mathrm{d}s \\
&\qquad + \; L e^{\omega t} \left| \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1 \right| \left(\|u_0\|_X + \int_0^t e^{-\omega s} \|f(s)\|_X \,\mathrm{d}s \right) \\
= &\left| \left(1+\tfrac{h}{t}\right) - \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \right| \; L \int_0^t e^{\omega(t-s)} \|f(s+\tfrac{h}{t}s)\|_X \,\mathrm{d}s \\
&\qquad + \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \; L \int_0^t e^{\omega(t-s)} \| f(s+\tfrac{h}{t}s) - f(s)\|_X \,\mathrm{d}s \\
&\qquad\qquad + \; L e^{\omega t} \left| \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1 \right| \left(2\,\|u_0\|_X + \int_0^t e^{-\omega s} \|f(s)\|_X \,\mathrm{d}s \right),
\end{aligned}
$$

which is (2.6). It is clear that (2.8)-(2.10) follow from (2.6). □

Examples of functions $f : [0,T] \to X$ for which $V_\omega(f,t)$ defined by (2.7) is finite at a.e. t and integrable on $L^1(0,T)$, are functions with *bounded variation* (cf., [13, Appendice, Section 2.]).

Definition 2.3. For a function $f : [0,T] \to X$, one calls

$$
\operatorname{Var}(f;[0,T]) := \sup \left\{ \sum_{i=1}^{N} \|f(t_i) - f(t_{i-1})\|_X \;\middle|\; \begin{array}{c} \text{all partitions :} \\ 0 = t_0 < \cdots < t_N = T \end{array} \right\}
$$

the *total variation of* f. Each X-valued function $f : [0,T] \to X$ is said to have *bounded variation* on $[0,T]$ if $\operatorname{Var}(f;[0,T])$ is finite. We denote by $BV(0,T;X)$ the space of all functions $f : [0,T] \to X$ of bounded variation and to simplify the notation, we set $V_f(t) = \operatorname{Var}(f;[0,t])$ for $t \in (0,T]$.

Functions of bounded variation have the following properties.

Proposition 2.4. *Let $f \in BV(0,T;X)$. Then the following statements hold.*

(1) $f \in L^\infty(0,T;X)$;
(2) At every $t \in [0,T]$, the left-hand side limit $f(t-) := \lim_{s\to t-} f(s)$ and right-hand side limit $f(t+) := \lim_{s\to t+} f(s)$ exist in X; and the set of discontinuity points in $[0,T]$ is at most countable;
(3) The mapping $t \mapsto V_f(t)$ is monotonically increasing on $[0,T]$, and

$$
\|f(t) - f(s)\|_X \le V_f(t) - V_f(s) \qquad \textit{for all } 0 \le s \le t \le T; \tag{2.13}
$$

(4) For $\omega \ge 0$, one has that

$$
\int_0^t e^{-\omega s} \frac{\|f(s+hs) - f(s)\|_X}{h} \mathrm{d}s \le t\, V_f(t) \qquad \textit{for all } h \in (0,t],\ 0 < t \le T.
$$

(5) For $\omega \ge 0$, let $V_\omega(f,t)$ be given by (2.7). Then $V_\omega(f,t)$ belongs to $L^\infty(0,T)$ satisfying

$$
V_\omega(f,t) \le t\ V_f(t) \qquad \textit{for all } t \in [0,T].
$$

The first three statements are standard and can be found, for example, in [13, Section 2., Lemme A.1]. Thus, we only outline the proof of statement (4) and (5).

Proof. Obviously, (5) follows from (4). Thus, it remains to show that for given $f \in BV(0,T;X)$, (4) holds. To see this, let $t \in (0,T)$, $h \in (0,t]$ such that $t+h \leq T$. Then, by (2.13) and since $\omega \geq 0$,

$$\int_0^t e^{-\omega s} \frac{\|f(s+hs)-f(s)\|_X}{h} \mathrm{d}s \leq \tfrac{1}{h} \int_0^t e^{-\omega s} \Big(V_f((1+h)s) - V_f(s) \Big) \mathrm{d}s$$
$$\leq \tfrac{1}{h} \int_0^t \Big(V_f((1+h)s) - V_f(s) \Big) \mathrm{d}s.$$

By using the substitution $r = (1+h)s$, we get

$$\begin{aligned} &\tfrac{1}{h} \int_0^t V_f((1+h)s)\,\mathrm{d}s - \tfrac{1}{h} \int_0^t V_f(s)\,\mathrm{d}s \\ &= \tfrac{1}{h(1+h)} \int_0^{(1+h)t} V_f(r)\,\mathrm{d}r - \tfrac{1}{h} \int_0^t V_f(s)\,\mathrm{d}s \\ &\leq \tfrac{1}{h} \int_t^{t+ht} V_f(s)\,\mathrm{d}s \end{aligned}$$

and by the monotonicity of $t \mapsto V_f(t)$,

$$\tfrac{1}{h} \int_t^{t+ht} V_f(s)\,\mathrm{d}s \leq t\, V_f(t).$$

This shows that (4) holds. $\square$

In the case $f \equiv 0$, we let $T = \infty$. Then the mapping T_t given by (2.2) only depends on the initial value u_0. In other words,

$$T_t u_0 = T_t(u_0, 0) \qquad \text{for every } u_0 \in C \text{ and } t \geq 0. \tag{2.14}$$

In this case Lemma 2.2 reads as follows (cf., [8]).

Corollary 2.5. *Let $\{T_t\}_{t\geq 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$, and suppose there are $\omega \in \mathbb{R}$, $L \geq 1$, and $\alpha \neq 1$ such that $\{T_t\}_{t\geq 0}$ satisfies*

$$\|T_t u_0 - T_t \hat{u}_0\|_X \leq L\, e^{\omega t}\, \|u_0 - \hat{u}_0\|_X \qquad \textit{for all } t \geq 0,\ u,\ \hat{u} \in C, \tag{2.15}$$

$$\lambda^{\frac{1}{\alpha-1}}\, T_{\lambda t} u_0 = T_t[\lambda^{\frac{1}{\alpha-1}} u_0] \qquad \textit{for all } \lambda > 0,\ t \geq 0 \textit{ and } u_0 \in C. \tag{2.16}$$

Further, suppose $T_t 0 \equiv 0$ for all $t \geq 0$. Then, for every $u_0 \in C$,

$$\|T_{t+h} u_0 - T_t u_0\|_X \leq 2\, L \left| 1 - \left(1 + \tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \right| e^{\omega\, t} \|u_0\|_X. \tag{2.17}$$

$t > 0$, $h \neq 0$ satisfying $1 + \frac{h}{t} > 0$. In particular, the family $\{T_t\}_{t\geq 0}$ satisfies

$$\limsup_{h \to 0+} \frac{\|T_{t+h} u_0 - T_t u_0\|_X}{h} \leq \frac{2L e^{\omega t}}{|1-\alpha|} \frac{\|u_0\|_X}{t} \qquad \textit{for every } t > 0,\ u_0 \in C. \tag{2.18}$$

Moreover, if for $u_0 \in C$, the right-hand side derivative $\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+$ exists (in X) at $t > 0$, then

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+} \right\|_X \leq \frac{2\, L\, e^{\omega t}}{|1-\alpha|} \frac{\|u_0\|_X}{t}. \tag{2.19}$$

Finally, we turn to the Cauchy problem governed by the operator $A + F$,

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) + F(u(t)) \ni f(t) & \text{on } (0,T), \\ u(0) = u_0, \end{cases} \tag{2.20}$$

for given $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$, involving a homogenous operator A in X of order $\alpha \neq 1$, and a Lipschitz continuous perturbation $F : X \to X$ with Lipschitz constant $\omega \geq 0$ satisfying $F(0) = 0$. We assume that Cauchy problem (2.20) is well-posed in X in the sense that for every $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$, there is a unique function $u \in C([0,T];X)$ satisfying $u(0) = u_0$ in X and (2.2) generates a semigroup $\{T_t\}_{t=0}^T$ of mappings $T_t : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ satisfying (2.5) for every $0 \leq s < t \leq T$.

One important idea to obtain global L^1 Aronson-Bénilan type estimates for the semigroup $\{T_t\}_{t=0}^T$ associated with (2.20) is the assumption that for given $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$, the unique solution $t \mapsto u(t) = T_t(u_0, f)$ of Cauchy problem (2.20) is, in particular, the unique solution of the unperturbed inhomogeneous Cauchy problem (2.1) for $\tilde{f} : [0,T] \to X$ given by

$$\tilde{f}(t) := f(t) - F(T_t(u_0, f)), \qquad (t \in [0,T]). \tag{2.21}$$

This property can be expressed by the identity

$$\tilde{T}_t(u_0, \tilde{f}) = T_t(u_0, f) \qquad \text{holds for every } t \in [0,T], \tag{2.22}$$

where $\{\tilde{T}_t\}_{t=0}^T$ denotes the semigroup associated with (2.1). The advantage of equation (2.22) is that one can employ inequality (2.5) satisfied by the family $\{\tilde{T}_t(\cdot, \tilde{f})\}_{t \geq 0}$. Thus, by Lemma 2.2, the following estimate holds.

Theorem 2.6. *Let $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz constant $\omega_F \geq 0$ satisfying $F(0) = 0$. Given $T > 0$ and a subset $C \subseteq X$, assume there are families $\{T_t\}_{t=0}^T$ and $\{\tilde{T}_t\}_{t=0}^T$ of mappings T_t, $\tilde{T}_t : C \times L^1(0,T;X) \to C$ satisfying (2.3) and related through (2.22) for every $u_0 \in C$ and $f \in L^1(0,T;X)$ with $\tilde{f}$ given by (2.21). Further suppose, $\{\tilde{T}_t\}_{t=0}^T$ satisfies (2.4) and (2.5) for some $\omega \geq 0$ and $L \geq 1$, and $\{T_t\}_{t=0}^T$ satisfies (2.5) with $\tilde{\omega} = \omega + \omega_F$ and L.*

Then, if for $u_0 \in C$ and $f \in BV(0,T;X)$, the function $t \mapsto T_t(u_0, f)$ is locally Lipschitz continuous on $[0,T)$, then one has that

$$\limsup_{h \to 0+} \frac{\|T_{t+h}(u_0,f) - T_t(u_0,f)\|}{h} \leq \frac{e^{\omega t}}{t}\Big[a(t) + L\omega_F \int_0^t a(s) e^{L\,\omega_F(t-s)}\mathrm{d}s\Big] \tag{2.23}$$

for a.e. $t \in (0,T)$, where

$$\begin{aligned} a(t) := L\,V_\omega(f,t) + \frac{L}{|1-\alpha|}\Bigg[\Big(2 + \omega_F\, L \int_0^t e^{\omega_F s}\mathrm{d}s\Big)\ \|u_0\|_X \\ + \int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s + \omega_F\, L \int_0^t \int_0^s e^{-\omega_F r}\|f(r)\|_X \mathrm{d}r\,\mathrm{d}s\Bigg]. \end{aligned} \tag{2.24}$$

and $V_\omega(f, \cdot)$ is given by (2.7).

For the proof of this theorem, we still need the following version of Gronwall's lemma.

Lemma 2.7 ([37, Lemma D.2]). *Suppose $v \in L^1_{loc}([0,T))$ satisfies*

$$v(t) \le a(t) + \int_0^t v(s)\, b(s) \mathrm{d}s \qquad \text{for a.e. } t \in (0,T), \tag{2.25}$$

where $b \in C([0,T))$ satisfying $b(t) \ge 0$, and $a \in L^1_{loc}([0,T))$. Then,

$$v(t) \le a(t) + \int_0^t a(s)\, b(s)\, e^{\int_s^t b(r)\mathrm{d}r}\, \mathrm{d}s \qquad \text{for a.e. } t \in (0,T). \tag{2.26}$$

We are now ready to give the proof of Theorem 2.6.

Proof of Theorem 2.6. Let $u_0 \in C$ and $f \in BV(0,T;X)$. Fix $t > 0$, and let $h > 0$ such that $t + h < T$. Then, by the assumption that there is a family $\{\tilde{T}_t\}_{t=0}^T$ of mappings $\tilde{T}_t$ satisfying (2.22) for every $u_0 \in C$ and $f \in L^1(0,T;X)$ with $\tilde{f}$ given by (2.21), and $\{\tilde{T}_t\}_{t=0}^T$ satisfies (2.3)-(2.5) for some $\omega \ge 0$, L, we can apply Lemma 2.2 to $\tilde{T}_t(u_0,\tilde{f})$. Then by (2.6), since $\tilde{f}$ is given by (2.21), by (2.22), and by the triangle inequality,

$$\begin{aligned}
&\|T_{t+h}(u_0,f) - T_t(u_0,f)\|_X\\
&= \|\tilde{T}_{t+h}(u_0,\tilde{f}) - \tilde{T}_t(u_0,\tilde{f})\|_X\\
&\le \left|\left(1+\tfrac{h}{t}\right) - \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}\right| L \int_0^t e^{\omega(t-s)} \|f(s+\tfrac{h}{t}s) - F(T_{s+\frac{h}{t}s}(u_0,f))\|_X \,\mathrm{d}s\\
&+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \|f(s+\tfrac{h}{t}s) - f(s)\|_X \,\mathrm{d}s\\
&+ \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \int_0^t e^{\omega(t-s)} \|F(T_{s+\frac{h}{t}s}(u_0,f)) - F(T_s(u_0,f))\|_X \,\mathrm{d}s\\
&+ L\, e^{\omega t} \left|\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} - 1\right| \left[2\,\|u_0\|_X + \int_0^t e^{-\omega s}\Big[\|f(s)\|_X + \|F(T_s(u_0,f))\|_X\Big] \mathrm{d}s\right]
\end{aligned}$$

Since F is globally Lipschitz continuous with constant ω_F, $F(0) = 0$, and since $\{T_t\}_{t=0}^T$ satisfies (2.3) and (2.5) with $\tilde{\omega} = \omega + \omega_F$ and L, one has that

$$\|F(T_s(u_0,f))\|_X \le \omega_F\, L \left[e^{\tilde{\omega} s}\|u_0\|_X + \int_0^s e^{\tilde{\omega}(s-r)} \|f(r)\|_X \,\mathrm{d}r\right].$$

We apply this to the last integral on the right-hand side of the previous estimate, and substitute $y = (1 + h/t)s$ into the first integral on the right-hand side of the previous estimate. Then, dividing by $h>0$ both sides in the resulting inequality yields that

$$\begin{aligned}
&\frac{\|T_{t+h}(u_0,f)-T_t(u_0,f)\|_X}{h}\\
&\le \left|\frac{(1+\frac{h}{t})-(1+\frac{h}{t})^{\frac{1}{1-\alpha}}}{\frac{1}{t}h}\right|\frac{1+\frac{h}{t}}{t}Le^{\omega t}\times\\
&\qquad\times\int_0^{t+h}e^{-\frac{\omega}{1+h/t}y}\|f(y)-F(T_y(u_0,f))\|_X\mathrm{d}y\\
&\quad+\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}L\,\tfrac{e^{\omega t}}{t}\int_0^t e^{-\omega s}\frac{\|f(s+\frac{h}{t}s)-f(s)\|_X}{\frac{h}{t}}\,\mathrm{d}s\\
&\quad+\left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}}L\,e^{\omega t}\omega_F\int_0^t e^{-\omega s}\frac{\|T_{s+\frac{h}{t}s}(u_0,f)-T_s(u_0,f)\|_X}{\frac{s}{t}h}\tfrac{s}{t}\,\mathrm{d}s\\
&\quad+\tfrac{Le^{\omega t}}{t}\left|\frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}}-1}{\frac{1}{t}h}\right|\Bigg[\left(2+\omega_F L\int_0^t e^{\omega_F s}\mathrm{d}s\right)\|u_0\|_X\\
&\qquad+\int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s+\omega_F\,L\int_0^t\int_0^s e^{-\omega_F r}\|f(r)\|_X\mathrm{d}r\,\mathrm{d}s\Bigg],
\end{aligned}\tag{2.27}$$

where we use twice that $e^{-\omega s}e^{\tilde{\omega}s}=e^{\omega_F s}$. Note that

$$\limsup_{h\to0+}\int_0^t e^{-\omega s}\frac{\|f(s+\frac{h}{t}s)-f(s)\|_X}{\frac{h}{t}}\,\mathrm{d}s=V_\omega(f,t)$$

and by Proposition 2.4, one has that $V_\omega(f,\cdot)\in L^\infty([0,T))$. Since $t\mapsto T_t(u_0,f)$ is locally Lipschitz continuous on $[0,T)$, for every $\varepsilon\in(0,T)$ there is a constant $C_\varepsilon>0$ such that

$$\frac{\left\|T_{s+\frac{h}{t}s}(u_0,f)-T_s(u_0,f)\right\|_X}{\frac{s}{t}h}\le C$$

for every $s\in[0,T-\varepsilon]$ and $h>0$ satisfying $s+\frac{h}{t}s<T-\varepsilon$. Thus, by the reverse version of Fatou's lemma, taking in (2.27) the limit-superior as $h\to0+$ gives

$$\begin{aligned}
&e^{-\omega t}t\limsup_{h\to0+}\frac{\|T_{t+h}(u_0,f)-T_t(u_0,f)\|}{h}\\
&\quad\le L\,V_\omega(f,t)+L\,\omega_F\int_0^t e^{-\omega s}s\left[\limsup_{h\to0+}\frac{\|T_{s+h}(u_0,f)-T_s(u_0,f)\|}{h}\right]\mathrm{d}s\\
&\qquad+\frac{L}{|1-\alpha|}\Bigg[\left(2+\omega_F\,L\int_0^t e^{\omega_F s}\mathrm{d}s\right)\|u_0\|_X\\
&\qquad\quad+\int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s+\omega_F\,L\int_0^t\int_0^s e^{-\omega_F r}\|f(r)\|_X\mathrm{d}r\,\mathrm{d}s\Bigg].
\end{aligned}$$

Now, applying Gronwall's lemma (Lemma 2.7) to $a(t)$ given by (2.24),

$$\begin{aligned}
&b(t)\equiv L\,\omega_F,\ \text{and}\\
&v(t)=e^{-\omega t}t\limsup_{h\to0+}\frac{\|T_{t+h}(u_0,f)-T_t(u_0,f)\|}{h},
\end{aligned}$$

then one obtains (2.23). This completes the proof. ∎

Next, we intend to extrapolate the regularity estimate (2.23) for $f \equiv 0$.

Corollary 2.8. *Let $\{T_t\}_{t\ge 0}$ be a semigroup of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ and suppose, there is a second vector space Y with semi-norm $\|\cdot\|_Y$ and constants M, γ, $\delta > 0$ and $\hat{\omega} \in \mathbb{R}$ such that $\{T_t\}_{t\ge0}$ satisfies the following Y-X-regularity estimate*

$$\|T_t u_0\|_X \le M\, e^{\hat{\omega} t} \frac{\|u_0\|_Y^\gamma}{t^\delta} \qquad \text{for every } t>0 \text{ and } u_0 \in C\cap Y. \tag{2.28}$$

If for $\alpha \neq 1$, ω, $\omega_F \in \mathbb{R}$ and $L \ge 1$, $\{T_t\}_{t\ge0}$ satisfies

$$\begin{aligned}&\limsup_{h\to0+} \frac{\|T_{t+h}u_0 - T_t u_0\|_X}{h}\\ &\qquad \le \frac{e^{\omega t}}{t}\frac{L}{|1-\alpha|}\Big[b(t) + L\omega_F \int_0^t b(s)\, e^{L\,\omega_F(t-s)}\mathrm{d}s\Big]\; \|u_0\|_X\end{aligned} \tag{2.29}$$

for a.e. $t>0$ and $u_0 \in C$, with $b(t) := 2 + \omega_F\, L \int_0^t e^{\omega_F s}\mathrm{d}s$, then

$$\begin{aligned}&\limsup_{h\to0+} \frac{\|T_{t+h}u_0 - T_t u_0\|_X}{h}\\ &\qquad \le \frac{2^{\delta+1}\, e^{\frac{\omega+\hat{\omega}}{2}t}}{t^{\delta+1}}\frac{L\,M}{|1-\alpha|}\Big[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\Big]\; \|u_0\|_Y^\gamma.\end{aligned} \tag{2.30}$$

In particular, if the right-hand side derivative $\frac{\mathrm{d}}{\mathrm{d}t_+}T_t u_0$ exists (in X) at $t>0$, then

$$\left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+}\right\|_X \le \frac{2^{\delta+1}\, e^{\frac{\omega+\hat{\omega}}{2}t}}{t^{\delta+1}}\frac{L\,M}{|1-\alpha|}\Big[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\Big]\; \|u_0\|_Y^\gamma.$$

Proof. Let $u_0 \in C$ and $t>0$. Note, if $u_0 \notin Y$ then (2.30) trivially holds. Thus, it is sufficient to consider the case $u_0 \in C\cap Y$. By the semigroup property of $\{T_t\}_{t\ge0}$ and by (2.29) and (2.28), one sees that

$$\begin{aligned}&\limsup_{h\to0+} \frac{\|T_{t+h}u_0 - T_t u_0\|_X}{h}\\ &\quad = \limsup_{h\to0+} \frac{\|T_{\frac{t}{2}+h}(T_{\frac{t}{2}}u_0) - T_{\frac{t}{2}}(T_{\frac{t}{2}}u_0)\|_X}{h}\\ &\quad \le \frac{2\,e^{\omega\frac{t}{2}}}{t}\frac{L}{|1-\alpha|}\Big[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\Big]\; \|T_{\frac{t}{2}}u_0\|_X\\ &\quad \le \frac{2^{\delta+1}\, e^{\frac{\omega+\hat{\omega}}{2}t}}{t^{\delta+1}}\frac{L\,M}{|1-\alpha|}\Big[b(\tfrac{t}{2}) + L\omega_F \int_0^{\frac{t}{2}} b(s)\, e^{L\,\omega_F(\frac{t}{2}-s)}\mathrm{d}s\Big]\; \|u_0\|_Y^\gamma.\end{aligned}$$

∎

Now, we suppose, there is a partial ordering "$\le$" on X such that $(X,\le)$ is an ordered vector space. Then, we can state the following theorem.

Theorem 2.9. *Let $(X,\leq)$ be an ordered vector space and $F : X \to X$ a Lipschitz continuous mapping satisfying $F(0) = 0$. Suppose, there is a subset $C \subseteq X$ and two families $\{T_t\}_{t\geq 0}$ and $\{\tilde{T}_t\}_{t\geq 0}$ of mappings $T_t : C \to C$ and $\tilde{T}_t : C \times L^1_{loc}([0,\infty);X) \to C$ related by the equation*

$$T_t u_0 = \tilde{T}_t(u_0,\tilde{f}) \qquad \text{for all } t \geq 0,\ u_0 \in C, \tag{2.31}$$

where $\tilde{f}$ is given by $\tilde{f}(t) = -F(T_t u_0)$. Further, suppose

$$\text{for every } u_0,\ \hat{u}_0 \in C \text{ satisfying } u_0 \leq \hat{u}_0, \text{ one has } T_t u_0 \leq T_t \hat{u}_0 \text{ for all } t \geq 0 \tag{2.32}$$

and $\{\tilde{T}_t\}_{t\geq 0}$ satisfies (2.3)-(2.5) for some $\omega \geq 0$ and $L \geq 1$. Then for every $u_0 \in C$ satisfying $u_0 \geq 0$, one has that

$$\frac{T_{t+h}u_0 - T_t u_0}{h} \geq \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h}\frac{T_t u_0}{t} + g_h(t) \tag{2.33}$$

for every $t, h > 0$ if $\alpha > 1$ and

$$\frac{T_{t+h}u_0 - T_t u_0}{h} \leq \frac{(1+\frac{h}{t})^{\frac{1}{1-\alpha}} - 1}{h}\frac{T_t u_0}{t} + g_h(t) \tag{2.34}$$

for every $t, h > 0$ if $\alpha < 1$, where for every $h > 0$, $g_h : (0,\infty) \to X$ is a continuous function satisfies

$$\begin{aligned} \|g_h(t)\|_X \leq \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} L \times \\ \times \int_0^t e^{\omega(t-r)} \left\| \frac{F(T_r u_0) - \left(1+\frac{h}{t}\right)^{\frac{\alpha}{\alpha-1}} F(T_{r+\frac{h}{t}r}u_0)}{h} \right\|_X \mathrm{d}r \end{aligned} \tag{2.35}$$

for every $t > 0$.

Before giving the proof of Theorem 2.9, we need to recall the following definition.

Definition 2.10. If $(X,\leq)$ is an ordered vector space then a family $\{T_t\}_{t\geq 0}$ of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ is called *order preserving* if $\{T_t\}_{t\geq 0}$ satisfies (2.32).

With this in mind, we can now give the proof above the preceding theorem.

Proof of Theorem 2.9. First, let $\{\tilde{T}_t\}_{t\geq 0}$ be the family of operators related to $\{T_t\}_{t\geq 0}$ by (2.31), and for $t, h > 0$, let $\lambda := \left(1+\frac{h}{t}\right)$. Since $\lambda > 1$, $\lambda^{\frac{1}{\alpha-1}} u_0 \leq u_0$ if $\alpha < 1$ and $\lambda^{\frac{1}{\alpha-1}} u_0 \geq u_0$ if $\alpha > 1$. Thus, if $\alpha < 1$, then by (2.11) and (2.32), one has that

$$\begin{aligned}
\tilde{T}_{t+h}(u_0,\tilde{f}) - \tilde{T}_t(u_0,\tilde{f}) &= \lambda^{\frac{1}{1-\alpha}} \tilde{T}_t\left[\lambda^{\frac{1}{\alpha-1}}u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right] - \tilde{T}_t(u_0,\tilde{f}) \\
&= \lambda^{\frac{1}{1-\alpha}}\left[\tilde{T}_t\left[\lambda^{\frac{1}{\alpha-1}}u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right]\right] \\
&\qquad + \lambda^{\frac{1}{1-\alpha}}\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right] - \tilde{T}_t(u_0,\tilde{f}) \\
&\leq \lambda^{\frac{1}{1-\alpha}}\left[\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0,\tilde{f}\right]\right] \\
&\qquad + \left[\lambda^{\frac{1}{1-\alpha}} - 1\right]\tilde{T}_t(u_0,\tilde{f})
\end{aligned}$$

and, similarly, if $\alpha > 1$, then

$$\tilde{T}_{t+h}(u_0,\tilde{f}) - \tilde{T}_t(u_0,\tilde{f}) \ge \lambda^{\frac{1}{1-\alpha}} \left[\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0,\tilde{f}\right]\right] + \left[\lambda^{\frac{1}{1-\alpha}} - 1\right]\tilde{T}_t(u_0,\tilde{f}).$$

Now, by replacing $\tilde{f}(t)$ by $-F(T_t u_0)$ and by (2.22), we can rewrite the above two inequalities and arrive to (2.33) and (2.34), where $g(t)$ is given by

$$g_h(t) = \left(1+\tfrac{h}{t}\right)^{\frac{1}{1-\alpha}} \frac{\tilde{T}_t\left[u_0, \lambda^{\frac{\alpha}{\alpha-1}}\tilde{f}(\lambda\cdot)\right] - \tilde{T}_t\left[u_0,\tilde{f}\right]}{h}.$$

Note, by (2.5), one has that g satisfies (2.35). □

By Theorem 2.9, if the derivative $\frac{\mathrm{d}}{\mathrm{d}t+}T_t u_0$ belongs to $L^1_{loc}(0,T;X)$ for $T>0$, then we can state the following.

Corollary 2.11. *Under the hypotheses of Theorem 2.9, suppose that for $u_0 \in C$ satisfying $u_0 \ge 0$, the right hand-side derivative $\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \in L^1_{loc}([0,T);X)$ for some $T>0$. Then, one has that*

$$(\alpha-1)\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \ge -\frac{T_t u_0}{t} + (\alpha-1)g_0(t),$$

for a.e. $t\in(0,T)$, where $g_0 : (0,T)\to X$ is a measurable function satisfying

$$\|g_0(t)\|_X \le \frac{L}{t}\int_0^t e^{\omega(t-r)}\left[\omega\left\|\frac{\mathrm{d}T_r u_0}{\mathrm{d}r}_+\right\|_X + \frac{|\alpha|}{|\alpha-1|}\|T_r u\|_X\right]\mathrm{d}r \tag{2.36}$$

for a.e. $t\in(0,T)$.

3. Homogeneous accretive operators

We begin this section with the following definition. Throughout this section, suppose X is a Banach space with norm $\|\cdot\|_X$.

Definition 3.1. An operator A on X is called *accretive* in X if for every (u,v), $(\hat{u},\hat{v})\in A$ and every $\lambda\ge 0$,

$$\|u-\hat{u}\|_X \le \|u-\hat{u}+\lambda(v-\hat{v})\|_X.$$

and A is called *m-accretive* in X if A is accretive and satisfies the *range condition*

$$Rg(I+\lambda A) = X \qquad \text{for some (or equivalently, for all) } \lambda>0,\, \lambda\,\omega<1, \tag{3.1}$$

More generally, an operator A on X is called *quasi (m-)accretive* in X if there is an $\omega\in\mathbb{R}$ such that $A+\omega I$ is (m-)accretive in X.

If A is quasi m-accretive in X, then the classical existence theorem [10, Theorem 6.5] (cf., [7, Corollary 4.2]) yields that for every $u_0 \in \overline{D(A)}^X$ and $f\in L^1(0,T;X)$, there is a unique *mild* solution $u\in C([0,T];X)$ of (2.1).

Definition 3.2. For given $u_0\in\overline{D(A)}^X$ and $f\in L^1(0,T;X)$, a function $u\in C([0,T];X)$ is called a *mild solution* of the inhomogeneous differential inclusion (2.1) with initial value u_0 if $u(0)=u_0$ and for every $\varepsilon>0$, there is a *partition* $\tau_\varepsilon : 0=t_0<t_1<\cdots<t_N=T$ and a *step function*

$$u_{\varepsilon,N}(t) = u_0\,\mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^N u_i\,\mathbb{1}_{(t_{i-1},t_i]}(t) \qquad \text{for every } t\in[0,T]$$

satisfying

$$\begin{aligned}
&t_i - t_{i-1} < \varepsilon \qquad \text{for all } i = 1, \dots, N,\\
&\sum_{N=1}^{N} \int_{t_{i-1}}^{t_i} \|f(t) - \overline{f}_i\| \,\mathrm{d}t < \varepsilon \qquad \text{where } \overline{f}_i := \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(t)\,\mathrm{d}t,\\
&\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Au_i \ni \overline{f}_i \qquad \text{for all } i = 1, \dots, N,
\end{aligned}$$

and

$$\sup_{t\in[0,T]} \|u(t) - u_{\varepsilon,N}(t)\|_X < \varepsilon.$$

Further, if A is quasi m-accretive, then the family $\{T_t\}_{t=0}^T$ of mappings $T_t : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ defined by (2.2) through the unique mild solution u of Cauchy problem (2.1) belongs to the following class.

Definition 3.3. Given a subset C of X, a family $\{T_t\}_{t=0}^T$ of mapping $T_t : C \times L^1(0,T;X) \to C$ is called a *strongly continuous semigroup of quasi-contractive mappings* T_t if $\{T_t\}_{t=0}^T$ satisfies the following three properties:

- (*semigroup property*) for every $(u_0, f) \in \overline{D(A)}^X \times L^1(0,T;X)$,

$$T_{t+s}(u_0, f) = T_t(T_s(u_0, f), f(s + \cdot)) \tag{3.2}$$

  for every $t, s \in [0,T]$ with $t + s \le T$;
- (*strong continuity*) for every $(u_0, f) \in \overline{D(A)}^X \times L^1(0,T;X)$,

$$t \mapsto T_t(u_0, f) \text{ belongs to } C([0,T];X);$$

- (*ω-quasi contractivity*) T_t satisfies (2.5) with $L = 1$.

Taking $f \equiv 0$ and only varying $u_0 \in \overline{D(A)}^X$, defines by

$$T_t u_0 = T_t(u_0, 0) \qquad \text{for every } t \ge 0, \tag{2.14}$$

a strongly continuous semigroup $\{T_t\}_{t\ge 0}$ on $\overline{D(A)}^X$ of ω-quasi contractions $T_t : \overline{D(A)}^X \to \overline{D(A)}^X$. Given a family $\{T_t\}_{t\ge 0}$ of ω-quasi contractions T_t on $\overline{D(A)}^X$, then the operator

$$A_0 := \left\{ (u_0, v) \in X \times X \,\middle|\, \lim_{h\downarrow 0} \frac{T_h(u_0, 0) - u_0}{h} = v \text{ in } X \right\} \tag{3.3}$$

is an ω-quasi accretive well-defined mapping $A_0 : D(A_0) \to X$ and called the *infinitesimal generator* of $\{T_t\}_{t\ge 0}$. If the Banach space X and its dual space X^* are both uniformly convex (see [7, Proposition 4.3]), then one has that

$$-A_0 = A^0,$$

where A° is the minimal selection of A defined by

$$A^\circ := \left\{ (u, v) \in A \,\middle|\, \|v\|_X = \inf_{\hat{v}\in Au} \|\hat{v}\|_X \right\}. \tag{3.4}$$

For simplicity, we ignore the additional geometric assumptions on the Banach space X, and refer to the two families $\{T_t\}_{t=0}^T$ defined by (2.2) on $\overline{D(A)}^X \times L^1(0,T;X)$ and $\{T_t\}_{t\ge 0}$ defined by (2.14) on $\overline{D(A)}^X$ as the *semigroup generated by* $-A$.

Further, for every $u_0 \in \overline{D(A)}^X$, if $f \in L^1(0,T;X)$ is given by the step function $f = \sum_{i=1}^N f_i \mathbb{1}_{(t_{i-1},t_i]}$, then the corresponding mild solution $u : [0,T] \to X$ of Cauchy problem (2.1) is given by

$$u(t) = u_0 \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_i(t)\mathbb{1}_{(t_{i-1},t_i]}(t) \tag{3.5}$$

where each u_i is the unique mild solution of the Cauchy problem (for constant $f \equiv f_i$)

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + A(u_i(t)) \ni f_i \quad \text{on } (t_{i-1},t_i), \text{and} \quad u_i(t_{i-1}) = u_{i-1}(t_{i-1}) \tag{3.6}$$

for every $i = 1,\dots,N$ (cf., [10, Chapter 4.3]). In particular, the semigroup $\{T_t\}_{t=0}^T$ is obtained by the *exponential formula*

$$T_t(u(t_{i-1}),f_i) = u_i(t) = \lim_{n\to\infty} \left[J^{A-f_i}_{\frac{t-t_{i-1}}{n}}\right]^n u(t_{i-1}) \qquad \text{in } C([t_{i-1},t_i];X) \tag{3.7}$$

iteratively for every $i = 1,\dots,N$, where for $\mu > 0$, $J_\mu^{A-f_i} = (I + \mu(A - f_i))^{-1}$ is the *resolvent operator* of $A - f_i$.

As for classical solutions, the fact that A is homogeneous of order $\alpha \neq 1$, is also reflected in the notion of mild solution and, in particular, in the semigroup $\{T_t\}_{t=0}^T$ as demonstrated in our next proposition.

Proposition 3.4 (Homogeneous accretive operators). *Let A be a quasi m-accretive operator on X and $\{T_t\}_{t=0}^T$ the semigroup generated by $-A$ on $\overline{D(A)}^X \times L^1(0,T;X)$. If A is homogeneous of order $\alpha \neq 1$, then for every $\lambda > 0$, $\{T_t\}_{t=0}^T$ satisfies equation*

$$\lambda^{\frac{1}{\alpha-1}} T_{\lambda t}(u_0,f) = T_t(\lambda^{\frac{1}{\alpha-1}}u_0, \lambda^{\frac{\alpha}{\alpha-1}} f(\lambda\cdot)) \qquad \text{for all } t \in [0,\tfrac{T}{\lambda}], \tag{2.4}$$

for every $(u_0,f) \in \overline{D(A)}^X \times L^1(0,T;X)$.

Proof. Let $\lambda > 0$ and $f \in X$. Then, for every $u, v \in X$ and $\mu > 0$,

$$J_\mu^{A-\lambda^{\frac{\alpha}{\alpha-1}}f}\left[\lambda^{\frac{1}{\alpha-1}}v\right] = u \qquad \text{if and only if} \qquad u + \mu(Au - \lambda^{\frac{\alpha}{\alpha-1}}f) \ni \lambda^{\frac{1}{\alpha-1}}v.$$

Now, the hypothesis that A is *homogeneous of order* $\alpha \neq 1$ implies that the right-hand side in the previous characterization is equivalent to

$$\lambda^{\frac{1}{1-\alpha}}u + \lambda\mu(A(\lambda^{\frac{1}{1-\alpha}}u) - f) \ni v, \qquad \text{or} \qquad J_{\lambda\mu}^{A-f}v = \lambda^{\frac{1}{1-\alpha}}u.$$

Therefore, one has that

$$\lambda^{\frac{1}{\alpha-1}} J_{\lambda\mu}^{A-f}v = J_\mu^{A-\lambda^{\frac{\alpha}{\alpha-1}}f}\left[\lambda^{\frac{1}{\alpha-1}}v\right] \qquad \text{for all } \lambda, \mu > 0, \text{ and } v \in X. \tag{3.8}$$

Now, let $u_0 \in \overline{D(A)}^X$, $\pi : 0 = t_0 < t_1 < \dots < t_N = T$ be a partition of $[0,T]$, and $f = \sum_{i=1}^N f_i \mathbb{1}_{(t_{i-1},t_i]} \in L^1(0,T;X)$ a step function. If u denotes the unique mild solution of (2.1) for this step function f, then u is given by (3.5), were on each subinterval $(t_{i-1},t_i]$, u_i is the unique mild solution of (3.6).

Next, let $\lambda > 0$ and set

$$v_\lambda(t) := \lambda^{\frac{1}{\alpha-1}}u(\lambda t) \qquad \text{for every } t \in [0,\tfrac{T}{\lambda}].$$

Then,

$$v_\lambda(t) = \lambda^{\frac{1}{\alpha-1}} u_0\, \mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} \lambda^{\frac{1}{\alpha-1}} u_i(\lambda t) \mathbb{1}_{\left(\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}\right]}(t)$$

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Obviously, $v_\lambda(0) = \lambda^{\frac{1}{\alpha-1}} u_0$. Thus, to show that (2.4) holds, it remains to verify that v_λ is a mild solution of

$$\tfrac{\mathrm{d}v_\lambda}{\mathrm{d}t} + A(v_\lambda(t)) \ni \lambda^{\frac{\alpha}{\alpha-1}} f(\lambda t) \qquad \text{on } \left(0, \tfrac{T}{\lambda}\right)$$

or, in other words,

$$v_\lambda(t) = T_t(\lambda^{\frac{1}{\alpha-1}} u_0, \lambda^{\frac{\alpha}{\alpha-1}} f(\lambda\cdot)) \tag{3.9}$$

for every $t \in \left[0, \frac{T}{\lambda}\right]$. Let $t \in (0, t_1/\lambda]$ and $n \in \mathbb{N}$. We apply (3.8) to

$$\mu = \frac{t}{n} \qquad \text{and} \qquad v = J_{\frac{\lambda t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}} f_1}[\lambda^{\frac{1}{\alpha-1}} u_0].$$

Then, one finds that

$$\left[J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}} f_1}\right]^2 [\lambda^{\frac{1}{\alpha-1}} u_0] = J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}} f_1} \left[\lambda^{\frac{1}{\alpha-1}} J_{\frac{\lambda t}{n}}^{A-f_1} u_0\right] = \lambda^{\frac{1}{\alpha-1}} \left[J_{\frac{\lambda t}{n}}^{A-f_1}\right]^2 u_0.$$

Applying (3.8) to $\lambda^{\frac{1}{\alpha-1}} \left[J_{\frac{\lambda t}{n}}^{A-f_1}\right]^i u_0$ iteratively for $i = 2, \dots, n$ yields

$$\lambda^{\frac{1}{\alpha-1}} \left[J_{\frac{\lambda t}{n}}^{A-f_1}\right]^n u_0 = \left[J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}} f_1}\right]^n \left[\lambda^{\frac{1}{\alpha-1}} u_0\right]. \tag{3.10}$$

By (3.7), sending $n \to +\infty$ in (3.10) yields on the one side

$$\lim_{n\to+\infty} \lambda^{\frac{1}{\alpha-1}} \left[J_{\frac{\lambda t}{n}}^{A-f_1}\right]^n u_0 = \lambda^{\frac{1}{\alpha-1}} u_1(\lambda t) = v_\lambda(t),$$

and on the other side

$$\lim_{n\to+\infty} \left[J_{\frac{t}{n}}^{A-\lambda^{\frac{\alpha}{\alpha-1}} f_1}\right]^n \left[\lambda^{\frac{1}{\alpha-1}} u_0\right] = T_t(\lambda^{\frac{1}{\alpha-1}} u_0, \lambda^{\frac{\alpha}{\alpha-1}} f_1),$$

showing that (3.9) holds for every $t \in [0, \frac{t_1}{\lambda}]$. Repeating this argument on each subinterval $(\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}]$ for $i = 2, \dots, N$, where one replaces in (3.10) u_0 by $u(t_{i-1})$, and f_1 by f_i, then one sees that v_λ satisfies (3.9) on the whole interval $[0, \frac{T}{\lambda}]$. □

By the preceding proposition and by Lemma 2.2, we can now state the following result.

Corollary 3.5. *Let A be a quasi m-accretive operator on a Banach space X and $\{T_t\}_{t=0}^T$ the semigroup generated by $-A$ on $L^1(0,T;X) \times \overline{D(A)}^X$. If A is homogeneous of order $\alpha \neq 1$, then for every $(u_0, f) \in \overline{D(A)}^X \times L^1(0,T;X)$, $t \mapsto T_t(u_0, f)$ satisfies*

$$\begin{aligned} &\limsup_{h\to 0+} \left\| \frac{T_{t+h}(u_0,f) - T_t(u_0,f)}{h} \right\|_X \\ &\qquad \le \frac{1}{t} e^{\omega t} \left[2\frac{\|u_0\|_X}{|1-\alpha|} + \frac{1}{|1-\alpha|} \int_0^t e^{-\omega s} \|f(s)\|_X \,\mathrm{d}s + V_\omega(f,t) \right], \end{aligned} \tag{3.11}$$

for a.e. $t \in (0,T]$, *where* $V_\omega(f,t)$ *is defined by* (2.7). *In particular, if* $f \in W^{1,1}(0,T;X)$ *and* $\frac{\mathrm{d}}{\mathrm{d}t}T_t(u_0,f)$ *exists in* X *at a.e.* $t \in (0,T)$, *then* $T_t(u_0,f)$ *satisfies*

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t_+}T_t(u_0,f)\right\|_X \le \frac{L}{t}\,e^{\omega t}\left[2\frac{\|u_0\|_X}{|1-\alpha|} + \frac{1}{|1-\alpha|}\int_0^t e^{-\omega s}\|f(s)\|_X\,\mathrm{d}s + \int_0^t e^{-\omega s}\|f'(s)\|_X\,s\,\mathrm{d}s\right] \tag{3.12}$$

for a.e. $t \in (0,T)$.

To consider the regularizing effect of mild solutions to the Cauchy problem (2.20) for the perturbed operator $A+F$, we recall the following well-known result from the literature.

Proposition 3.6 ([10, Lemma 7.8])**.** *If* $A+\omega I$ *is accretive in* X *and* $f \in BV(0,T;X)$, *then for every* $u_0 \in D(A)$, *the mild solution* $u(t) := T_t(u_0,f)$, $(t \in [0,T])$, *of Cauchy problem* (2.1) *is Lipschitz continuous on* $[0,T]$ *satisfying*

$$\limsup_{h\to 0+}\frac{\|u(t+h)-u(t)\|_X}{h} \le e^{\omega t}\|f(0+)-y\|_X + \tilde{V}(f,t+) + \omega\int_0^t e^{\omega(t-s)}\tilde{V}(f,s+)\,\mathrm{d}s$$

for every $t \in [0,T]$ *and* $v \in Au_0$, *where*

$$\tilde{V}(f,t+) := \limsup_{h\to 0+}\int_0^t \frac{\|f(s+h)-f(s)\|_X}{h}\,\mathrm{d}s.$$

With the preceding Theorem 2.6, Proposition 3.4, and Proposition 3.6 in mind, we are now in the position to outline the proof of our main Theorem 1.1.

Proof of Theorem 1.1. We begin by noting that if A is m-accretive in X and F is a Lipschitz continuous mapping with Lipschitz constant ω, then the operator $A+F$ is ω-quasi m-accretive in X; or in other words, $A+F+\omega I$ is m-accretive in X. Hence, for every $T>0$, there is a semigroup $\{T_t\}_{t=0}^T$ of mappings $T_t : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ satisfying (2.3) and (2.5) with ω and $L=1$. Further, the semigroup $\{\tilde{T}_t\}_{t=0}^T$ generated by $-A$ satisfies (2.3) and (2.5) with $\omega=0$ and $L=1$, (2.22) for every $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$ with $\tilde{f}$ given by (2.21), and by Proposition 3.4, $\{\tilde{T}_t\}_{t=0}^T$ satisfies (2.4). Now, let $u_0 \in D(A)$ and $f \in BV(0,T;X)$. Then by Proposition 3.6, the mild solution $u(t) := T_t(u_0,f)$, $(t \in [0,T])$, of Cauchy problem (2.1) is Lipschitz continuous on $[0,T]$. Thus, we can apply Theorem 2.6 and obtain that u satisfies (1.3). □

In order the semigroup $\{T_t\}_{t=0}^T$ generated by $-A$ satisfies regularity estimate (2.10) (respectively, (2.19)), one requires that each mild solution u of (2.1) (respectively, of (1.8)) is *differentiable* at a.e. $t \in (0,T)$, or in other words, u is a *strong solution* of (2.1). The next definition is taken from [10, Definition 1.2] (cf [7, Chapter 4]).

Definition 3.7. A locally absolutely continuous function $u[0,T] :\to X$ is called a *strong solution* of the differential inclusion

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + A(u(t)) \ni f(t) \qquad \text{for a.e. } t \in (0,T), \tag{3.13}$$

if u is differentiable a.e. on $(0,T)$, and for a.e. $t \in (0,T)$, $u(t) \in D(A)$ and $f(t) - \frac{du}{dt}(t) \in A(u(t))$. Further, for given $u_0 \in X$ and $f \in L^1(0,T;X)$, a function u is called a *strong solution* of Cauchy problem (2.1) if $u \in C([0,T];X)$, u is strong solution of (3.13) and $u(0) = u_0$.

The next characterization of strong solutions of (3.13) highlights the important point of *a.e. differentiability*.

Proposition 3.8 ([10, Theorem 7.1]). *Let X be a Banach space, $f \in L^1(0,T;X)$ and A be quasi m-accretive in X. Then u is a strong solution of the differential inclusion (3.13) on $[0,T]$ if and only if u is a mild solution on $[0,T]$ and u is "absolutely continuous" on $[0,T]$ and differentiable a.e. on $(0,T)$.*

Of course, every strong solution u of (3.13) is a mild solution of (3.13), and u is absolutely continuous and differentiable a.e. on $[0,T]$. The differential inclusion (3.13) admits mild and Lipschitz continuous solutions if A is ω-quasi m-accretive in X (cf [10, Lemma 7.8]). But, in general, an absolutely continuous functions $u : [0,T] \to X$ is not necessarily differentiable a.e. on $(0,T)$. Only if one assumes additional geometric properties on X, then the latter implication holds true. Our next definition is taken from [10, Definition 7.6] (cf [3, Chapter 1]).

Definition 3.9. A Banach space X is said to have the *Radon-Nikodým property* if every absolutely continuous function $F : [a,b] \to X$, (a, $b \in \mathbb{R}$, $a < b$), is differentiable almost everywhere on (a,b).

Known examples of Banach spaces X admitting the Radon-Nikodým property are:

- **(Dunford-Pettis)** if $X = Y^*$ is separable, where Y^* is the dual space of a Banach space Y;
- if X is *reflexive*.

We emphasize that $X_1 = L^1(\Sigma,\mu)$, $X_2 = L^\infty(\Sigma,\mu)$, or $X_3 = C(\mathcal{M})$ for a σ-finite measure space (Σ,μ), or respectively, for a compact metric space $(\mathcal{M},d)$ don't have, in general, the Radon-Nikodým property (cf [3]). Thus, it is quite surprising that there is a class of operators A (namely, the class of *completely accretive operators*, see Section 4 below), for which the differential inclusion (2.1) nevertheless admits strong solutions (with values in $L^1(\Sigma,\mu)$ or $L^\infty(\Sigma,\mu)$).

Now, by Corollary 3.5 and Proposition 3.8, we can conclude the following results. We emphasize that one crucial point in the statement of Corollary 3.10 below is that due to the uniform estimate (2.9), one has that for all initial values $u_0 \in \overline{D(A)}^X$, the unique mild solution u of (2.1) is strong.

Corollary 3.10. *Suppose A is a quasi m-accretive operator on a Banach space X admitting the Radon-Nikodým property, and $\{T_t\}_{t=0}^T$ is the semigroup generated by $-A$ on $\overline{D(A)}^X \times L^1(0,T;X)$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $f \in W^{1,1}(0,T;X)$, the unique mild solution $u(t) := T_t(u_0,f)$ of (2.1) is strong and $\{T_t\}_{t=0}^T$ satisfies (3.12) for a.e. $t \in (0,T)$.*

We omit the proof of Corollary 3.10 since it is straightforward. Now, we are ready to show that the statement of Corollary 1.2 holds.

Proof of Corollary 1.2. First, let $u_0 \in D(A)$. Then by Proposition 3.6, the mild solution $u(t) = T_t(u_0, f)$ of Cauchy problem (1.1) is Lipschitz continuous on $[0, T)$, and since every reflexive Banach space X admits the Radon-Nikodým property, u is differentiable a.e. on $(0, T)$. Thus, by Theorem 1.1, there is a function $\psi \in L^\infty(0, T)$ such that u satisfies

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)\right\|_X \le \frac{1}{t}\left[\frac{e^{\omega t}+1}{|1-\alpha|}\|u_0\|_X + \psi(t) + \omega\int_0^t \left[\frac{e^{\omega s}+1}{|1-\alpha|}\|u_0\|_X + \psi(s)\right] e^{\omega(t-s)}\mathrm{d}s\right]$$

for a.e. $t \in (0, T)$. Next, we square both sides of the last inequality and subsequently integrate over (a, b) for given $0 < a < b \le T$. Then, one finds that

$$\begin{aligned}\int_a^b \left\|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right\|_X^2 \mathrm{d}t \le \int_a^b \frac{1}{t^2}\bigg\{&\frac{e^{\omega t}+1}{|1-\alpha|}\|u_0\|_X + \psi(t)\\ &+\omega\int_0^t \left[\frac{e^{\omega s}+1}{|1-\alpha|}\|u_0\|_X + \psi(s)\right] e^{\omega(t-s)}\mathrm{d}s\bigg\}^2 \mathrm{d}t\end{aligned} \tag{3.14}$$

Due to this estimate, we can now show that also for $u_0 \in \overline{D(A)}^X$, the corresponding mild solution u of (1.1) is strong. To see this let $(u_{0,n})_{n\ge1} \subseteq D(A)$ such that $u_{0,n} \to u_0$ in X as $n \to \infty$ and set $u_n = T_t u_{0,n}$ for all $n \ge 1$. By (2.5) (which is satisfied with $L = 1$ by all u_n), $(u_n)_{n\ge1}$ is a Cauchy sequence in $C([0, T]; X)$. Hence, there is a function $u \in C([0, T]; X)$ satisfying $u(0) = u_0$ and $u_n \to u$ in $C([0, T]; X)$. In particular, one can show that u is the unique mild solution of Cauchy problem (1.1). Now, by inequality (3.14), $(\mathrm{d}u_n/\mathrm{d}t)_{n\ge1}$ is bounded in $L^2(a, b; X)$ for any $0 < a < b \le T$. Since X is reflexive, also $L^2(a, b; X)$ is reflexive and hence, there is a $v \in L^2(a, b; X)$ and a subsequence of $(u_{0,n})_{n\ge1}$, which, for simplicity, we denote again by $(u_{0,n})_{n\ge1}$, such that $\frac{\mathrm{d}u_n}{\mathrm{d}t} \rightharpoonup v$ weakly in $L^2(a, b; X)$ as $n \to +\infty$. Since $u_n \to u$ in $C([a, b]; X)$, it follows by a standard argument that $v = \frac{\mathrm{d}u}{\mathrm{d}t}$ in the sense of vector-valued distributions $\mathcal{D}'((a, b); X)$. Since $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(a, b; X)$, the mild solution u of Cauchy problem (1.1) is absolutely continuous on (a, b), and since X is reflexive, u is differentiable a.e. on (a, b). Since $0 < a < b < \infty$ were arbitrary, $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^1_{loc}((0, \infty); X)$.

To see that u satisfies inequality (1.5), note that for $\varepsilon > 0$, the function $t \mapsto \tilde{u}(t) := u(t + \varepsilon)$ on $[0, T - \varepsilon]$ satisfies the hypotheses of Theorem 1.1 with $\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} \in L^1([0, T - \varepsilon); X)$ and so, we find that

$$\begin{aligned}\left\|\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t_+}(t)\right\|_X \le \frac{1}{t}\bigg[&\frac{e^{\omega t}+1}{|1-\alpha|}\|u(\varepsilon)\|_X + \psi(t)\\ &+\omega\int_0^t \left[\frac{e^{\omega s}+1}{|1-\alpha|}\|u(\varepsilon)\|_X + \psi(s)\right] e^{\omega(t-s)}\mathrm{d}s\bigg]\end{aligned}$$

for every $t \in (0, T - \varepsilon]$ and $\varepsilon \in (0, t)$. Sending $\varepsilon \to 0+$ shows that u satisfies (1.5). Since $u_0 \in \overline{D(A)}^X$ was arbitrary, this completes the proof of this corollary. $\square$

If the Banach space X and its dual space X^* are *uniformly convex* and $A + F$ is quasi m-accretive in X, then (cf., [7, Theorem 4.6]) for every $u_0 \in D(A)$, $f \in W^{1,1}(0, T; X)$, the mild solution $u(t) = T_t(u_0, f)$, $(t \in [0, T])$, of Cauchy problem (1.1) is a strong one, u is everywhere differentiable from the right, $\frac{\mathrm{d}u}{\mathrm{d}t_+}$

is right continuous, and

$$\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) + (A + F - f(t))^\circ u(t) = 0 \qquad \text{for every } t \geq 0,$$

where for every $t \in [0, T]$, $(A + F - f(t))^\circ$ denotes the *minimal selection* of $A + F - f(t)$ defined by (3.4). Thus, under those assumptions on X and by Corollary 1.2, we can conclude the following two corollaries. We begin by stating the inhomogeneous case.

Corollary 3.11. *Suppose X and its dual space X^* are uniformly convex, for $\omega \in \mathbb{R}$, A is an ω-quasi m-accretive operator on X, and $\{T_t\}_{t=0}^T$ is the semigroup on $\overline{D(A)}^X \times L^1(0, T; X)$ generated by $-A$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $f \in W^{1,1}(0, T; X)$, the mild solution $u(t) = T_t(u_0, f)$, $(t \in [0, T])$ of Cauchy problem (2.1) is strong and*

$$\|(A + F - f(t))^\circ T_t(u_0, f)\|_X \leq \frac{1}{t}\Big[a(t) + \omega \int_0^t a(s) e^{\omega(t-s)} \mathrm{d}s\Big]$$

for every $t \in (0, T]$, where $a(t)$ is defined by (1.4).

Our next corollary considers the homogeneous case of Cauchy problem (2.1).

Corollary 3.12. *Suppose X and its dual space X^* are uniformly convex, for $\omega \in \mathbb{R}$, A is an ω-quasi m-accretive operator on X, and $\{T_t\}_{t\geq 0}$ is the semigroup on $\overline{D(A)}^X$ generated by $-A$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$, the mild solution $u(t) = T_t u_0$, $(t \geq 0)$ of Cauchy problem (1.8) (for $f \equiv 0$) is a strong solution satisfying*

$$\|(A + F - f(t))^\circ T_t u_0\|_X \leq \frac{e^{\omega t} + 1}{|1 - \alpha|\, t} \left[1 + \omega \int_0^t e^{\omega(t-s)}\, \mathrm{d}s\right] \|u_0\|_X$$

for every $t > 0$.

To conclude this section, we briefly outline the proof of Theorem 1.3.

Proof of Theorem 1.3. In the case A is an m-accretive operator on a Banach lattice X and F a Lipschitz continuous perturbation with constant $\omega \geq 0$, then the statement of Theorem 1.3 immediately follow from Theorem 2.9 and Corollary 2.11 with constants $L = 1$. □

4. Homogeneous completely accretive operators

In [8], Bénilan and Crandall introduced the class of *completely accretive* operators A and showed: even though the underlying Banach spaces does not admit the Radon-Nikodým property, but if A is completely accretive and homogeneous of order $\alpha > 0$ with $\alpha \neq 1$, then the mild solutions of differential inclusion (1.8) involving A are strong. This was extended in [27] to the zero-order case. Here, we provide a generalization to the case of completely accretive operators which are homogeneous of order $\alpha \neq 1$ and perturbed by a Lipschitz nonlinearity.

4.1. **General framework.** In order to keep this paper self-contained, we provide a brief introduction to the class of completely accretive operators, where we mainly follow [8] and the monograph [18].

For the rest of this paper, suppose $(\Sigma, \mathcal{B}, \mu)$ is a σ-finite measure space, and $M(\Sigma,\mu)$ the space of μ-a.e. equivalent classes of measurable functions $u : \Sigma \to \mathbb{R}$. For $u \in M(\Sigma,\mu)$, we write $[u]^+$ to denote $\max\{u,0\}$ and $[u]^- = -\min\{u,0\}$. We denote by $L^q(\Sigma,\mu)$, $1 \le q \le \infty$, the corresponding standard Lebesgue space with norm

$$\|\cdot\|_q = \begin{cases} \left(\int_\Sigma |u|^q \,\mathrm{d}\mu\right)^{1/q} & \text{if } 1 \le q < \infty, \\ \inf\left\{k \in [0,+\infty] \;\middle|\; |u| \le k \ \mu\text{-a.e. on } \Sigma\right\} & \text{if } q = \infty. \end{cases}$$

For $1 \le q < \infty$, we identify the dual space $(L^q(\Sigma,\mu))'$ with $L^{q'}(\Sigma,\mu)$, where q' is the conjugate exponent of q given by $1 = \frac{1}{q} + \frac{1}{q'}$.

Next, we first briefly recall the notion of *Orlicz spaces* (cf [36, Chapter 3]). A continuous function $\psi : [0,+\infty) \to [0,+\infty)$ is an *N-function* if it is convex, $\psi(s) = 0$ if and only if $s = 0$, $\lim_{s\to 0+} \psi(s)/s = 0$, and $\lim_{s\to\infty} \psi(s)/s = \infty$. Given an N-function ψ, the *Orlicz space* is defined as follows

$$L^\psi(\Sigma,\mu) := \left\{ u \in M(\Sigma,\mu) \;\middle|\; \int_\Sigma \psi\left(\frac{|u|}{\alpha}\right) \mathrm{d}\mu < \infty \text{ for some } \alpha > 0 \right\}$$

and equipped with the *Orlicz-Minkowski norm*

$$\|u\|_\psi := \inf\left\{ \alpha > 0 \;\middle|\; \int_\Sigma \psi\left(\frac{|u|}{\alpha}\right) \mathrm{d}\mu \le 1 \right\}.$$

With these preliminaries in mind, we are now in the position to recall the notation of *completely accretive* operators introduced in [8] and further developed to the ω*-quasi* case in [18].

Let J_0 be the set given by

$$J_0 = \left\{ j : \mathbb{R} \to [0,+\infty] \;\middle|\; j \text{ is convex, lower semicontinuous, } j(0) = 0 \right\}.$$

Then, for every $u, v \in M(\Sigma,\mu)$, we write

$$u \ll v \quad \text{if and only if} \quad \int_\Sigma j(u)\,\mathrm{d}\mu \le \int_\Sigma j(v)\,\mathrm{d}\mu \quad \text{for all } j \in J_0.$$

Remark 4.1. Due to the interpolation result [8, Proposition 1.2], for given u, $v \in M(\Sigma,\mu)$, the relation $u \ll v$ is equivalent to the two conditions

$$\begin{cases} \int_\Sigma [u-k]^+ \,\mathrm{d}\mu & \le \int_\Sigma [v-k]^+ \,\mathrm{d}\mu \qquad \text{for all } k > 0 \text{ and} \\ \int_\Sigma [u+k]^- \,\mathrm{d}\mu & \le \int_\Sigma [v+k]^- \,\mathrm{d}\mu \qquad \text{for all } k > 0. \end{cases}$$

By this characterization, it is clear that for every $u, v, w \in M(\Sigma,\mu)$,

$$\text{if } u \ll v \text{ and } 0 \le w \le u \text{ then } w \ll v. \tag{4.1}$$

Thus, the relation $\ll$ is closely related to the theory of rearrangement-invariant function spaces (cf [11]). Another, useful characterization of the relation $\ll$ is the following (cf [8, Remark 1.5]): for every $u, v \in M(\Sigma,\mu)$, one has that

$$u \ll v \text{ if and only if } u^+ \ll v^+ \text{ and } u^- \ll v^-.$$

Further, the relation $\ll$ on $M(\Sigma,\mu)$ has the following properties. We omit the easy proof of this proposition.

Proposition 4.2. *For every $u, v, w \in M(\Sigma,\mu)$, one has that*
(1) $u^+ \ll u$, $u^- \ll -u$;
(2) $u \ll v$ if and only if $u^+ \ll v^+$ and $u^- \ll v^-$;
*(3) (*positive homogeneity*) if $u \ll v$ then $\alpha u \ll \alpha v$ for all $\alpha > 0$;*
*(4) (*transitivity*) if $u \ll v$ and $v \ll w$ then $u \ll w$;*
(5) if $u \ll v$ then $|u| \ll |v|$;
*(6) (*convexity*) for every $u \in M(\Sigma,\mu)$, the set $\{w \mid w \ll u\}$ is convex.*

With these preliminaries in mind, we can now state the following definitions.

Definition 4.3. A mapping $S : D(S) \to M(\Sigma,\mu)$ with domain $D(S) \subseteq M(\Sigma,\mu)$ is called a *complete contraction* if

$$Su - S\hat{u} \ll u - \hat{u} \qquad \text{for every } u, \hat{u} \in D(S).$$

More generally, for $L \geq 1$, we call S to be an *L-complete contraction* if

$$L^{-1}Su - L^{-1}S\hat{u} \ll u - \hat{u} \qquad \text{for every } u, \hat{u} \in D(S),$$

and for some $\omega \in \mathbb{R}$, S is called to be *ω-quasi completely contractive* if S is an L-complete contraction with $L = e^{\omega t}$ for some $t \geq 0$.

Remark 4.4. Note, for every $1 \leq q < \infty$, $j_q(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$, $j_\infty(\cdot) = [[\cdot]^+ - k]^+ \in \mathcal{J}_0$ for every $k \geq 0$ (and for large enough $k > 0$ if $q = \infty$), and for every N-function ψ and $\alpha > 0$, $j_{\psi,\alpha}(\cdot) = \psi(\frac{[\cdot]^+}{\alpha}) \in \mathcal{J}_0$. This shows that for every L-complete contraction $S : D(S) \to M(\Sigma,\mu)$ with domain $D(S) \subseteq M(\Sigma,\mu)$, the mapping $L^{-1}S$ is order-preserving and contractive respectively for every L^q-norm ($1 \leq q \leq \infty$), and every L^ψ-norm with N-function ψ.

Now, we can state the definition of completely accretive operators.

Definition 4.5. An operator A on $M(\Sigma,\mu)$ is called *completely accretive* if for every $\lambda > 0$, the resolvent operator J_λ of A is a complete contraction, or equivalently, if for every $(u_1, v_1), (u_2, v_2) \in A$ and $\lambda > 0$, one has that

$$u_1 - u_2 \ll u_1 - u_2 + \lambda(v_1 - v_2).$$

If X is a linear subspace of $M(\Sigma,\mu)$ and A an operator on X, then A is said to be *m-completely accretive on* X if A is completely accretive and satisfies the *range condition* (3.1). Further, for $\omega \in \mathbb{R}$, an operator A on a linear subspace $X \subseteq M(\Sigma,\mu)$ is called *ω-quasi (m)-completely accretive* in X if $A + \omega I$ is (m)-completely accretive in X. Finally, an operator A on a linear subspace $X \subseteq M(\Sigma,\mu)$ is called *quasi (m-)completely accretive* if there is an $\omega \in \mathbb{R}$ such that $A + \omega I$ is (m-)completely accretive in X.

Remark 4.6. For $\omega \in \mathbb{R}$, the fact that A is ω-quasi (m)-completely accretive in X implies that the resolvent operator J_λ^A of A is L-completely contractive for

$L = (1-\lambda\omega)^{-1}$ for every $\lambda > 0$ satisfying $\lambda\omega < 1$. Indeed, if A is ω-quasi (m)-completely accretive in X then by taking $L = (1-\lambda\omega)^{-1}$, one sees that

$$\begin{aligned}\int_\Sigma j\Big(L^{-1}(u_1-u_2)\Big)\,\mathrm{d}\mu &= \int_\Sigma j_{L^{-1}}(u_1-u_2)\,\mathrm{d}\mu\\ &\le \int_\Sigma j_{L^{-1}}\Big(u_1-u_2+\tfrac{\lambda}{1-\lambda\omega}(\omega(u_1-u_2)+v_1-v_2)\Big)\,\mathrm{d}\mu\\ &= \int_\Sigma j(u_1-u_2+\lambda(v_1-v_2))\,\mathrm{d}\mu\end{aligned}$$

for every (u_1,v_1), $(u_1,v_1) \in A$ and $\lambda > 0$ satisfying $\lambda\omega < 1$, where we used that $j_{L^{-1}}(s) := j((1-\lambda w)s)$ belongs to J_0.

This property transfers as follows to the semigroup $\{T_t\}_{t\ge 0}$
Before stating a useful characterization of quasi completely accretive operators, we first need to introduce the following function spaces. Let

$$L^{1+\infty}(\Sigma,\mu) := L^1(\Sigma,\mu) + L^\infty(\Sigma,\mu) \quad\text{and}\quad L^{1\cap\infty}(\Sigma,\mu) := L^1(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)$$

be the *sum* and the *intersection space* of $L^1(\Sigma,\mu)$ and $L^\infty(\Sigma,\mu)$, which are equipped, respectively, with the norms

$$\begin{aligned}&\|u\|_{1+\infty} := \inf\Big\{\|u_1\|_1 + \|u_2\|_\infty \Big| u = u_1+u_2,\ u_1\in L^1(\Sigma,\mu), u_2 \in L^\infty(\Sigma,\mu)\Big\},\\ &\|u\|_{1\cap\infty} := \max\{\|u\|_1, \|u\|_\infty\}\end{aligned}$$

are Banach spaces. In fact, $L^{1+\infty}(\Sigma,\mu)$ and and $L^{1\cap\infty}(\Sigma,\mu)$ are respectively the largest and the smallest of the rearrangement-invariant Banach function spaces (cf., [11, Chapter 3.1]). If $\mu(\Sigma)$ is finite, then $L^{1+\infty}(\Sigma,\mu) = L^1(\Sigma,\mu)$ with equivalent norms, but if $\mu(\Sigma) = \infty$ then $L^{1+\infty}(\Sigma,\mu)$ contains $\bigcup_{1\le q\le\infty} L^q(\Sigma,\mu)$. Further, we will employ the space

$$L_0(\Sigma,\mu) := \left\{u \in M(\Sigma,\mu) \;\Big|\; \int_\Sigma \left[|u|-k\right]^+ \mathrm{d}\mu < \infty \text{ for all } k > 0\right\},$$

which equipped with the $L^{1+\infty}$-norm is a closed subspace of $L^{1+\infty}(\Sigma,\mu)$. In fact, one has (cf., [8]) that $L_0(\Sigma,\mu) = \overline{L^1(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)}^{1+\infty}$. Since for every $k \ge 0$, $T_k(s) := [|s|-k]^+$ is a Lipschitz mapping $T_k : \mathbb{R} \to \mathbb{R}$ and by Chebyshev's inequality, one sees that $L^q(\Sigma,\mu) \hookrightarrow L_0(\Sigma,\mu)$ for every $1 \le q < \infty$ (and $q = \infty$ if $\mu(\Sigma) < +\infty$), and $L^\psi(\Sigma,\mu) \hookrightarrow L_0(\Sigma,\mu)$ for every N-function ψ.

Proposition 4.7 ([18]). *Let P_0 denote the set of all functions $T \in C^\infty(\mathbb{R})$ satisfying $0 \le T' \le 1$ such that T' is compactly supported, and $x = 0$ is not contained in the support supp(T) of T. Then for $\omega \in \mathbb{R}$, an operator $A \subseteq L_0(\Sigma,\mu)\times L_0(\Sigma,\mu)$ is ω-quasi completely accretive if and only if*

$$\int_\Sigma T(u-\hat{u})(v-\hat{v})\,\mathrm{d}\mu + \omega\int_\Sigma T(u-\hat{u})(u-\hat{u})\,\mathrm{d}\mu \ge 0$$

for every $T \in P_0$ and every (u,v), $(\hat{u},\hat{v}) \in A$.

Remark 4.8. For convenience, we denote the unique extension of $\{T_t\}_{t\ge0}$ on $L^\psi(\Sigma,\mu)$ or $L^1(\Sigma,\mu)$ again by $\{T_t\}_{t\ge0}$.

Definition 4.9. A Banach space $X \subseteq M(\Sigma,\mu)$ with norm $\|\cdot\|_X$ is called *normal* if the norm $\|\cdot\|_X$ has the following property:

$$\begin{cases} \text{for every } u \in X, v \in M(\Sigma,\mu) \text{ satisfying } \quad v \ll u, \\ \text{one has that } \quad v \in X \quad \text{ and } \quad \|v\|_X \le \|u\|_X. \end{cases}$$

Typical examples of normal Banach spaces $X \subseteq M(\Sigma,\mu)$ are Orlicz-spaces $L^{\psi}(\Sigma,\mu)$ for every N-function ψ, $L^q(\Sigma,\mu)$, $(1 \le q \le \infty)$, $L^{1\cap\infty}(\Sigma,\mu)$, $L_0(\Sigma,\mu)$, and $L^{1+\infty}(\Sigma,\mu)$.

Remark 4.10. It is important to point out that if X is a normal Banach space, then for every $u \in X$, one always has that u^+, u^- and $|u| \in X$. To see this, recall that by (1) Proposition 4.2, if $u \in X$, then $u^+ \ll u$ and $u^- \ll -u$. Thus, u^+ and $u^- \in X$ and since $|u| = u^+ + u^-$, one also has that $|u| \in X$.

The dual space $(L_0(\Sigma,\mu))'$ of $L_0(\Sigma,\mu)$ is isometrically isomorphic to the space $L^{1\cap\infty}(\Sigma,\mu)$. Thus, a sequence $(u_n)_{n\ge1}$ in $L_0(\Sigma,\mu)$ is said to be *weakly convergent* to u in $L_0(\Sigma,\mu)$ if

$$\langle v, u_n\rangle := \int_\Sigma v\, u_n \,\mathrm{d}\mu \to \int_\Sigma v\, u \,\mathrm{d}\mu \qquad \text{for every } v \in L^{1\cap\infty}(\Sigma,\mu).$$

For the rest of this paper, we write $\sigma(L_0, L^{1\cap\infty})$ to denote the *weak topology* on $L_0(\Sigma,\mu)$. For this weak topology, we have the following compactness result.

Proposition 4.11 ([8, Proposition 2.11])**.** *Let $u \in L_0(\Sigma,\mu)$. Then, the following statements hold.*

(1) The set $\Big\{v \in M(\Sigma,\mu) \;\Big|\; v \ll u\Big\}$ is $\sigma(L_0, L^{1\cap\infty})$-sequentially compact in $L_0(\Sigma,\mu)$;

(2) Let $X \subseteq M(\Sigma,\mu)$ be a normal Banach space satisfying $X \subseteq L_0(\Sigma,\mu)$ and

$$\begin{cases} \text{for every } u \in X,\ (u_n)_{n\ge1} \subseteq M(\Sigma,\mu) \text{ with } u_n \ll u \text{ for all } n \ge 1 \\ \text{and } \lim_{n\to+\infty} u_n(x) = u(x)\ \mu\text{-a.e. on } \Sigma, \text{ yields } \quad \lim_{n\to+\infty} u_n = u \text{ in } X. \end{cases} \tag{4.2}$$

Then for every $u \in X$ and sequence $(u_n)_{n\ge1} \subseteq M(\Sigma,\mu)$ satisfying

$$u_n \ll u \text{ for all } n \ge 1 \quad \text{and} \quad \lim_{n\to+\infty} u_n = u\ \sigma(L_0, L^{1\cap\infty})\text{-weakly in } X,$$

one has that

$$\lim_{n\to+\infty} u_n = u \qquad \text{in } X.$$

Note, examples of normal Banach spaces $X \subseteq L_0(\Sigma,\mu)$ satisfying (4.2) are $X = L^p(\Sigma,\mu)$ for $1 \le p < \infty$ and $L_0(\Sigma,\mu)$.

To complete this preliminary section, we state the following Proposition summarizing statements from [18], which we will need in the sequel (cf., [8] for the case $\omega = 0$).

Proposition 4.12. *For $\omega \in \mathbb{R}$, let A be ω-quasi completely accretive in $L_0(\Sigma,\mu)$.*

(1.) If there is a $\lambda_0 > 0$ such that $Rg(I+\lambda A)$ is dense in $L_0(\Sigma,\mu)$, then for the closure $\overline{A}^{L_0}$ of A in $L_0(\Sigma,\mu)$ and every normal Banach space with $X \subseteq L_0(\Sigma,\mu)$, the restriction $\overline{A}^{L_0}_X := \overline{A}^{L_0} \cap (X \times X)$ of A on X is the unique ω-quasi m-completely accretive extension of the part $A_X = A \cap (X \times X)$ of A in X.

(2.) *For a given normal Banach space $X \subseteq L_0(\Sigma,\mu)$, and $\omega \in \mathbb{R}$, suppose A is ω-quasi m-completely accretive in X, and $\{T_t\}_{t\ge 0}$ be the semigroup generated by $-A$ on $\overline{D(A)}^X$. Further, let $\{S_t\}_{t\ge 0}$ be the semigroup generated by $-\overline{A}^{L_0}$, where $\overline{A}^{L_0}$ denotes the closure of A in $\overline{X}^{L_0}$. Then, the following statements hold.*

(a) *The semigroup $\{S_t\}_{t\ge 0}$ is ω-quasi completely contractive on $\overline{D(A)}^{L_0}$, T_t is the restriction of S_t on $\overline{D(A)}^X$, S_t is the closure of T_t in $L_0(\Sigma,\mu)$, and*

$$S_t u_0 = L_0 - \lim_{n\to+\infty}\left(I + \frac{t}{n}A\right)^{-n} u_0 \quad \textit{for all } u_0 \in \overline{D(A)}^{L_0} \cap X; \tag{4.3}$$

(b) *If there exists $u \in L^{1\cap\infty}(\Sigma,\mu)$ such that the orbit $\{T_t u \,|\, t \ge 0\}$ is locally bounded on $\mathbb{R}_+$ with values in $L^{1\cap\infty}(\Sigma,\mu)$, then, for every N-function ψ, the semigroup $\{T_t\}_{t\ge 0}$ can be extrapolated to a strongly continuous, order-preserving semigroup of ω-quasi contractions on $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\psi}$ (respectively, on $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^1}$), and to an order-preserving semigroup of ω-quasi contractions on $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\infty}$. We denote each extension of T_t on on those spaces again by T_t.*

(c) *The restriction $A_X := \overline{A}^{L_0} \cap (X \times X)$ of $\overline{A}^{L_0}$ on X is the unique ω-quasi m-complete extension of A in X; that is, $A = A_X$.*

(d) *The operator A is sequentially closed in $X \times X$ equipped with the relative $(L_0(\Sigma,\mu) \times (X, \sigma(L_0, L^{1\cap\infty})))$-topology.*

(e) *The domain of A is characterized by*

$$D(A) = \left\{ u \in \overline{D(A)}^{L_0} \cap X \;\middle|\; \begin{array}{l} \exists\, v \in X \text{ such that} \\ e^{-\omega t}\frac{S_t u - u}{t} \ll v \text{ for all small } t > 0 \end{array} \right\};$$

(f) *For every $u \in D(A)$, one has that*

$$\lim_{t\to 0+} \frac{S_t u - u}{t} = -A^\circ u \qquad \textit{strongly in } L_0(\Sigma,\mu). \tag{4.4}$$

4.2. **Regularizing effect of the associated semigroup.** It is worth recalling that the Banach space $L^1(\Sigma,\mu)$ does not admit the Radon-Nikodým property. Thus, the time-derivative $\frac{\mathrm{d}}{\mathrm{d}t_+} T_t u_0(t)$, $u_0 \in L^1(\Sigma,\mu)$, of a given semigroup $\{T_t\}_{t\ge 0}$ on $L^1(\Sigma,\mu)$ does not need to exist in $L^1(\Sigma,\mu)$. But, in this section, we show that even though the underlying Banach space X is not reflexive, if the infinitesimal generator $-A$ is homogeneous of order $\alpha \neq 1$ and A is quasi-completely accretive, then the time-derivative $\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)$ exists in X. This fact follows from the following compactness result generalizing the one in [8] for $\omega = 0$.

Here, the partial ordering "$\le$" is the standard one defined by $u \le v$ for u, $v \in M(\Sigma,\mu)$ if $u(x) \le v(x)$ for μ-a.e. $x \in \Sigma$, and we write$X \hookrightarrow Y$ for indicating that the space X is continuously embedded into the space Y.

Lemma 4.13. *Let $X \subseteq L_0(\Sigma,\mu)$ be a normal Banach space satisfying (4.2). For $\omega \in \mathbb{R}$, let $\{T_t\}_{t\ge 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ of ω-quasi complete contractions satisfying (2.16) and $T_t 0 = 0$ for all $t \ge 0$. Then, for every $u_0 \in C$ and $t > 0$, the set*

$$\left\{ \frac{T_{t+h}u_0 - T_t u_0}{h} \;\middle|\; h \neq 0, t+h > 0 \right\} \tag{4.5}$$

is $\sigma(L_0, L^{1\cap\infty})$-weakly sequentially compact in $L_0(\Sigma,\mu)$.

The proof of this lemma is essentially the same as in the case $\omega = 0$ (cf., [8]). For the convenience of the reader, we include here the proof.

Proof. Let $u_0 \in C$, $t > 0$, and $h \neq 0$ such that $t + h > 0$. Then by taking $\lambda = 1 + \frac{h}{t}$ in (2.16), one sees that

$$\begin{aligned}|T_{t+h}u_0 - T_t u_0| &= |\lambda^{\frac{1}{1-\alpha}}\, T_t\left[\lambda^{\frac{1}{\alpha-1}} u_0\right] - T_t u_0| \\ &\le \lambda^{\frac{1}{1-\alpha}}\,\left|T_t\left[\lambda^{\frac{1}{\alpha-1}} u_0\right] - T_t u_0\right| + |\lambda^{\frac{1}{1-\alpha}} - 1|\,|T_t u_0|.\end{aligned}$$

Since T_t is an ω-quasi complete contraction and since $T_t 0 = 0$, $(t \geq 0)$, claim (3) and (5) of Proposition 4.2 imply that

$$\lambda^{\frac{1}{1-\alpha}}\, e^{-\omega t}\left|T_t\left[\lambda^{\frac{1}{\alpha-1}} u_0\right] - T_t u_0\right| \ll |1 - \lambda^{\frac{1}{1-\alpha}}|\,|u_0|$$

and

$$|\lambda^{\frac{1}{1-\alpha}} - 1|\, e^{-\omega t}|T_t u_0| \ll |\lambda^{\frac{1}{1-\alpha}} - 1|\,|u_0|.$$

Since the set $\{w \,|\, w \ll |\lambda^{\frac{1}{1-\alpha}} - 1|\,|u_0|\}$ is convex (cf., (6) of Proposition 4.2), the previous inequalities imply that

$$\frac{1}{2} e^{-\omega t}|T_{t+h}u_0 - T_t u_0| \ll |\lambda^{\frac{1}{1-\alpha}} - 1|\,|u_0|.$$

Using again (3) of Proposition 4.2, gives

$$\frac{|T_{t+h}u_0 - T_t u_0|}{|\lambda^{\frac{1}{1-\alpha}} - 1|} \ll 2\, e^{\omega t}\,|u_0|. \tag{4.6}$$

Since for every $u \in M(\Sigma,\mu)$, one always has that $u^+ \ll |u|$, the transitivity of "$\ll$" ((4) of Proposition 4.2) implies that

$$f_h := \frac{T_{t+h}u_0 - T_t u_0}{\lambda^{\frac{1}{1-\alpha}} - 1} \quad \text{satisfies} \quad f_h^+ \ll 2\, e^{\omega t}\,|u_0|.$$

Therefore and since $|u_0| \in X$, (1) of Proposition 4.11 yields that the two sets $\{f_h^+ | h \neq 0, t + h > 0\}$ and $\{|f_h| | h \neq 0, t + h > 0\}$ are $\sigma(L_0, L^{1\cap\infty})$- weakly sequentially compact in $L_0(\Sigma,\mu)$. Since $f_h^- = |f_h| - f_h^+$ and $f_h = f_h^+ - f_h^-$, and since $(\lambda^{\frac{1}{1-\alpha}} - 1)/h = ((1 + \frac{h}{t})^{\frac{1}{1-\alpha}} - 1)/h \to 1/t(1-\alpha) \neq 0$ as $h \to 0$, we can conclude that the claim of this lemma holds. □

With these preliminaries in mind, we can now state the regularization effect of the semigroup $\{T_t\}_{t\geq 0}$ generated by a ω-quasi m-completely accretive operator of homogeneous order $\alpha \neq 1$.

Theorem 4.14. *Let $X \subseteq L_0(\Sigma,\mu)$ be a normal Banach space satisfying (4.2), and $\|\cdot\|$ denote the norm on X. For $\omega \in \mathbb{R}$, let A be ω-quasi m-completely accretive in X, and $\{T_t\}_{t\geq 0}$ be the semigroup generated by $-A$ on $\overline{D(A)}^X$. If A is homogeneous of order $\alpha \neq 1$, then for every $u_0 \in \overline{D(A)}^X$ and $t > 0$, $\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}$ exists in X and*

$$|A^\circ T_t u_0| \le \frac{2e^{\omega t}}{|\alpha - 1|}\,\frac{|u_0|}{t} \qquad \mu\text{-a.e. on } \Sigma. \tag{4.7}$$

In particular, for every $u_0 \in \overline{D(A)}^X$,

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right\| \le \frac{2e^{\omega t}}{|\alpha - 1|} \frac{\|u_0\|}{t} \qquad \text{for every } t > 0, \tag{4.8}$$

and

$$\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \ll \frac{2e^{\omega t}}{|\alpha - 1|} \frac{|u_0|}{t} \qquad \text{for every } t > 0. \tag{4.9}$$

Proof. Let $u_0 \in \overline{D(A)}^X$, $t > 0$, and $(h_n)_{n \ge 1} \subseteq \mathbb{R}$ be a zero sequence such that $t + h_n > 0$ for all $n \ge 1$. Then, by Proposition 3.4, we can apply the compactness result stated in Lemma 4.13. Thus, there is a $z \in L_0(\Sigma, \mu)$ and a subsequence $(h_{k_n})_{n \ge 1}$ of $(h_n)_{n \ge 1}$ such that

$$\lim_{n \to \infty} \frac{T_{t+h_{k_n}} u_0 - T_t u_0}{h_{k_n}} = z \qquad \text{weakly in } L_0(\Sigma, \mu). \tag{4.10}$$

By (2e) of Proposition 4.12, one has that $(T_t u_0, -z) \in A$. Thus (2f) of Proposition 4.12 yields that $z = -A^\circ T_t u_0$ and

$$\lim_{n \to \infty} \frac{T_{t+h_{k_n}} u_0 - T_t u_0}{h_{k_n}} = -A^\circ T_t u_0 \qquad \text{strongly in } L_0(\Sigma, \mu). \tag{4.11}$$

After possibly passing to another subsequence, the limit (4.11) also holds μ-a.e. on Σ. The argument shows that the limit (4.11) is independent of the choice of the initial zero sequence $(h_n)_{n \ge 1}$. Thus

$$\lim_{h \to 0} \frac{T_{t+h} u_0 - T_t u_0}{h} = -A^\circ T_t u_0 \qquad \text{exists } \mu\text{-a.e. on } \Sigma. \tag{4.12}$$

Since $2e^{-\omega t} |u_0| \in X$, by (4.6), and since $(\lambda^{\frac{1}{1-\alpha}} - 1)/h = ((1 + \frac{h}{t})^{\frac{1}{1-\alpha}} - 1)/h \to 1/t(1-\alpha) \neq 0$ as $h \to 0$, it follows from (2) of Proposition 4.11 that

$$\lim_{h \to 0} \frac{T_{t+h} u_0 - T_t u_0}{h} = -A^\circ T_t u_0 \qquad \text{exists in } X \tag{4.13}$$

and with $\lambda = 1 + \frac{h}{t}$,

$$\frac{|T_{t+h} u_0 - T_t u_0|}{|\lambda^{\frac{1}{1-\alpha}} - 1|} \le 2\, e^{-\omega t} |u_0|$$

for all $h \neq 0$ satisfying $t + h > 0$. Sending $h \to 0$ in the last inequality and applying (4.13) gives (4.7). In particular, by Corollary 2.5, one has that (4.8) holds for the norm $\|\cdot\|_X$ on X. Moreover, (4.6) is equivalent to

$$\int_\Sigma j\left(\frac{|T_{t+h} u_0 - T_t u_0|}{|\lambda^{\frac{1}{1-\alpha}} - 1|} \right) \mathrm{d}\mu \le \int_\Sigma j\left(2\, e^{-\omega t} |u_0|\right) \mathrm{d}\mu \tag{4.14}$$

for all $h \neq 0$ satisfying $t + h > 0$, and every $j \in J_0$. By the lower semicontinuity of $j \in J_0$ and by the μ-a.e. limit (4.12), we have that

$$j\left(\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}(x)\, |\alpha - 1|\, t \right) \le \liminf_{h \to 0} j\left(\frac{|T_{t+h} u_0(x) - T_t u_0(x)|}{|\lambda^{\frac{1}{1-\alpha}} - 1|} \right)$$

for μ-a.e. $x \in \Sigma$. Thus, taking the limit inferior as $h \to 0+$ in (4.14) and applying Fatou's lemma yields

$$\int_{\Sigma} j\left(\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}(x)\,|\alpha-1|\,t\right)\,\mathrm{d}\mu \le \int_{\Sigma} j\left(2\,e^{-\omega t}\,|u_0|\right)\,\mathrm{d}\mu$$

Since $j \in J_0$ was arbitrary and by (3) of Proposition 4.2, this shows that (4.9) holds and thereby completes the proof of this theorem. ☐

5. Application

5.1. **An elliptic-parabolic boundary-value problem.** Our aim in this section is to derive global L^1 Aronson-Bénilan estimate (1.5) for $X = L^q(\partial M)$, $(1 \le q \le \infty)$, and point-wise Aronson-Bénilan estimate (1.7) on the time-derivative $\frac{\mathrm{d}u}{\mathrm{d}t}$ of any solutions u to the elliptic-parabolic boundary-value problem

$$\begin{cases} -\Delta_p u + m\,|u|^{p-2}u = 0 & \text{in } M \times (0,\infty),\\ \partial_t u + |\nabla u|_g^{p-2}\nabla u\cdot\nu + f(x,u) = 0 & \text{on } \partial M \times (0,\infty),\\ u(0) = u_0 & \text{on } \partial M. \end{cases} \tag{5.1}$$

Here, we assume that $1 < p < \infty$, Δ_p denotes the celebrated *p-Laplace-Beltrami operator*

$$\Delta_p u := \mathrm{div}\,\left(|\nabla u|_g^{p-2}\nabla u\right) \qquad \text{in } \mathcal{D}'(M) \tag{5.2}$$

for $u \in W^{1,p}(M)$ on a compact, smooth, N-dimensional Riemannian manifold (M,g) with a Lipschitz continuous boundary ∂M, $m > 0$ and $f : \partial M \times \mathbb{R} \to \mathbb{R}$ a Lipschitz-continuous Carathéodory function (see (5.7)-(5.9) below).

For applying the theory developed in the previous sections of this paper, it is worth noting that the elliptic-parabolic problem (5.1) can be rewritten in the form of the perturbed Cauchy problem (1.6) in the Banach space $X = L^q(\partial M)$, $(1 \le q \le \infty)$, where the operator A is the *Dirichlet-to-Neumann operator* realized in X associated with the operator $-\Delta_p + m\,|\cdot|^{p-2}\cdot$; that is, A assigns Dirichlet data φ on ∂M to the *co-normal derivative* $|\nabla u|_g^{p-2}\nabla u\cdot\nu$ on ∂M, where u is the unique weak solution of the *Dirichlet problem*

$$\begin{cases} -\Delta_p u + m\,|u|^{p-2}u = 0 & \text{in } M,\\ u = \varphi & \text{on } \partial M. \end{cases} \tag{5.3}$$

In the (flat) case $M = \Omega$ is a bounded domain in $\mathbb{R}^N$ with a Lipschitz-continuous boundary $\partial\Omega$, the Dirichlet-to-Neumann operator A associated with the p-Laplace-Beltrami operator Δ_p and its semigroup $\{T_t\}_{t\ge 0}$ were studied in the past by several authors (see, for instance, in [25, 17, 18] and the references therein).

5.2. **Framework.** Throughout this section, let (M,g) denote a compact, smooth, (orientable), N-dimensional Riemannian manifold with a Lipschitz continuous boundary ∂M. Let $g = \{g(x)\}_{x\in M}$ denote the corresponding Riemannian metric tensor and for every $x \in M$, T_x be the tangent space and TM the tangent bundle of M. We write $|\xi|_g = \sqrt{\langle \xi,\xi\rangle_{g(x)}}$, $(\xi \in T_x)$, to denote the induced norm of the inner product $\langle\cdot,\cdot\rangle_{g(x)}$ on the tangent space T_x. If for given $f \in C^\infty(M)$,

df is the *differential* at $x \in M$ and for every chart (Ω, ϕ), $\boldsymbol{g} = (g_{ij})_{i,j=1}^N$ is the matrix of the Riemannian metric g on Ω with inverse $\boldsymbol{g}^{-1}$, then the corresponding gradient of f at x is given by $\nabla f(x) = \boldsymbol{g}^{-1}(x) df(x)$, and for every C^1-vector field $X = (X^1, \dots, X^N)$ on M, the *divergence*

$$\operatorname{div}(X) := \tfrac{1}{\sqrt{\det(\boldsymbol{g})}} \tfrac{\partial}{\partial x_i} \left(\sqrt{\det(\boldsymbol{g})} X^i \right).$$

For given C^1-curve κ with parametrization $\gamma_\kappa : [0,1] \to M$, the length $L(\kappa)$ of κ is defined by

$$L(\kappa) = \int_0^1 \left| \tfrac{\mathrm{d}\gamma_\kappa}{\mathrm{d}t}(t) \right|_{g(\gamma_\kappa(t))} \mathrm{d}t.$$

If we denote by $C^1_{x,y}$ the space of all piecewise C^1-curves κ with starting point $\gamma_\kappa(0) = x \in M$ and end point $\gamma_\kappa(1) = y \in M$, then $\mathrm{d}_g(x,y) := \inf_{\kappa \in C^1_{x,y}} L(\kappa)$ defines a distance (called Riemannian distance) whose induced topology τ_g coincides with the original one by M. There exists a unique Borel measure μ_g defined on the Borel σ-Algebra $\sigma(\tau_g)$ such that on any chart (Ω, ϕ) of M, one has that $\mathrm{d}\mu_g = \sqrt{\det(\boldsymbol{g})}\,\mathrm{d}x$, where $\mathrm{d}x$ refers to the Lebesgue measure in Ω.

For the measure space (M, μ_g), and $1 \le q \le \infty$, we denote by $L^q(M) = L^q(M, \mu_g)$ (respectively, $L^q_{loc}(M) = L^q_{loc}(M, \mu_g)$) the classical Lebesgue space of (locally) q-integrable functions, and we denote by $\|\cdot\|_q$ its standard norm on $L^q(M)$. Since a vector field v on M is *measurable* if and only if every component of v is measurable on all charts U of M, one defines similarly for every $1 \le q \le \infty$, the space $\vec{L}^q(M) = \vec{L}^q(M, \mu_g)$ (respectively, $\vec{L}^q_{loc}(M) = \vec{L}^q_{loc}(M, \mu_g)$) of all measurable vector fields v on M such that $|v| \in L^q(M, \mu_g)$ (respectively, $|v| \in L^q_{loc}(M, \mu_g)$).

The *space of test functions* $\mathcal{D}(M)$ be the set $C^\infty_c(M)$ of smooth compactly supported functions equipped with the following type of convergence: given a sequence $(\varphi_n)_{n\ge1}$ in $C^\infty_c(M)$ and $\varphi \in C^\infty_c(M)$, we say $\varphi_n \to \varphi$ in $\mathcal{D}(M)$ if there is a compact subset K of M such that the support $\mathrm{supp}(\varphi_n) \subseteq K$ for all n, and for every chart U, and all multi-index α, one has $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on U. Then the *space of distributions* $\mathcal{D}'(M)$ is the topological dual space of $\mathcal{D}(M)$. Similarly, one defines the *space of test vector fields* $\vec{\mathcal{D}}(M)$ on M and corresponding dual space $\vec{\mathcal{D}}'(M)$ of *distributional vector fields*. Given a distribution $T \in \mathcal{D}'(M)$, the *distributional gradient* $\nabla T \in \vec{\mathcal{D}}'(M)$ is defined by

$$\langle \nabla T, \psi \rangle_{\vec{\mathcal{D}}'(M), \vec{\mathcal{D}}(M)} = -\langle T, \operatorname{div} \psi \rangle_{\mathcal{D}'(M), \mathcal{D}(M)} \qquad \text{for every } \psi \in \vec{\mathcal{D}}(M).$$

For given $u \in L^1_{loc}(M)$,

$$\langle u, \varphi \rangle_{\mathcal{D}'(M), \mathcal{D}(M)} := \int_M u\, \varphi \mathrm{d}\mu_g, \qquad \varphi \in \mathcal{D}(M),$$

defines a distribution (called *regular distribution*) on M. If the distributional gradient ∇u of the distribution u belongs to $\vec{L}^1_{loc}(M)$, then ∇u is called a *weak gradient* of u. The *first Sobolev space* $W^{1,q}(M) = W^{1,q}(M, \mu_g)$ is the space of all $u \in L^q(M)$ such that for the weak gradient ∇u of u belongs to $\vec{L}^q(M)$. The space $W^{1,q}(M)$ is a Banach space equipped with the norm

$$\|u\|_{W^{1,q}(M)} := \left(\|u\|_q + \| |\nabla u|_g \|_q \right)^{1/q}, \qquad (u \in W^{1,q}(M)), \tag{5.4}$$

and $W^{1,q}(M)$ is reflexive if $1 < q < \infty$ (cf., [28, Proposition 2.4]). Further, we denote by $W^{1,q}_0(M) = W^{1,q}_0(M,\mu_g)$ the closure of $C^\infty_c(M)$ in $W^{1,q}(M)$. Since we have assumed that (M,g) is compact, the volume $\mu_g(M)$ is finite. Hence by the compactness result of Rellich-Kondrakov (see, e.g., [28, Corollary 3.7]), we know that a Poincaré inequality on $W^{1,q}_0(M)$ is available. Thus, $\||\nabla\cdot|_g\|_q$ defines an equivalent norm to (5.4) on $W^{1,q}_0(M)$.

Further, let s_g denote the surface measure on ∂M induced by the outward pointing unit normal on ∂M. Then for $1 \le q < \infty$ and $0 < s < 1$, let $W^{s,q}(\partial M) := W^{s,q}(\partial M, s_g)$ be the Sobolev-Slobodečki space given by all measurable functions $u \in L^q(\partial M) = L^q(\partial M, s_g)$ with finite Gagliardo semi-norm

$$[u]^q_{W^{s,q}(\partial M)} := \int_{\partial M}\int_{\partial M} \tfrac{|u(x)-u(y)|^q}{d_g^{N-2+sq}(x,y)} \mathrm{d}s_g(x)\mathrm{d}s_g(y).$$

The space $W^{s,q}(\partial M)$ equipped with the norm

$$\|u\|_{W^{s,q}(\partial M)} := \left(\|u\|^q_{L^q(\partial M)} + [u]^q_{W^{s,q}(\partial M)}\right)^{1/q}$$

is a Banach space, which is reflexive if $1 < q < \infty$.

Since M is compact, M can be covered by a finite family $((\Omega_l,\phi_l))_{l=1}^K$ of charts (Ω_l,ϕ_l) such that for every $l \in \{1,\dots,K\}$, each component g_{ij} of the matrix $\mathbf{g}$ of the Riemannian metric g satisfies

$$\frac{c_l}{2}\,\delta_{ij} \le g_{ij} \le 2c_l\,\delta_{ij} \qquad \text{on } \Omega_l \tag{5.5}$$

as bilinear forms, for some constant $c_l > 0$. By using (5.5) together with a partition of unity, one can conclude from the Euclidean case (see, e.g., [32, Théorème 5.5 & Théorème 5.7]) that for $1 < q < \infty$, there is a linear bounded *trace operator* $T : W^{1,q}(M) \to W^{1-1/q,q}(\partial M)$ with kernel $\ker(T) = W^{1,q}_0(M)$ with bounded right inverse $Z : W^{1-1/q,q}(\partial M) \to W^{1,q}(M)$. For simplicity, we also write $u_{|\partial M}$ for the trace $T(u)$ of $u \in W^{1,q}(M)$ and $\|u_{|\partial M}\|_q$ instead of $\|T(u)\|^q_{L^q(\partial M)}$.

Similarly, one transfers from the Euclidean case (cf., [32, Théorème 4.2]) the *Sobolev-trace inequality*

$$\|u_{|\partial M}\|_{\frac{q(N-1)}{(N-q)}} \lesssim \|u\|_{W^{1,q}(M)}, \qquad u \in W^{1,q}(M). \tag{5.6}$$

5.3. **Construction of the Dirichlet-to-Neumann operator.** Let $1 < p < \infty$ and $m \ge 0$. Then, by the classical theory of convex minimization (see, e.g., [25]), for every boundary data $\varphi \in W^{1-1/p,p}(\partial M)$, there is a unique *weak solution* $u \in (M)$ of the Dirichlet problem (5.3) (cf., [25]).

Definition 5.1. For given boundary data $\varphi \in W^{1-1/p,p}$, a function $u \in W^{1,p}(M)$ is called a *weak solution* of Dirichlet problem (5.3) if $Z\varphi - u \in W^{1,p}_0$ and

$$\int_M |\nabla u|_g^{p-2}\nabla u \nabla\psi + m\,|u|^{p-2}u\psi \mathrm{d}\mu_g = 0$$

for every $\psi \in C^\infty_c(M)$.

Now, we are in the position to define the nonlocal *Dirichlet-to-Neumann operator* A in $L^2 := L^2(\partial M)$ associated with the p-Laplace Beltrami operator Δ_p by

$$A = \left\{ (\varphi, h) \in L^2 \times L^2 \;\middle|\; \begin{array}{l} \exists\, u \in V_{p,2}(M, \partial M) \text{ with trace } u_{|\partial M} = \varphi \\ \text{satisfying } \forall\, \psi \in V_{p,2}(M,\partial M) : \\ \displaystyle\int_M |\nabla u|_g^{p-2} \nabla u \nabla \psi + m\, |u|^{p-2} u \psi \,\mathrm{d}\mu_g = \int_{\partial M} h\, \psi_{|\partial M} \,\mathrm{d}s_g \end{array} \right\}.$$

In the operator A, we denote by $V_{p,2}(M, \partial M)$ the set of all $u \in W^{1,p}(M)$ with trace $u_{|\partial M} \in L^2(\partial M)$. Note, the space $V_{p,2}(M, \partial M)$ contains the function space $C^\infty(\overline{M})$. It follows from the theory developed in [17] that A is the T-subdifferential operator $\partial_T \mathcal{E}$ in L^2 (cf., [17]) of the convex, continuously differentiable, and T-elliptic functional $\mathcal{E} : W^{1,p}(M) \to [0, +\infty)$ defined by

$$\mathcal{E}(u) := \tfrac{1}{p} \int_M \left(|\nabla u|_g^p + m\, |u|^p \right) \,\mathrm{d}\mu_g$$

for every $u \in V_{p,2}(M, \partial M)$. Thus, A is a maximal monotone operator with dense domain in the Hilbert space $L^2(\partial M)$. One immediately sees that A is homogeneous of order $\alpha = p - 1$.

Next, suppose $f : \partial M \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous *Carathéodory* function, that is, f satisfies the following three properties:

(5.7) • $f(\cdot, u) : \partial M \to \mathbb{R}$ is measurable on ∂M for every $u \in \mathbb{R}$,

(5.8) • $f(x, 0) = 0$ for a.e. $x \in \partial M$, and

• there is a constant $\omega \geq 0$ such that

$$|f(x,u) - f(x,\hat{u})| \leq \omega\, |u - \hat{u}| \qquad \text{for all } u, \hat{u} \in \mathbb{R}, \text{ a.e. } x \in \partial M. \tag{5.9}$$

Then, for every $1 \leq q \leq \infty$, $F : L^q(\partial M) \to L^q(\partial M)$ defined by

$$F(u)(x) := f(x, u(x)) \qquad \text{for every } u \in L^q(\partial M)$$

is the associated *Nemytskii operator* on $L^q := L^q(\partial M)$. Moreover, by (5.9), F is globally Lipschitz continuous on $L^q(\partial M)$ with constant $\omega \geq 0$ and $F(0)(x) = 0$ for a.e. $x \in \partial M$.

Under these assumptions, it follows from Proposition 4.7 that the perturbed operator $A + F$ in $L^2(\partial M)$ is an ω-quasi m-completely accretive operator with dense domain $D(A + F) = D(A)$ in $L^2(\partial M)$ (see [25] or [18] for the details in the Euclidean case). Thus, $-(A + F)$ generates a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ of Lipschitz-continuous mappings T_t on $L^2(\partial M)$ with Lipschitz constant $e^{\omega t}$. For every $1 \leq q < \infty$, each T_t admits a unique Lipschitz-continuous extension $T_t^{(q)}$ on $L^q(\partial M)$ with Lipschitz constant $e^{\omega t}$ such that $\{T_t^{(q)}\}_{t\geq 0}$ is a strongly continuous semigroup on $L^q(\partial M)$, and each $T_t^{(q)}$ is Lipschitz-continuous on $\overline{L^2(\partial M) \cap L^\infty(\partial M)}^{L^\infty}$ with respect to the L^∞-norm. According to Proposition 4.12, for

$$\overline{A}^{L_0}_{L^q(\partial M)} := \overline{A}^{L_0} \cap (L^q(\partial M) \times L^q(\partial M))$$

with $L_0 := L_0(\partial M, s_g)$, the operator $-(\overline{A}^{L_0}_{L^q(\partial M)} + F)$ is the unique infinitesimal generator of $\{T_t^{(q)}\}_{t\ge 0}$ in L^q. Since

$$\overline{A}^{L_0}_{L^q(\partial M)} u = Au \cap L^q(\partial M) \tag{5.10}$$

for every $u \in D(A) \cap L^q(\partial M)$, we call $\overline{A}^{L_0}_{L^q(\partial M)}$ the *realization in* $L^q(\partial M)$ of the *Dirichlet-to-Neumann operator* A associated with the p-Laplace Beltrami operator Δ_p.

For simplicity, we denote the extension $T_t^{(q)}$ on $L^q(\partial M)$ of T_t again by T_t.

It is worth noting that for $1 < p < N$, the semigroup $\{T_t\}_{t\ge 0}$ generated by $-(A+F)$ has an immediate regularization effect. Indeed, by the Sobolev-trace inequality (5.6), the operator $A+F$ satisfies the inequality

$$\begin{aligned}
\left[u_{|\partial M}, (A+F)u_{|\partial M}\right]_2 + \omega \, \|u_{|\partial M}\|_2^2 &= \int_\Omega |\nabla u|_g^p + m\,|u|^p \, \mathrm{d}\mu_g \\
&\quad + \int_{\partial M} f(x,u)u + \omega\,|u|^2 \, \mathrm{d}s_g \\
&\ge \int_\Omega |\nabla u|_g^p + m\,|u|^p \, \mathrm{d}\mu_g \\
&\ge \min\{1,m\}\, \|u\|^p_{W^{1,p}(M)} \\
&\ge C\, \|u_{|\partial M}\|^p_{\frac{p(N-1)}{(N-p)}}
\end{aligned}$$

for every $u \in D(A)$, where $[\cdot,\cdot]_2$ denotes the duality brackets on $L^2(\partial M)$, and $C > 0$ is a constant including $\min\{1,m\}$ and the constant of the Sobolev-trance inequality. By [18, Theorem 1.2], the semigroup $\{T_t\}_{t\ge 0}$ satisfies

$$\|T_t u_0\|_{L^{\frac{p(N-1)}{(N-p)}}(\partial M)} \le \left(\tfrac{C}{2}\right)^{\frac{1}{p}} t^{\frac{1}{p}}\, e^{\omega(\frac{2}{p}+1)t} \|u_0\|^{\frac{2}{p}}_{L^2(\partial M)} \tag{5.11}$$

for all $t > 0$ and $u_0 \in L^2(\partial M)$. Moreover, by (5.11) and since $\{T_t\}_{t\ge 0}$ has unique Lipschitz-continuous extension on $L^1(\partial M)$, the same theorem infers that the semigroup $\{T_t\}_{t\ge 0}$ satisfies for every $1 \le q \le (N-1)\, q_0/(N-p)$ satisfying $q > (2-p)(N-1)/(p-1)$ the following L^q-L^∞*-regularity estimate*

$$\|T_t u_0\|_{L^\infty(\partial M)} \lesssim t^{-\alpha_q}\, e^{\omega\beta_q t}\, \|u_0\|^{\gamma_q}_{L^q(\partial M)} \tag{5.12}$$

for every $t > 0$, $u_0 \in L^q(\partial M)$, with exponents

$$\alpha_q = \frac{\alpha^*}{1 - \gamma^*\left(1 - \frac{q(N-p)}{(N-1)q_0}\right)}, \qquad \beta_q = \frac{\frac{\beta^*}{2} + \gamma^* \frac{q(N-p)}{(d-1)q_0}}{1 - \gamma^*\left(1 - \frac{q(N-p)}{(N-1)q_0}\right)},$$
$$\gamma_q = \frac{\gamma^*\, q(N-p)}{(N-1)q_0\left(1 - \gamma^*\left(1 - \frac{q(N-p)}{(N-1)q_0}\right)\right)},$$

where $q_0 \ge p$ is chosen (minimal) such that $\left(\frac{N-1}{N-p}-1\right) q_0 + p - 2 > 0$ and

$$\alpha^* := \frac{N-p}{(p-1)\,q_0 + (N-p)(p-2)}, \quad \beta^* := \frac{(\frac{2}{p}-1)N + p - \frac{2}{p}}{(p-1)q_0 + (N-p)(p-2)} + 1,$$
$$\gamma^* := \frac{(p-1)\,q_0}{(p-1)\,q_0 + (N-p)(p-2)}.$$

5.4. **Global regularity estimates on $\frac{du}{dt}$.** Throughout this subsection, let $p \in (1,\infty) \setminus \{2\}$. Since, the Dirichlet-to-Neumann operator A in $L^2(\partial M)$ is homogeneous of order $\alpha = p-1$, identity (5.10) yields that for every $1 \le q < \infty$, the realization $\overline{A}^{L_0}_{L^q(\partial M)}$ of A in $L^q(\partial M)$ is also homogeneous of order $p-1$. Thus, by Corollary 1.2, for every $1 < q < \infty$ and $u_0 \in L^q(\partial M)$, the function $u(t) = T_t u_0$is differentiable a.e. on $(0,\infty)$ and satisfies

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right\|_{L^q(\partial M)} \le \frac{\|u_0\|_{L^q(\partial M)}}{|p-2|\,t} \left[1 + e^{\omega t} + \omega \int_0^t (1+e^{\omega s})\, e^{\omega(t-s)} \mathrm{d}s\right]$$

for every $t > 0$. Note, the right hand side of this estimate can be rearranged as follows

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right\|_{L^q(\partial M)} \le \frac{[2+\omega\, t]\; e^{\omega t}}{|p-2|\,t} \|u_0\|_{L^q(\partial M)} \tag{5.13}$$

for every $t > 0$. Since the boundary ∂M is compact, Hölder's inequality gives

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right\|_{L^1(\partial M)} \le \boldsymbol{s}_g^{1/q'}(\partial M) \frac{[2+\omega\, t]\; e^{\omega t}}{|p-2|\,t} \|u_0\|_{L^q(\partial M)}$$

for every $q > 1$ and hence, if we fix $u_0 \in L^2(\partial M)$, then sending $q \to 1+$ in the above inequality shows that (5.13), in particular, holds for $q = 1$. By (5.12), for either $p > 2$ or $(2N-1)/N < p < 2$, one has that $u_0 \in L^1(\partial M)$ yields that $T_t u_0 \in L^\infty(\partial M) \hookrightarrow L^2(\partial M)$ for all $t > 0$. Hence, for this range of p, we get that (5.13) hold for $q = 1$ and $u_0 \in L^1(\partial M)$.

Next, let $u_0 \in L^\infty(\partial M)$ and $t > 0$. We assume $\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\|_{L^\infty(\partial M)} > 0$ (otherwise, there is nothing to show). Then, for every $s \in (0, \|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\|_{L^\infty(\partial M)})$ and $2 \le q < \infty$, Chebyshev's inequality yields

$$\boldsymbol{s}_g\left(\left\{\left|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\right| \ge s\right\}\right)^{1/q} \le \frac{\left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\right\|_{L^q(\partial M)}}{s}$$

and so, by (5.13),

$$s\,\boldsymbol{s}_g\left(\left\{\left|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}\right| \ge s\right\}\right)^{1/q} \le \frac{[2+\omega\, t]\; e^{\omega t}}{|p-2|\,t} \|u_0\|_{L^q(\partial M)}$$

Thus and since $\lim_{q\to\infty} \|u_0\|_{L^q(\partial M)} = \|u_0\|_{L^\infty(\partial M)}$, sending $q \to +\infty$ in the last inequality, yields

$$s \le \frac{[2+\omega\, t]\; e^{\omega t}}{|p-2|\,t} \|u_0\|_{L^\infty(\partial M)}$$

and since $s \in (0, \left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\right\|_{L^\infty(\partial M)})$ was arbitrary, we have thereby shown that (5.13) also holds for $q = \infty$.

Finally, for $p \in (1, N) \setminus \{2\}$, we can apply Corollary 2.8 or, alternatively, combine (5.12) with (5.13) for $q = \infty$. Then, we find that

$$\left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\right\|_{L^\infty(\partial M)} \lesssim \frac{2\,\left[2 + \frac{\omega}{2}\,t\right]\,e^{\omega\,(1+\frac{\beta_q}{2})t}}{|p-2|\,t^{\alpha_q+1}}\,\|u_0\|_{L^q(\partial M)}^{\gamma_q} \tag{5.14}$$

for every $t > 0$, $u_0 \in L^q(\partial M)$, and $1 \le q \le (N-1)\,q_0/(N-p)$ satisfying $q > (2-p)(N-1)/(p-1)$.

By this computation together with Theorem 1.3, we can state the following regularity result on mild solutions to the elliptic-parabolic problem (5.1).

Theorem 5.2. *Let $N \ge 2$ and $1 < p < \infty$. Then every mild solution u of the elliptic-parabolic problem (5.1) admits the following additional regularity.*

(1) (L^1 Aronson-Bénilan type estimates) *If either $(2N-1)/N < p < 2$ or $p > 2$, then for every $1 \le q \le \infty$ and $u_0 \in L^q(\partial\Omega)$, the mild solution $u(t) := T_t u_0$ of the elliptic-parabolic problem (5.1) is differentiable for a.e. $t > 0$, is a* strong solution *in $L^q(\partial\Omega)$ of (5.1), and satisfies*

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)\right\|_{L^q(\partial M)} \le \frac{[2+\omega\,t]\;e^{\omega t}}{|p-2|\,t}\|u_0\|_{L^q(\partial M)} \qquad \text{for every } t > 0.$$

(2) (Extrapolated L^1 Aronson-Bénilan type estimates) *Let $p \in (1, N) \setminus \{2\}$. Then, in addition to statement (1), for every $1 \le q \le (N-1)\,q_0/(N-p)$ satisfying $q > (2-p)(N-1)/(p-1)$ and $u_0 \in L^q(\partial\Omega)$, the mild solution $u(t) := T_t u_0$ of the elliptic-parabolic problem (5.1) satisfies*

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)\right\|_{L^\infty(\partial M)} \lesssim \frac{2\,\left[2 + \frac{\omega}{2}\,t\right]\,e^{\omega\,(1+\frac{\beta_q}{2})t}}{|p-2|\,t^{\alpha_q+1}}\,\|u_0\|_{L^q(\partial M)}^{\gamma_q} \qquad \text{for every } t > 0.$$

(3) (Point-wise Aronson-Bénilan type estimates) *If either $(2N-1)/N < p < 2$ or $p > 2$, then for every $1 \le q \le \infty$ and positive $u_0 \in L^q(\partial\Omega)$, the strong solution u of problem (5.1) satisfies*

$$(p-2)\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) \ge -\frac{u(t)}{t} + (p-2)\,g_0(t),$$

for a.e. $t > 0$, where $g_0 : (0, \infty) \to L^q(\partial\Omega)$ is a measurable function.

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