$\mathcal{W}$-algebras associated with centralizers in type A

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Abstract

We introduce a new family of affine $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{a})$ associated with the centralizers of arbitrary nilpotent elements in $\mathfrak{gl}_N$. We define them by using a version of the BRST complex of the quantum Drinfeld–Sokolov reduction. A family of free generators of $\mathcal{W}^k(\mathfrak{a})$ is produced in an explicit form. We also give an analogue of the Fateev–Lukyanov realization for the new $\mathcal{W}$-algebras by applying a Miura-type map.

1 Introduction

The *affine $\mathcal{W}$-algebra* $\mathcal{W}^k(\mathfrak{g})$ *at the level* $k \in \mathbb{C}$ associated with a simple Lie algebra $\mathfrak{g}$ is a vertex algebra defined by a quantum Drinfeld–Sokolov reduction [8]. These algebras originate in conformal field theory and first appeared in the work of Zamolodchikov [17] and Fateev and Lukyanov [7]. They were intensively studied both in mathematics and physics literature; see e.g. [1], [2], [5], [9, Ch. 15] for detailed reviews. More general $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ were introduced in [11], which depend on a simple Lie (super)algebra $\mathfrak{g}$ and an (even) nilpotent element $f \in \mathfrak{g}$ so that $\mathcal{W}^k(\mathfrak{g})$ corresponds to a principal nilpotent element f. Their counterparts for odd nilpotent elements f were studied in [12] and [15] from the viewpoint of quantum hamiltonian reduction.

Our goal in this paper is to introduce and describe some basic properties of $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{a})$, where the underlying Lie algebra $\mathfrak{a}$ is the centralizer of a nilpotent element e in $\mathfrak{gl}_N$. In the case $e = 0$ the corresponding algebra coincides with the principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{gl}_N)$.

We follow [4] to equip the Lie algebra $\mathfrak{a}$ with an invariant symmetric bilinear form and introduce the corresponding affine Kac–Moody algebra $\widehat{\mathfrak{a}}$. Its vacuum module $V^k(\mathfrak{a})$ at the level k is a vertex algebra. The Lie algebra $\mathfrak{a}$ admits a triangular decomposition $\mathfrak{a} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ which gives rise to a Clifford algebra associated with $\mathfrak{n}_+$ and we let $\mathcal{F}$ be its vacuum module. As with the case of simple Lie algebras [9, Ch. 15], the vertex algebra $C^k(\mathfrak{a}) = V^k(\mathfrak{a}) \otimes \mathcal{F}$ acquires a structure of a BRST complex of the quantum Drinfeld–Sokolov reduction. We show that its cohomology $H^k(\mathfrak{a})^i$ is zero for all degrees $i \neq 0$ and define the $\mathcal{W}$-algebra by setting $\mathcal{W}^k(\mathfrak{a}) = H^k(\mathfrak{a})^0$.

Furthermore, we give an explicit construction of free generators of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$. In the particular case $e = 0$ they coincide with those previously found in [3]. Similar to this particular case, by taking the limit $k \to \infty$ we get a commutative algebra isomorphic to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{a})$ introduced in [14], which is also isomorphic to the center of the vertex algebra $V^k(\mathfrak{a})$ at the critical level $k = -N$ as described in [4] and [13]. On the other hand, the quantum Miura map applied to the generators of $\mathcal{W}^k(\mathfrak{a})$ yields its realization as a subalgebra of the vertex algebra $V^{k+N}(\mathfrak{h})$ associated with the diagonal subalgebra $\mathfrak{h}$ of $\mathfrak{a}$. In the case $e = 0$ we recover the corresponding realization [7] of $\mathcal{W}^k(\mathfrak{gl}_N)$ as in [3]; see also [2].

Note that in the particular case where all Jordan blocks of the nilpotent e are of the same size, the Lie algebra $\mathfrak{a}$ is isomorphic to a truncated polynomial current algebra of the form $\mathfrak{gl}_n[v]/(v^p = 0)$, which is also known as the Takiff algebra. This leads to a natural generalization of our definition of the $\mathcal{W}$-algebras to the class of Takiff algebras $\mathfrak{g}[v]/(v^p = 0)$ associated with an arbitrary simple Lie algebra $\mathfrak{g}$.

2 BRST cohomology for centralizers

Here we adapt the well-known BRST construction of vertex algebras to the case of centralizers in type A. We generally follow [1, Sec. 4] and [9, Ch. 15] with some straightforward modifications.

Let $e \in \mathfrak{gl}_N$ be a nilpotent matrix and let $\mathfrak{a}$ be the centralizer of e in $\mathfrak{gl}_N$. Suppose that the Jordan canonical form of e has Jordan blocks of sizes $\lambda_1, \dots, \lambda_n$, where $\lambda_1 \leqslant \dots \leqslant \lambda_n$ and $\lambda_1 + \dots + \lambda_n = N$. The corresponding *pyramid* is a left-justified array of rows of unit boxes such that the top row contains λ_1 boxes, the next row contains λ_2 boxes, etc. Denote by $q_1 \geqslant \dots \geqslant q_l$ the column lengths of the pyramid (with $l = \lambda_n$). The *row-tableau* is obtained by writing the numbers $1, \dots, N$ into the boxes of the pyramid consecutively by rows from left to right. For instance, the row-tableau

1	2		
3	4	5	
6	7	8	9

corresponds to the pyramid with the rows of lengths $2, 3, 4$; its column lengths are $3, 3, 2, 1$. We let $\mathrm{row}(a)$ and $\mathrm{col}(a)$ denote the row and column number of the box containing the entry a. Let e_{ab} be the standard basis elements of $\mathfrak{gl}_N$. For any $1 \leqslant i, j \leqslant n$ and $\lambda_j - \min(\lambda_i, \lambda_j) \leqslant r < \lambda_j$ set

$$E_{ij}^{(r)} = \sum_{\substack{\mathrm{row}(a)=i,\ \mathrm{row}(b)=j \\ \mathrm{col}(b)-\mathrm{col}(a)=r}} e_{ab}, \tag{2.1}$$

summed over $a, b \in \{1, \dots, N\}$. It is well-known that the elements $E_{ij}^{(r)}$ form a basis of the Lie algebra $\mathfrak{a}$; see e.g. [6] and [16]. The commutation relations are given by

$$\left[E_{ij}^{(r)}, E_{hl}^{(s)}\right] = \delta_{hj}\, E_{il}^{(r+s)} - \delta_{il}\, E_{hj}^{(r+s)},$$

assuming that $E_{ij}^{(r)} = 0$ for $r \geqslant \lambda_j$.

2.1 Affine vertex algebra

The Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$ gets a $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{r\in\mathbb{Z}} \mathfrak{g}_r$ determined by e such that the degree of the basis element e_{ab} equals $\mathrm{col}(b) - \mathrm{col}(a)$. We thus get an induced $\mathbb{Z}$-gradation $\mathfrak{a} = \bigoplus_{r\in\mathbb{Z}} \mathfrak{a}_r$

on the Lie algebra $\mathfrak{a}$, where $\mathfrak{a}_r = \mathfrak{a} \cap \mathfrak{g}_r$. Note that the element (2.1) is homogeneous of degree r. The subalgebra $\mathfrak{g}_0$ is isomorphic to the direct sum

$$\mathfrak{g}_0 \cong \mathfrak{gl}_{q_1} \oplus \cdots \oplus \mathfrak{gl}_{q_l}. \tag{2.2}$$

Equip this subalgebra with the normalized Killing form

$$\langle X, Y \rangle = \frac{1}{2N} \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y), \qquad X, Y \in \mathfrak{g}_0. \tag{2.3}$$

Now define an invariant symmetric bilinear form on $\mathfrak{a}$ following [4]. The value $\langle X, Y \rangle$ for homogeneous elements $X, Y \in \mathfrak{a}$ is found by (2.3) for $X, Y \in \mathfrak{a}_0$, and is zero otherwise. Writing $X = X_1 + \cdots + X_l$ and $Y = Y_1 + \cdots + Y_l$ in accordance with the decomposition (2.2), we get

$$\langle X, Y \rangle = \frac{1}{N} \sum_{i=1}^{l} \Big(q_i \operatorname{tr} X_i Y_i - \operatorname{tr} X_i \operatorname{tr} Y_i \Big).$$

Therefore, if $\lambda_i = \lambda_j$ for some $i \neq j$ then

$$\Big\langle E_{ij}^{(0)}, E_{ji}^{(0)} \Big\rangle = \frac{1}{N} (q_1 + \cdots + q_{\lambda_i}) = \frac{1}{N} \Big(\lambda_1 + \cdots + \lambda_{i-1} + (n - i + 1)\lambda_i \Big),$$

and for all i and j we have

$$\Big\langle E_{ii}^{(0)}, E_{jj}^{(0)} \Big\rangle = \frac{1}{N} \Big(\delta_{ij}(\lambda_1 + \cdots + \lambda_{i-1} + (n - i + 1)\lambda_i) - \min(\lambda_i, \lambda_j) \Big),$$

whereas all remaining values of the form on the basis vectors are zero.

The affine Kac–Moody algebra $\widehat{\mathfrak{a}}$ is the central extension $\widehat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}K$, where $\mathfrak{a}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in $\mathfrak{a}$. For any $r \in \mathbb{Z}$ and $X \in \mathfrak{g}$ we will write $X[m] = X\, t^m$. The commutation relations of the Lie algebra $\widehat{\mathfrak{a}}$ have the form

$$\Big[X[m], Y[p] \Big] = [X, Y][m + p] + m\, \delta_{m,-p} \langle X, Y \rangle\, K, \qquad X, Y \in \mathfrak{a},$$

and the element K is central in $\widehat{\mathfrak{a}}$. The vacuum module at the level $k \in \mathbb{C}$ over $\widehat{\mathfrak{a}}$ is the quotient

$$V^k(\mathfrak{a}) = \mathrm{U}(\widehat{\mathfrak{a}})/\mathrm{I},$$

where I is the left ideal of $\mathrm{U}(\widehat{\mathfrak{a}})$ generated by $\mathfrak{a}[t]$ and the element $K - k$. This module is equipped with a vertex algebra structure and is known as the (*universal*) *affine vertex algebra* associated with $\mathfrak{a}$; see [9], [10]. The vacuum vector is the image of the element 1 in the quotient and we will denote it by $|0\rangle$. Furthermore, introduce the fields

$$E_{ij}^{(r)}(z) = \sum_{m \in \mathbb{Z}} E_{ij}^{(r)}[m]\, z^{-m-1} \in \operatorname{End} V^k(\mathfrak{a})[[z, z^{-1}]]$$

so that under the state-field correspondence map we have

$$Y : E_{ij}^{(r)}[-1]|0\rangle \mapsto E_{ij}^{(r)}(z).$$

The map Y extends to the whole of $V^k(\mathfrak{a})$ with the use of normal ordering. The translation operator T on $V^k(\mathfrak{a})$ is determined by the properties

$$T : |0\rangle \mapsto 0 \qquad \text{and} \qquad \Big[T, X[m] \Big] = -m X[m-1], \quad X \in \mathfrak{a}, \quad m < 0, \tag{2.4}$$

where $X[m]$ is understood as the operator of left multiplication by $X[m]$.

2.2 Affine Clifford algebra

Consider the following triangular decomposition of the Lie algebra $\mathfrak{a}$,

$$\mathfrak{a} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{2.5}$$

where the subalgebras are defined by

$$\mathfrak{n}_- = \text{span of } \{E_{ij}^{(r)} \,|\, i > j\}, \quad \mathfrak{n}_+ = \text{span of } \{E_{ij}^{(r)} \,|\, i < j\} \quad \text{and} \quad \mathfrak{h} = \text{span of } \{E_{ii}^{(r)}\},$$

with the superscript r ranging over all admissible values. Denote by $\mathcal{C}l$ the Clifford algebra associated with $\mathfrak{n}_+[t, t^{-1}]$, so it is generated by odd elements $\psi_{ij}^{(r)}[m]$ and $\psi_{ij}^{(r)*}[m]$ with the parameters satisfying the conditions $1 \leqslant i < j \leqslant n$ together with $\lambda_j - \lambda_i \leqslant r \leqslant \lambda_j - 1$ and $m \in \mathbb{Z}$. The defining relations are given by the anti-commutation relations

$$\Big[\psi_{ij}^{(r)}[m], \psi_{ij}^{(r)*}[-m]\Big] = 1,$$

while all other pairs of generators anti-commute. Let $\mathcal{F}$ be the Fock representation of $\mathcal{C}l$ generated by a vector $\mathbf{1}$ such that

$$\psi_{ij}^{(r)}[m]\,\mathbf{1} = 0 \quad \text{for} \quad m \geqslant 0 \qquad \text{and} \qquad \psi_{ij}^{(r)*}[m]\,\mathbf{1} = 0 \quad \text{for} \quad m > 0.$$

The space $\mathcal{F}$ is a vertex algebra with the vacuum vector $\mathbf{1}$, and the translation operator T is determined by the properties $T : \mathbf{1} \mapsto 0$ and

$$\Big[T, \psi_{ij}^{(r)}[m]\Big] = -m\,\psi_{ij}^{(r)}[m-1], \qquad \Big[T, \psi_{ij}^{(r)*}[m]\Big] = -(m-1)\,\psi_{ij}^{(r)*}[m-1].$$

The fields are defined by

$$\psi_{ij}^{(r)}(z) = \sum_{m \in \mathbb{Z}} \psi_{ij}^{(r)}[m]\, z^{-m-1} \qquad \text{and} \qquad \psi_{ij}^{(r)*}(z) = \sum_{m \in \mathbb{Z}} \psi_{ij}^{(r)*}[m]\, z^{-m}$$

so that

$$Y : \psi_{ij}^{(r)}[-1]\,\mathbf{1} \mapsto \psi_{ij}^{(r)}(z) \qquad \text{and} \qquad Y : \psi_{ij}^{(r)*}[0]\,\mathbf{1} \mapsto \psi_{ij}^{(r)*}(z).$$

The vertex algebra $\mathcal{F}$ has a $\mathbb{Z}$-gradation $\mathcal{F} = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}^i$, defined by

$$\deg \mathbf{1} = 0, \qquad \deg \psi_{ij}^{(r)}[m] = -1 \qquad \text{and} \qquad \deg \psi_{ij}^{(r)*}[m] = 1.$$

2.3 BRST complex

Introduce the vertex algebra $C^k(\mathfrak{a})$ as the tensor product

$$C^k(\mathfrak{a}) = V^k(\mathfrak{a}) \otimes \mathcal{F}.$$

We will use notation $|0\rangle$ for its vacuum vector $|0\rangle \otimes \mathbf{1}$. The vertex algebra $C^k(\mathfrak{a})$ is $\mathbb{Z}$-graded, its i-th component has the form

$$C^k(\mathfrak{a})^i = V^k(\mathfrak{a}) \otimes \mathcal{F}^i.$$

Consider the fields $Q(z)$ and $\chi(z)$ defined by

$$Q(z)=\sum_{i<j}E_{ij}^{(a)}(z)\psi_{ij}^{(a)*}(z)-\sum_{i<j<h}\psi_{ij}^{(a)*}(z)\psi_{jh}^{(b)*}(z)\psi_{ih}^{(a+b)}(z), \tag{2.6}$$

and

$$\chi(z)=\sum_{i=1}^{n-1}\psi_{i\,i+1}^{(\lambda_{i+1}-1)*}(z). \tag{2.7}$$

To simplify the formulas, here and throughout the paper we use the convention that summation over all admissible values of repeated superscripts of the form a,b,c is assumed. For instance, summation over a running over the values $\lambda_j-\lambda_i,\dots,\lambda_j-1$ is assumed within the first sum in (2.6). Define the odd endomorphisms $d_{\rm st}$ and χ of $C^k(\mathfrak{a})$ as the residues (coefficients of z^{-1}) of the fields (2.6) and (2.7),

$$d_{\rm st}=Q_{(0)}\qquad\text{and}\qquad \chi=\sum_{i=1}^{n-1}\psi_{i\,i+1}^{(\lambda_{i+1}-1)*}[1].$$

Lemma 2.1. *We have the relations*

$$d_{\rm st}^2=\chi^2=[d_{\rm st},\chi]=0.$$

Proof. The relations are verified by the standard OPE calculus with the use of the Taylor formula and Wick theorem [10]. Using the basic OPEs

$$E_{ij}^{(r)}(z)E_{hl}^{(s)}(w)\sim\frac{1}{z-w}\Big(\delta_{hj}\,E_{il}^{(r+s)}(w)-\delta_{il}\,E_{hj}^{(r+s)}(w)\Big)+\frac{k\,\langle E_{ij}^{(r)},E_{hl}^{(s)}\rangle}{(z-w)^2}, \tag{2.8}$$

and

$$\psi_{ij}^{(r)}(z)\,\psi_{ij}^{(r)*}(w)\sim\frac{1}{z-w},\qquad \psi_{ij}^{(r)*}(z)\,\psi_{ij}^{(r)}(w)\sim\frac{1}{z-w}, \tag{2.9}$$

we find that the OPE $Q(z)Q(w)$ is regular, thus implying that $d_{\rm st}^2=0$. The remaining relations are straightforward to verify. □

By Lemma 2.1, the odd endomorphism $d=d_{\rm st}+\chi$ of $C^k(\mathfrak{a})$ has the properties $d^2=0$ and $d:C^k(\mathfrak{a})^i\to C^k(\mathfrak{a})^{i+1}$. We thus get an analogue $(C^k(\mathfrak{a})^{\bullet},d)$ of the BRST complex of the quantum Drinfeld–Sokolov reduction, associated with the Lie algebra $\mathfrak{a}$; cf. [9, Ch. 15]. Since d is a residue of a vertex operator, the cohomology $H^k(\mathfrak{a})^{\bullet}$ of the complex is a vertex algebra which we will use to define and describe the $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{a})$.

3 $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{a})$

Introduce another $\mathbb{Z}$-gradation on $C^k(\mathfrak{a})^{\bullet}$ by defining the (conformal) degrees by

$$\deg' E_{ij}^{(r)}[m]=\deg'\psi_{ij}^{(r)}[m]=-m+i-j\qquad\text{and}\qquad \deg'\psi_{ij}^{(r)*}[m]=-m+j-i.$$

Observe that the differential d has degree 0 and so it preserves this gradation thus defining a $\mathbb{Z}$-gradation on the cohomology $H^k(\mathfrak{a})^{\bullet}$.

Definition 3.1. The $\mathbb{Z}$-graded vertex algebra $H^k(\mathfrak{a})^0$ is called the *$\mathcal{W}$-algebra associated with the centralizer $\mathfrak{a}$ at the level k* and denoted by $\mathcal{W}^k(\mathfrak{a})$. □

Our next goal is to prove the following analogue of [9, Thm 15.1.9] describing the structure of principal $\mathcal{W}$-algebras associated with simple Lie algebras.

Theorem 3.2. *The $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$ is strongly generated by elements $w_1, \dots, w_N$ of the respective degrees*

$$\underbrace{1, \dots, 1}_{\lambda_n}, \underbrace{2, \dots, 2}_{\lambda_{n-1}}, \dots, \underbrace{n, \dots, n}_{\lambda_1}.$$

Moreover, $H^k(\mathfrak{a})^i = 0$ for all $i \neq 0$.

The proof relies on essentially the same arguments as in [9, Ch. 15] (see also [1, Sec. 4]) which we will outline in the rest of this section. A family of generators $w_1, \dots, w_N$ will be produced in Sec. 4.

For all $1 \leqslant i < j \leqslant n$ and $r = \lambda_j - \lambda_i, \dots, \lambda_j - 1$ introduce the fields

$$e_{ij}^{(r)}(z) = E_{ij}^{(r)}(z) + \sum_{h>j} \psi_{ih}^{(a)}(z)\psi_{jh}^{(a-r)*}(z) - \sum_{h<i} \psi_{hj}^{(a)}(z)\psi_{hi}^{(a-r)*}(z), \tag{3.1}$$

where we keep using the convention on the summation over a as in (2.6). Similarly, for $i \geqslant j$ and $r = 0, 1, \dots, \lambda_j - 1$ set

$$e_{ij}^{(r)}(z) = E_{ij}^{(r)}(z) + \sum_{h>i} : \psi_{ih}^{(a)}(z)\psi_{jh}^{(a-r)*}(z) : - \sum_{h<j} : \psi_{hj}^{(a)}(z)\psi_{hi}^{(a-r)*}(z) : . \tag{3.2}$$

Note that by the defining relations in the Clifford algebra $\mathcal{C}l$, the normal ordering is necessary only for the case where $i = j$ and $r = 0$. Introduce Fourier coefficients $e_{ij}^{(r)}[m]$ of the fields (3.1) and (3.2) by setting

$$e_{ij}^{(r)}(z) = \sum_{m \in \mathbb{Z}} e_{ij}^{(r)}[m]\, z^{-m-1}.$$

In the formulas of the next lemmas we assume that the fields with out-of-range parameters are equal to zero.

Lemma 3.3. *(i) For $i \geqslant j$ and $h < l$ we have*

$$\Big[e_{ij}^{(r)}[m], \psi_{hl}^{(s)*}[p]\Big] = \delta_{lj}\, \psi_{hi}^{(s-r)*}[m+p] - \delta_{hi}\, \psi_{jl}^{(s-r)*}[m+p]. \tag{3.3}$$

Moreover, if $i \geqslant j$ and $h \geqslant l$ then

$$\Big[e_{ij}^{(r)}[m], e_{hl}^{(s)}[p]\Big] = \delta_{hj}\, e_{il}^{(r+s)}[m+p] - \delta_{il}\, e_{hj}^{(r+s)}[m+p] + m\, \delta_{m,-p}(k+N)\langle E_{ij}^{(r)}, E_{hl}^{(s)}\rangle.$$

(ii) For $i < j$ and $h < l$ we have

$$\Big[e_{ij}^{(r)}[m], \psi_{hl}^{(s)}[p]\Big] = \delta_{hj}\, \psi_{il}^{(r+s)}[m+p] - \delta_{il}\, \psi_{kj}^{(r+s)}[m+p]$$

and

$$\Big[e_{ij}^{(r)}[m], e_{hl}^{(s)}[p]\Big] = \delta_{hj}\, e_{il}^{(r+s)}[m+p] - \delta_{il}\, e_{hj}^{(r+s)}[m+p].$$

Proof. All relations are easily verified with the use of the OPEs (2.8) and (2.9). □

For all $i = 1, \dots, n$ set

$$\alpha_i = -\lambda_i + \frac{k+N}{N}\big(\lambda_1 + \dots + \lambda_{i-1} + (n-i+1)\lambda_i\big). \tag{3.4}$$

Lemma 3.4. *The following relations hold for all $i \geqslant j$:*

$$\begin{aligned}\Big[d_{\text{st}}, e_{ij}^{(r)}(z)\Big] &= \sum_{h=j}^{i-1} :e_{hj}^{(a+r)}(z)\,\psi_{hi}^{(a)*}(z): - \sum_{h=j+1}^{i} :\psi_{jh}^{(a)*}(z)\,e_{ih}^{(a+r)}(z): + \alpha_j\,\delta_{r0}\,\partial_z\psi_{ji}^{(0)*}(z),\\ \Big[\chi, e_{ij}^{(r)}(z)\Big] &= \psi_{j\,i+1}^{(\lambda_{i+1}-r-1)*}(z) - \psi_{j-1\,i}^{(\lambda_j-r-1)*}(z).\end{aligned} \tag{3.5}$$

Moreover, for all $i < j$ we have

$$\Big[d_{\text{st}}, e_{ij}^{(r)}(z)\Big] = 0, \qquad \Big[\chi, e_{ij}^{(r)}(z)\Big] = 0,$$
$$\Big[d_{\text{st}}, \psi_{ij}^{(r)}(z)\Big] = e_{ij}^{(r)}(z), \qquad \Big[\chi, \psi_{ij}^{(r)}(z)\Big] = \delta_{i\,j-1}\,\delta_{r\,\lambda_j-1},$$

and

$$\Big[d_{\text{st}}, \psi_{ij}^{(r)*}(z)\Big] = -\sum_{i<h<j} \psi_{ih}^{(a)*}(z)\,\psi_{hj}^{(r-a)*}(z), \qquad \Big[\chi, \psi_{ij}^{(r)*}(z)\Big] = 0.$$

Proof. All relations are verified by using the OPEs (2.8) and (2.9). We give some details for the proof of the first relation. As a first step, by a direct computation with the use of the Wick theorem we get the OPE

$$\begin{aligned}Q(z)\,e_{ij}^{(r)}(w) \sim \frac{1}{z-w}\Big(\sum_{h=j}^{i-1} :e_{hj}^{(a+r)}(w)\,\psi_{hi}^{(a)*}(w): - \sum_{h=j+1}^{i} :e_{ih}^{(a+r)}(w)\,\psi_{jh}^{(a)*}(w):\Big)\\ + \frac{1}{(z-w)^2}\,\delta_{r0}\,\Big(k\langle E_{ij}^{(0)}, E_{ji}^{(0)}\rangle + \lambda_1 + \dots + \lambda_{j-1} + (n-i)\lambda_j\Big)\,\psi_{ji}^{(0)*}(z),\end{aligned}$$

where the term $\psi_{ji}^{(0)*}(z)$ is nonzero only if $j < i$ and $\lambda_i = \lambda_j$. Relation (3.3) of Lemma 3.3 implies (assuming summation over a) that

$$:e_{ih}^{(a+r)}(w)\,\psi_{jh}^{(a)*}(w): = :\psi_{jh}^{(a)*}(w)\,e_{ih}^{(a+r)}(w): + \delta_{r0}\,\lambda_j\,\partial_w\psi_{ji}^{(0)*}(w).$$

The required relation now follows by applying the Taylor formula to $\psi_{ji}^{(0)*}(z)$ to write

$$\psi_{ji}^{(0)*}(z) = \psi_{ji}^{(0)*}(w) + (z-w)\,\partial_w\psi_{ji}^{(0)*}(w) + \dots,$$

and then by taking the residue over z in the resulting expressions. □

Denote by $C^k(\mathfrak{a})_0$ the subspace of $C^k(\mathfrak{a})$ spanned by all vectors of the form

$$e_{i_1j_1}^{(r_1)}[m_1]\dots e_{i_qj_q}^{(r_q)}[m_q]\,\psi_{h_1l_1}^{(s_1)*}[p_1]\dots\psi_{h_tl_t}^{(s_t)*}[p_t]\,|0\rangle, \qquad i_a \geqslant j_a, \qquad h_a < l_a,$$

and by $C^k(\mathfrak{a})_+$ the subspace of $C^k(\mathfrak{a})$ spanned by all vectors of the form

$$e_{i_1j_1}^{(r_1)}[m_1]\dots e_{i_qj_q}^{(r_q)}[m_q]\,\psi_{h_1l_1}^{(s_1)}[p_1]\dots\psi_{h_tl_t}^{(s_t)}[p_t]\,|0\rangle, \qquad i_a < j_a, \qquad h_a < l_a.$$

By Lemma 3.3, both $C^k(\mathfrak{a})_0$ and $C^k(\mathfrak{a})_+$ are vertex subalgebras of $C^k(\mathfrak{a})$. Furthermore, by Lemma 3.4 each of the subalgebras is preserved by the differential $d = d_{\text{st}} + \chi$. This implies the tensor product decomposition of complexes

$$C^k(\mathfrak{a})^\bullet \cong C^k(\mathfrak{a})_0^\bullet \otimes C^k(\mathfrak{a})_+^\bullet.$$

Hence the cohomology of $C^k(\mathfrak{a})^\bullet$ is isomorphic to the tensor product of the cohomologies of $C^k(\mathfrak{a})_0^\bullet$ and $C^k(\mathfrak{a})_+^\bullet$.

By Lemma 3.4, for $i < j$ we have

$$\big[d, e_{ij}^{(r)}[m]\big] = 0, \qquad \big[d, \psi_{ij}^{(r)}[m]\big] = e_{ij}^{(r)}[m] + \delta_{i\,j-1}\,\delta_{r\,\lambda_j-1}\delta_{m,-1}.$$

Therefore, the complex $C^k(\mathfrak{a})_+^\bullet$ has no higher cohomologies, while its zeroth cohomology is one-dimensional; see [9, Sec 15.2.6]. So the cohomology of $C^k(\mathfrak{a})^\bullet$ is isomorphic to the cohomology of the complex $C^k(\mathfrak{a})_0^\bullet$. To calculate the latter, equip this complex with a double gradation by setting

$$\operatorname{bideg} e_{ij}^{(r)}[m] = (i-j, j-i), \qquad \operatorname{bideg} \psi_{ij}^{(r)*}[m] = (j-i, i-j+1).$$

Then $C^k(\mathfrak{a})_0^\bullet$ acquires a structure of bicomplex with $\operatorname{bideg}\chi = (1,0)$ and $\operatorname{bideg} d_{\text{st}} = (0,1)$. Take χ as the zeroth differential of the associated spectral sequence and d_{st} as the first. Next we compute the cohomology of $C^k(\mathfrak{a})_0^\bullet$ with respect to χ.

Consider the linear span of all fields $e_{ij}^{(r)}(z)$ with $i \geqslant j$ and $r = 0, 1, \dots, \lambda_j - 1$. We will choose a new basis of this vector space which is formed by the fields

$$P_l^{(r)}(z) = e_{n\,n-l+1}^{(r)}(z) + e_{n-1\,n-l}^{(r+\lambda_n-\lambda_2)}(z) + \dots + e_{l\,1}^{(r+\lambda_n+\dots+\lambda_{l+1}-\lambda_{n-l+1}-\dots-\lambda_2)}(z)$$

for $l = 1, \dots, n$ and $r = 0, 1, \dots, \lambda_{n-l+1} - 1$ together with

$$I_{ij}^{(r)}(z) = \sum_{h=1}^{i} e_{j-h\,i-h+1}^{(r+\lambda_{j-1}+\dots+\lambda_{j-h+1}-\lambda_i-\dots-\lambda_{i-h+2})}(z)$$

for $i < j$ and $r = 0, 1, \dots, \lambda_i - 1$. The following properties of the new basis vectors are immediate from (3.5).

Lemma 3.5. *We have the relations*

$$\big[\chi, P_l^{(r)}(z)\big] = 0 \qquad \textit{and} \qquad \big[\chi, I_{ij}^{(r)}(z)\big] = \psi_{ij}^{(\lambda_j-r-1)*}(z).$$

□

Lemma 3.5 allows us to apply the arguments of [9, Sec. 15.2.9] to conclude that all higher cohomologies of the complex $C^k(\mathfrak{a})_0^\bullet$ with respect to χ vanish, while the zeroth cohomology is the commutative vertex subalgebra of $C^k(\mathfrak{a})_0^\bullet$ spanned by all monomials

$$P_{l_1}^{(r_1)}[m_1]\dots P_{l_q}^{(r_q)}[m_q]\,|0\rangle, \tag{3.6}$$

where we use the Fourier coefficients $P_l^{(r)}[m]$ defined by

$$P_l^{(r)}(z) = \sum_{m\in\mathbb{Z}} P_l^{(r)}[m]\,z^{-m-1}. \tag{3.7}$$

By a standard procedure outlined in [9, Sec. 15.2.11], each element of this subalgebra gives rise to a cocycle in the complex $C^k(\mathfrak{a})_0^\bullet$ with the differential d. Moreover, the cocycles $W_l^{(r)}$ corresponding to the vectors $P_l^{(r)}[-1]\,|0\rangle$ with $l = 1,\dots,n$ and $r = 0,1,\dots,\lambda_{n-l+1}-1$ strongly generate the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$. The proof of Theorem 3.2 is completed by the observation that the conformal degree of the generator $W_l^{(r)}$ equals l.

4 Generators of $\mathcal{W}^k(\mathfrak{a})$

For an $n\times n$ matrix $A = [a_{ij}]$ with entries in a ring we will consider its *column-determinant* defined by

$$\operatorname{cdet} A = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma\cdot a_{\sigma(1)\,1}\dots a_{\sigma(n)\,n}.$$

We will produce generators of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$ as elements of the vertex algebra $C^k(\mathfrak{a})_0$. Combine the Fourier coefficients $e_{ij}^{(r)}[-1]\in\operatorname{End} C^k(\mathfrak{a})_0$ into polynomials in a variable u by setting

$$e_{ij}(u) = \sum_{r=0}^{\lambda_j-1} e_{ij}^{(r)}[-1]\,u^r, \qquad i\geqslant j.$$

Let x be another variable and consider the matrix

$$\mathcal{E} = \begin{bmatrix} x+\alpha_1T+e_{11}(u) & -u^{\lambda_2-1} & 0 & \dots & 0\\ e_{21}(u) & x+\alpha_2T+e_{22}(u) & -u^{\lambda_3-1} & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & -u^{\lambda_n-1}\\ e_{n1}(u) & e_{n2}(u) & \dots & \dots & x+\alpha_nT+e_{nn}(u)\end{bmatrix},$$

where the constants α_i are defined in (3.4). Its column-determinant is a polynomial in x of the form

$$\operatorname{cdet}\mathcal{E} = x^n + w_1(u)\,x^{n-1}+\dots+w_n(u), \qquad w_l(u) = \sum_r w_l^{(r)}\,u^r, \tag{4.1}$$

so that the coefficients $w_l^{(r)}$ are endomorphisms of $C^k(\mathfrak{a})_0$.

The particular case $e = 0$ of the following theorem (that is, with $\lambda_1 = \dots = \lambda_n = 1$) is contained in [3, Thm 2.1].

Theorem 4.1. *All elements* $w_l^{(r)}|0\rangle$ *with* $l = 1, \dots, n$ *and*

$$\lambda_{n-l+2} + \dots + \lambda_n < r + l \leqslant \lambda_{n-l+1} + \dots + \lambda_n \tag{4.2}$$

belong to the $\mathcal{W}$*-algebra* $\mathcal{W}^k(\mathfrak{a})$. *Moreover, the* $\mathcal{W}$*-algebra is strongly generated by these elements.*

Proof. The first part of the theorem will follow if we show that the elements $w_l^{(r)}|0\rangle \in C^k(\mathfrak{a})_0$ are annihilated by the differential d. To verify this property, it will be convenient to identify $C^k(\mathfrak{a})_0$ with an isomorphic vertex algebra $\widetilde{V}^k(\mathfrak{a})$ defined as follows; cf. [3]. Consider the Lie superalgebra

$$\Big(\mathfrak{b}[t,t^{-1}] \oplus \mathbb{C}K\Big) \oplus \mathfrak{m}[t,t^{-1}], \tag{4.3}$$

where the Lie algebra $\mathfrak{b}$ is spanned by the vectors $e_{ij}^{(r)}$ with $i \geqslant j$ and $r = 0, 1, \dots, \lambda_j - 1$ understood as basis elements of the low triangular part $\mathfrak{n}_- \oplus \mathfrak{h}$ in the decomposition (2.5) via the identification $e_{ij}^{(r)} \rightsquigarrow E_{ij}^{(r)}$, the even element K is central and $\mathfrak{m}$ is the supercommutative Lie superalgebra spanned by (abstract) odd elements $\psi_{ij}^{(r)*}$ with $i < j$ and $r = \lambda_j - \lambda_i, \dots, \lambda_j - 1$. The even component of the Lie superalgebra (4.3) is the Kac–Moody affinization $\mathfrak{b}[t,t^{-1}] \oplus \mathbb{C}K$ of $\mathfrak{b}$ with the commutation relations given by

$$\Big[e_{ij}^{(r)}[m], e_{hl}^{(s)}[p]\Big] = \delta_{hj}\, e_{il}^{(r+s)}[m+p] - \delta_{il}\, e_{hj}^{(r+s)}[m+p] + m\, \delta_{m,-p} K \,\langle E_{ij}^{(r)}, E_{hl}^{(s)}\rangle,$$

where the element $e_{ij}^{(r)}[m]$ is now understood as the vector $e_{ij}^{(r)} t^m$. The remaining commutation relations coincide with those in (3.3), where $\psi_{ij}^{(r)*}[m]$ is understood as the vector $\psi_{ij}^{(r)*} t^{m-1}$. Now define $\widetilde{V}^k(\mathfrak{a})$ as the representation of the Lie superalgebra (4.3) induced from the one-dimensional representation of $(\mathfrak{b}[t] \oplus \mathbb{C}K) \oplus \mathfrak{m}[t]$ on which $\mathfrak{b}[t]$ and $\mathfrak{m}[t]$ act trivially and K acts as $k + N$. Then $\widetilde{V}^k(\mathfrak{a})$ is a vertex algebra isomorphic to $C^k(\mathfrak{a})_0$ so that the fields with the same names respectively correspond to each other. Moreover, the cyclic span of the vacuum vector over the Lie algebra $\mathfrak{b}[t,t^{-1}] \oplus \mathbb{C}K$ is a subalgebra of the vertex algebra $\widetilde{V}^k(\mathfrak{a})$ isomorphic to the vacuum module $V^{k+N}(\mathfrak{b})$.

Observe that the coefficients $w_l^{(r)}$ defined in (4.1) can now be understood as elements of the universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{b}[t^{-1}])$. As a vertex algebra, $\widetilde{V}^k(\mathfrak{a})$ is equipped with the (-1)-product, and each Fourier coefficient $e_{ij}^{(r)}[m]$ with $m < 0$ can be regarded as the operator of left (-1)-multiplication by the vector $e_{ij}^{(r)}[m]|0\rangle$, and which is the same as the left multiplication by the element $e_{ij}^{(r)}[m]$ in the algebra $\mathrm{U}(t^{-1}\mathfrak{b}[t^{-1}])$. Therefore, the monomials in the elements $e_{ij}^{(r)}[m]$ which occur in the expansion of the column-determinant $\operatorname{cdet}\mathcal{E}$ can be regarded as the corresponding (-1)-products calculated consecutively from right to left, starting from the vacuum vector.

By Lemma 3.4, for $i \geqslant j$ we have the relations

$$\Big[d, e_{ij}^{(r)}[-1]\Big] = \sum_{h=j}^{i-1} e_{hj}^{(a+r)}[-1]\psi_{hi}^{(a)*}[0] - \sum_{h=j+1}^{i} \psi_{jh}^{(a)*}[0] e_{ih}^{(a+r)}[-1]$$
$$+ \psi_{j\,i+1}^{(\lambda_{i+1}-r-1)*}[0] - \psi_{j-1\,i}^{(\lambda_j-r-1)*}[0] + \alpha_j\, \delta_{r0}\, \psi_{ji}^{(0)*}[-1].$$

Introducing the Laurent polynomials

$$\phi_{ij} = \sum_{r=\lambda_i-\lambda_j}^{\lambda_i-1} \psi_{ji}^{(r)*}[0]\, u^{-r}, \qquad i > j,$$

we can write the relations in the form

$$\Big[d, e_{ij}(u)\Big] = \Big\{ \sum_{h=j}^{i-1} e_{hj}(u)\, \phi_{ih} - \sum_{h=j+1}^{i} \phi_{hj}\, e_{ih}(u) + \phi_{i+1\, j}\, u^{\lambda_{i+1}-1} - \phi_{i\, j-1}\, u^{\lambda_j-1} + \alpha_j T\, \phi_{ij} \Big\}_+ ,$$

where the symbol $\{\dots\}_+$ indicates the component of a Laurent polynomial containing only nonnegative powers of u,

$$\Big\{ \sum_i c_i\, u^i \Big\}_+ = \sum_{i\geqslant 0} c_i\, u^i.$$

Let $\mathcal{E}_{ij}$ denote the (i, j) entry of the matrix $\mathcal{E}$. Since d commutes with the translation operator T, we come to the commutation relations

$$\Big[d, \mathcal{E}_{ij}\Big] = \Big\{ \sum_{h=j}^{i-1} \mathcal{E}_{hj}\, \phi_{ih} - \sum_{h=j+1}^{i} \phi_{hj}\, \mathcal{E}_{ih} + \phi_{i+1\, j}\, u^{\lambda_{i+1}-1} - \phi_{i\, j-1}\, u^{\lambda_j-1} \Big\}_+ , \tag{4.4}$$

which hold for $i \geqslant j$. The column-determinant of $\mathcal{E}$ can be written explicitly in the form[1]

$$\operatorname{cdet} \mathcal{E} = \sum_{p=0}^{n-1} \sum_{0=i_0<i_1<\dots<i_p<i_{p+1}=n} \mathcal{E}_{i_1 i_0+1}\, \mathcal{E}_{i_2\, i_1+1} \dots \mathcal{E}_{i_{p+1}\, i_p+1}\, u^{\lambda_{j_1}-1+\dots+\lambda_{j_q}-1},$$

where $\{j_1, \dots, j_q\}$ is the complement to the subset $\{i_0+1, \dots, i_p+1\}$ in the set $\{1, \dots, n\}$. Since d is the residue of a vertex operator, d is a derivation of the (-1)-product on $\widetilde{V}^k(\mathfrak{a})$. Hence, using (4.4), we get

$$\begin{aligned}\Big[d, \operatorname{cdet} \mathcal{E}\Big] = \sum_{p=0}^{n-1} \sum_{0=i_0<i_1<\dots<i_p<i_{p+1}=n} \sum_{s=0}^{p} \mathcal{E}_{i_1 i_0+1} \dots \mathcal{E}_{i_s\, i_{s-1}+1}& \\ \times \Big\{ \sum_{i_s<i'_{s+1}<i_{s+1}} \mathcal{E}_{i'_{s+1}\, i_s+1}\, \phi_{i_{s+1}\, i'_{s+1}} - \sum_{i_s<i'_s<i_{s+1}} \phi_{i'_s+1\, i_s+1}\, \mathcal{E}_{i_{s+1}\, i'_s+1}& \\ + \phi_{i_{s+1}+1\, i_s+1}\, u^{\lambda_{i_{s+1}+1}-1} - \phi_{i_{s+1}\, i_s}\, u^{\lambda_{i_s+1}-1} \Big\}_+& \\ \times \mathcal{E}_{i_{s+2}\, i_{s+1}+1} \dots \mathcal{E}_{i_{p+1}\, i_p+1}\, u^{\lambda_{j_1}-1+\dots+\lambda_{j_q}-1}&.\end{aligned}$$

Now apply the quasi-associativity property of the (-1)-product [10, Ch. 4],

$$(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c) + \sum_{j\geqslant 0} a_{(-j-2)}(b_{(j)}c) + \sum_{j\geqslant 0} b_{(-j-2)}(a_{(j)}c),$$

[1] This also shows that it coincides with the row-determinant of $\mathcal{E}$ defined in a similar way.

to bring the expression to the right-normalized form, where the consecutive (-1)-products are calculated from right to left. Note that by Lemma 3.3(i) the additional terms coming from the sums over $j \geqslant 0$ annihilate the vacuum vector because all arising commutators involve elements with distinct subscripts.

Regarding the above expansion of $[d, \operatorname{cdet} \mathcal{E}]$ as being written in the right-normalized form, observe that if we ignore all symbols $\{\dots\}_+$, then it would turn into a telescoping sum and so would be identically zero.

As a next step, for a fixed value $l \in \{1, \dots, n\}$ consider the terms in the expansion of $[d, \operatorname{cdet} \mathcal{E}]$ containing the variable x with the powers at least $n - l$. Such terms can occur only in those summands where the cardinality of the subset $\{i_0 + 1, \dots, i_p + 1\}$ is at least $n - l + 1$. Therefore, the maximum value of the powers $\lambda_{j_1} - 1 + \dots + \lambda_{j_q} - 1$ of the variable u which occur in these terms in the expansion, equals $\lambda_{n-l+2} + \dots + \lambda_n - l + 1$. This means that the coefficients of the powers of u exceeding $\lambda_{n-l+2} + \dots + \lambda_n - l$ can be calculated from the expansion $[d, \operatorname{cdet} \mathcal{E}]$ with all symbols $\{\dots\}_+$ omitted. However, as we observed above, this expansion is identically zero. It is clear from (4.1) that the degree of the polynomial $w_l(u)$ equals $\lambda_{n-l+1} + \dots + \lambda_n - l$ so that the relations $d w_l^{(r)} |0\rangle = 0$ hold for the parameters r and l satisfying the conditions of the theorem.

To show that the vectors $w_l^{(r)} |0\rangle$ are strong generators of $\mathcal{W}^k(\mathfrak{a})$, consider the gradation on $\mathrm{U}(t^{-1}\mathfrak{b}[t^{-1}])$ defined by setting the degree of $e_{ij}^{(r)}[m]$ equal to $j - i$. It is clear from the formulas for the column-determinant $\operatorname{cdet} \mathcal{E}$ that the lowest degree component of the vector $w_l^{(r)} |0\rangle$ with $r = r' + \lambda_{n-l+2} + \dots + \lambda_n - l + 1$ coincides with $P_l^{(r')}[-1] |0\rangle$ for all $r' = 0, 1, \dots, \lambda_{n-l+1} - 1$, as defined in (3.7). Therefore, by the argument completing the proof of Theorem 3.2 at the end of Sec. 3, the vector $w_l^{(r)} |0\rangle$ coincides with the respective cocycle $W_l^{(r')}$. □

As was observed in the proof of Theorem 4.1, the lowest degree components of the generators $w_l^{(r)} |0\rangle$ generate a commutative vertex subalgebra of $C^k(\mathfrak{a})_0^{\bullet}$ spanned by all monomials (3.6). Hence, using the terminology of [1, Sec. 3.6], we come to the following; cf. [1, Prop. 4.12.1].

Corollary 4.2. *The linear span of vectors $w_l^{(r)} |0\rangle$ satisfying (4.2) generates a PBW basis of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$.* □

5 Miura map and Fateev–Lukyanov realization

Consider the affine Kac–Moody algebra $\widehat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C} K$ associated with $\mathfrak{h}$ and the bilinear form defined in Sec. 2.1. Its generators are elements $e_{ii}^{(r)}[m]$ with $i = 1, \dots, n$, where r runs over the set $0, 1, \dots, \lambda_i - 1$ and m runs over $\mathbb{Z}$. The element K is central and the commutation relations are given by the OPEs

$$e_{ii}^{(r)}(z) e_{jj}^{(s)}(w) \sim \frac{K \langle E_{ii}^{(r)}, E_{jj}^{(s)} \rangle}{(z - w)^2},$$

where we set

$$e_{ii}^{(r)}(z) = \sum_{m \in \mathbb{Z}} e_{ii}^{(r)}[m] \, z^{-m-1}.$$

Define the vacuum module $V^{k+N}(\mathfrak{h})$ over $\widehat{\mathfrak{h}}$ as the representation induced from the one-dimensional representation of $\mathfrak{h}[t] \oplus \mathbb{C}K$ on which $\mathfrak{h}[t]$ acts trivially and K acts as $k+N$. Then $V^{k+N}(\mathfrak{h})$ is a vertex algebra with the vacuum vector $|0\rangle$ and translation operator T defined as in (2.4) for $X \in \mathfrak{h}$. Recalling the constants α_i introduced in (3.4), expand the product

$$\big(x + \alpha_1 T + e_{11}(u)\big) \dots \big(x + \alpha_n T + e_{nn}(u)\big) = x^n + v_1(u)x^{n-1} + \dots + v_n(u)$$

and define the coefficients $v_l^{(r)}$ by writing $v_l(u) = \sum_r v_l^{(r)} u^r$.

The particular case $e = 0$ of the following proposition is the realization of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{gl}_n)$ given by Fateev and Lukyanov [7]; see also [3].

Proposition 5.1. *The elements $v_l^{(r)}|0\rangle$ with $l = 1, \dots, n$ and r satisfying (4.2) generate a subalgebra of the vertex algebra $V^{k+N}(\mathfrak{h})$, isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$.*

Proof. The Lie algebra projection $\mathfrak{b} \to \mathfrak{h}$ with the kernel $\mathfrak{n}_-$ induces the vertex algebra homomorphism $V^{k+N}(\mathfrak{b}) \to V^{k+N}(\mathfrak{h})$. As we have seen in the previous section, the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$ can be regarded a subalgebra of the vertex algebra $V^{k+N}(\mathfrak{b})$. Hence, we get a vertex algebra homomorphism

$$\Upsilon : \mathcal{W}^k(\mathfrak{a}) \to V^{k+N}(\mathfrak{h}),$$

obtained by restriction, which we can call the *Miura map*; cf. [2, Sec. 5.9], [9, Sec. 15.4]. For the image of the column-determinant we have

$$\Upsilon : \operatorname{cdet} \mathcal{E} \mapsto \big(x + \alpha_1 T + e_{11}(u)\big) \dots \big(x + \alpha_n T + e_{nn}(u)\big).$$

Therefore, the images of the generators of $\mathcal{W}^k(\mathfrak{a})$ under the Miura map are found by

$$\Upsilon : w_l^{(r)}|0\rangle \mapsto v_l^{(r)}|0\rangle.$$

It was shown in the proof of [14, Prop. 4.3] that all elements $T^s v_l^{(r)} \in \mathrm{U}(t^{-1}\mathfrak{h}[t^{-1}])$, where $s \geqslant 0$ and $l = 1, \dots, n$ with r satisfying conditions (4.2), are algebraically independent. In view of Corollary 4.2, this implies that the Miura map is injective. Therefore, its image is a vertex subalgebra of $V^{k+N}(\mathfrak{b})$ isomorphic to $\mathcal{W}^k(\mathfrak{a})$ which is strongly generated by the elements $v_l^{(r)}|0\rangle$ satisfying conditions (4.2). □

After re-scaling the elements $e_{ij}^{(r)}[m] \mapsto k^{-1} e_{ij}^{(r)}[m]$ and letting $k \to \infty$ the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{a})$ turns into a commutative vertex algebra isomorphic to the classical $\mathcal{W}$-algebra introduced in [14].

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